On Decentralized Control Algorithms for Multipacket Aloha

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ABSTRACT. Decentralized control algorithms for the infinite-user slotted collision channel with multipacket reception capability are considered. In order to find the optimal control, the case with perfect state information where the users know the value of the backlog is studied first, and the optimum throughput achievable by decentralized control protocols is obtained. Those results are then applied to derive control schemes when the backlog is unknown, which is usually the case in practice. Assuming that the receiver cannot demodulate correctly more than K simultaneous transmissions, we analyze an optimal algorithm which is based on binary feedback.

1. The model

Consider an infinite-user slotted collision channel with the multipacket reception capability described in [4]. This model is useful to study multiuser communication systems where the simultaneous transmission of several packets does not necessarily result in the destruction of all the transmitted information. These include systems with capture ([5], [10]), code division multiple access [13] and multiuser detectors ([16]). It is assumed that the number of successful transmissions in each slot is a random variable, which, given the number of packets simultaneously transmitted, is independent of the backlog and of the number of attempts at transmission packets might have made. Given that n packets are simultaneously transmitted in a slot, ϵ_{nk} is the probability that k are correctly received $(0 \le k \le n)$; the reception matrix is defined as $(\epsilon_{nk})_{n\ge 1, 0\le k\le n}$. It has been proved in [4] that the Aloha random access algorithm with such a channel has a maximum stable throughput $\eta_0 = \lim_{n\to\infty} C_n$, where $C_n = \sum_{k=1}^{n} k\epsilon_{nk}$ is the average number of packets correctly received in collisions of size n, and the limit η_0 is assumed to exist.

Decentralized control strategies have been shown ([6], [7], [14]) to stabilize the slotted Aloha algorithm with the usual collision channel, and in this paper we study the maximum stable throughput achievable by those strategies in the multipacket reception channel. Let us consider schemes of the form

$$\begin{aligned} p_n &= F(S_n) \\ S_{n+1} &= G(S_n, Z_n) \end{aligned} \tag{1}$$

where p_n is the retransmission probability in slot n, S_n is an estimate of the backlog X_n at the beginning of slot n, and Z_n is the feedback at the end of slot n. The number of new packets arriving during slot n, $(A_n)_{n\ge 0}$, is assumed to be a sequence of i.i.d. random variables with probability distribution $P(A_n = k) = \lambda_k \quad (k \ge 0)$, such that the mean arrival rate $\lambda = \sum_{n=1}^{\infty} n\lambda_n$ is finite. Each of the A_{n-1} new packets that arrived during slot $n-1$ is transmitted in slot n with probability p_n.

As in the case of conventional collision channels, it is useful to study first the case of control with perfect state information where the value of the backlog is known prior to the selection of the retransmission probability. Even though this situation is not relevant in practical systems, its study, which is carried out in Section 2, is motivated by the fact that in all random acces systems analyzed so far, the throughput achievable with channel feedback equals that achievable with perfect state information. In Section 3, we drop the assumption on perfect state information and restrict our attention to multipacket channels where the number of successful transmissions per slot cannot exceed some fixed integer K. The estimate of the backlog studied in Section 3 is recursively updated using a binary feedback which indicates only whether each slot was empty or not. It is shown that the throughput achievable with this type of feedback is the same as the perfect state information throughput.

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2. Control of multipacket Aloha with perfect state information

In this section we let the retransmission probability be a function of the exact value of the backlog at the beginning of slot n, i.e., $p_n = F(X_n)$. In this ideal case, the system is much simpler to analyze, since $(X_n)_{n\geq 0}$ is a homogeneous Markov chain. Our goal is to determine the optimal control F^* that maximizes the stability region, and the corresponding throughput η_c. For instance, it is well known [3] that for the usual collision channel with the delayed first transmission rule that we are using here, $F^*(X_n) = 1/X_n$ is the optimal retransmission probability, resulting in an ideal throughput of $\eta_c = e^{-1}$. In reality the users do not know the value of the backlog, but the results of this section give an upper bound which is actually tight on the maximum stable throughput of any control scheme of the form (1). They also suggest the use of certainty-equivalence retransmission probability of the form $p_n = F^*(S_n)$ in the partial state information case.

We first determine the throughput of the system if we simply set $p_n = A/X_n$, which is analogous to the optimal retransmission probability for the usual collision channel, and then we show that this is in fact optimal, provided that the constant A is properly chosen.

Lemma 1. If A is any positive constant and if $p_n = A/X_n$ for $X_n \geq A$, then the system is stable for $\lambda < t(A)$ and unstable for $\lambda > t(A)$ with $t(A) = e^{-A} \sum_{n=1}^{\infty} C_n \frac{A^n}{n!}$.

Proof. $(X_n)_{n\geq 0}$ is a homogeneous Markov chain which evolves according to

$$X_{n+1} = X_n + A_n - \Sigma_n \tag{2}$$

where Σ_n is the number of packets successfully transmitted in slot n. The system is defined to be stable if $(X_n)_{n\geq 0}$ is ergodic and unstable otherwise. Let d_i be the drift of X_n at state i: $d_i = E[X_{n+1} - X_n \mid X_n = i]$. We have $0 \leq \Sigma_n \leq X_n$, and for $i \geq \max\{1, A\}$

$$P[\Sigma_n = k \mid X_n = i] = \sum_{j=k}^{i} \binom{i}{j} \left(\frac{A}{i}\right)^j \left(1-\frac{A}{i}\right)^{i-j} \epsilon_{jk} \quad (0 \leq k \leq i)$$

It then follows from (2) that for $i \geq \max\{1, A\}$

$$d_i = \lambda - \sum_{k=1}^{i} k \sum_{j=k}^{i} \binom{i}{j} \left(\frac{A}{i}\right)^j \left(1-\frac{A}{i}\right)^{i-j} \epsilon_{jk} = \lambda - \sum_{j=1}^{i} \binom{i}{j} \left(\frac{A}{i}\right)^j \left(1-\frac{A}{i}\right)^{i-j} C_j \tag{3}$$

We now show that the drift limit is $\lambda - t(A)$. We have for $i \geq M+1$:

$$|d_i - (\lambda - t(A))| \leq \sum_{j=1}^{M} \left| \frac{i(i-1)\cdots(i-j+1)}{i^j} \left(1-\frac{A}{i}\right)^{i-j} - e^{-A} \right| \frac{A^j}{j!} C_j$$
$$+ \sum_{j=M+1}^{i} \frac{i(i-1)\cdots(i-j+1)}{i^j} \left(1-\frac{A}{i}\right)^{i-j} \frac{A^j}{j!} C_j + e^{-A} \sum_{j=M+1}^{\infty} \frac{A^j}{j!} C_j$$

Let T_1, T_2 and T_3 be the three terms on the right hand side of this equation, and ϵ any positive real. Since $T_2 + T_3 \leq 2 \sum_{j=M+1}^{\infty} \frac{A^j}{j!} C_j$, we can fix M such that $T_2 + T_3 < \epsilon$. Then each of the M terms in T_1 goes to zero when i goes to infinity. Therefore $\lim_{i\to\infty} d_i = \lambda - t(A)$. Since it is clear from (3) that the drifts are bounded, it follows from Pakes Lemma [11] that $(X_n)_{n\geq 0}$ is ergodic if $\lambda < t(A)$. It is shown in Appendix A that Kaplan's condition holds and therefore if $\lambda > t(A)$ Kaplan's result [8] applies and $(X_n)_{n\geq 0}$ is nonergodic. Note that the value of p_n for $X_n < A$ is left unspecified because it does not affect the throughput.

If $\lim_{n\to\infty} C_n = 0$, the control $p_n = A/X_n$ stabilizes the system. But for some reception matrices, there may not exist a constant A such that $t(A) > \eta_0$, i.e., no throughput improvement can be obtained with a retransmission probability which is inversely proportional to the backlog. In particular, if $C_n \geq \eta_0$ for all $n \geq 1$, (for instance, this is the case for a system with perfect capture and no noise, or also for the model with mobile users and pairwise transmissions developed in [4]), it can be proved with simple drift analysis considerations that no decentralized control protocol of the form (1) can achieve a throughput larger than η_0. In the remainder of this section, we will assume that the sequence $(C_n)_{n\geq 1}$ is nonincreasing and nonconstant, which is true in most cases of interest. Then there does exist an A such that $t(A) > \eta_0$ and furthermore it is easy to see that $t(A)$ is maximized by the unique solution A^* of the equation

$$C_1 = \sum_{n=1}^{\infty} A^n \frac{C_n - C_{n+1}}{n!} \tag{4}$$

It follows from the theorem below that this simple control scheme is optimal.

Theorem 1. If $(C_n)_{n\geq 1}$ is nonincreasing and nonconstant, there exists a unique retransmission probability $p_n^* = F^*(X_n)$ minimizing the drift d_i at state i. With the control $p_n^* = F^*(X_n)$, the system is stable for $\lambda < \eta_c$ and unstable for $\lambda > \eta_c$, with $\eta_c = e^{-y} \sum_{n=1}^{\infty} C_n \frac{y^n}{n!}$ where y is the unique solution of $C_1 = \sum_{n=1}^{\infty} y^n \frac{C_n - C_{n+1}}{n!}$.

Proof. If $X_n = i \geq 1$ and if we denote by p the retransmission probability used in slot n, we have

$$d_i(p) = \lambda - \sum_{j=1}^{i} \binom{i}{j} p^j (1-p)^{i-j} C_j \tag{5}$$

and

$$\frac{1}{i} d_i'(p) = \sum_{j=1}^{i-1} \binom{i-1}{j} p^j (1-p)^{i-j-1} (C_j - C_{j+1}) - (1-p)^{i-1} C_1 \tag{6}$$

so that if $p \neq 1$, $d_i'(p) = 0$ iff

$$\sum_{j=1}^{i-1} \binom{i-1}{j} x^j (C_j - C_{j+1}) = C_1 \tag{7}$$

where we have defined $x = \frac{p}{1-p}$. Since $(C_n)_{n\geq 1}$ is nonincreasing and nonconstant, $C_1 > 0$ and the left hand side of (7) is a strictly increasing function of x with range $[0, +\infty[$. Therefore (7) has a unique solution x_i^* corresponding to a unique zero of $d_i'(p)$, $p_i^* = \frac{x_i^*}{1+x_i^*}$. If $p=1$, then $d_i'(p) = 0$ iff $C_{i-1} = C_i$ but this corresponds to a local maxima.

Let us now study some properties of the sequence $(p_i^*)_{i\geq 1}$. We start by showing that $\lim_{i\to\infty} p_i^* = 0$, which is intuitively obvious. Since $p_i^* = \frac{x_i^*}{1+x_i^*}$, it is enough to prove that $x_i^* = \frac{p_i^*}{1-p_i^*}$ decreases to zero. Define $h_i(x) = \sum_{j=1}^{i-1} \binom{i-1}{j} (C_j - C_{j+1}) x^j$. Then from (7), x_i^* is the only solution of $h_i(x) = C_1$. It is easily checked that for all $x \geq 0$, $h_{i+1}(x) \geq h_i(x)$, and since both functions are strictly increasing, we can conclude that $(x_i^*)_{i\geq 1}$ is a decreasing sequence. Denoting by l the smallest integer such that $C_l - C_{l+1} > 0$, we get for $i > l+1$: $h_i(x) \geq \binom{i-1}{l} (C_l - C_{l+1}) x^l$, so that $C_1 \geq \binom{i-1}{l} (C_l - C_{l+1}) (x_i^*)^l$, which can be written $x_i^* \leq \frac{D}{(i-1)^{1/l}}$, where D is a constant. Therefore $\lim_{i\to\infty} x_i^* = 0$.

The next step of the proof is to show that $(ip_i^*)_{i\geq 1}$ has a limit y which is the only solution of $C_1 = \sum_{n=1}^{\infty} y^n \frac{C_n - C_{n+1}}{n!}$. Let us define $y_i = ix_i^* = \frac{ip_i^*}{1-p_i^*}$. Then (y_i) has a limit iff (ip_i^*) has one, and these limits are equal. From (7), y_i is the only solution of $g_i(z) = C_1$, where we have defined

$$g_i(z) = \sum_{j=1}^{i-1} \frac{(i-1)\cdots(i-j)}{i^j} \frac{C_j - C_{j+1}}{j!} z^j \tag{8}$$

It is easily shown from (8) that $g_{i+1}(z) \geq g_i(z)$. Since both functions are increasing, it follows that $y_{i+1} \leq y_i$ and therefore that $(y_i)_{i\geq 1}$ has a limit y. From (6), $d_i'(p) < 0$ for $p < \frac{1}{i}$, so we necessarily have $ip_i^* \geq 1$ and $y \geq 1$. Next we show that $\lim_{i\to\infty} g_i(y_i) = \sum_{n=1}^{\infty} y^n \frac{C_n - C_{n+1}}{n!}$, and therefore we will have $\sum_{n=1}^{\infty} y^n \frac{C_n - C_{n+1}}{n!} = C_1$. Define $\Delta_i = g_i(y_i) - \sum_{n=1}^{\infty} y^n \frac{C_n - C_{n+1}}{n!}$. Then for $M \leq i-2$:

$$|\Delta_i| \leq \sum_{j=1}^{M} \left| \frac{(i-1)\cdots(i-j)}{i^j} y_i^j - y^j \right| \frac{C_j - C_{j+1}}{j!} + 2 \sum_{j=M+1}^{\infty} y_1^j \frac{C_j - C_{j+1}}{j!}$$

M can be fixed so that the second term is smaller than any $\epsilon > 0$, and then each term in the first summation goes to zero as i goes to infinity.

Now we proceed to find the maximum stable throughput of the system if p_i^* is the retransmission probability, by computing again the drift limit, $\lim_{i\to\infty} d_i(p_i^*) = \lambda - e^{-y} \sum_{n=1}^{\infty} y^n \frac{C_n}{n!} = \lambda - \eta_c$. From (5), and with

$z_i = ip_i^*$:

$$d_i(p_i^*) = \lambda - \sum_{j=1}^{i} \frac{(i-1)\cdots(i-j+1)}{i^j} z_i^j (1-\frac{z_i}{i})^{i-j} \frac{C_j}{j!}$$

We have

$$| d_i(p_i^*) - \eta_c | \le \sum_{j=1}^{M} | \frac{i(i-1)\cdots(i-j+1)}{i^j} z_i^j (1-\frac{z_i}{i})^{i-j} - e^{-y} y^j | \frac{C_j}{j!}$$

$$+ \sum_{j=M+1}^{i} \frac{i(i-1)\cdots(i-j+1)}{i^j} z_i^j (1-\frac{z_i}{i})^{i-j} \frac{C_j}{j!} + e^{-y} \sum_{j=M+1}^{\infty} y^n \frac{C_n}{n!}.$$

If y, $z_i \le Z$, the last two terms on the right hand side of this equation are smaller than $\sum_{j=M+1}^{\infty} Z^j \frac{C_j}{j!} < \epsilon$ for M large enough. Having fixed such an M, each term in the first sum goes to zero. It is shown in Appendix A that Kaplan's condition holds for this system; then the Theorem follows from the results in [8] and [11].

The main conclusion of this section is that it is not necessary to compute the exact value of $F^*(i)$, since we have $t(A^*) = \eta_c$, meaning that if we set $p_n = A^*/X_n$ for $X_n \ge A^*$, the maximum stable throughput of the system is η_c. It is intuitively obvious that no decentralized control algorithm of the form (1) can have a maximum stable throughput larger than η_c. This can indeed be proved by using a result of [15]. Consider the Markov chain (X_t, S_t), and the Lyapunov function $V(n,s)=n$. Assume that $\lambda > \eta_c$. Then

$$E[V(X_{t+1},S_{t+1}) - V(X_t,S_t) \mid X_t = n,\ S_t = s] = \lambda - \sum_{j=1}^{n} \binom{n}{j} F(s)^j (1-F(s))^{n-j} C_j \ge d_n(p_n^*) \ge \frac{\lambda-\eta_c}{2}$$

for all n large enough and all s. Therefore the drift of V is strictly positive outside a finite subset of the state space. Since it is shown in Appendix A that the generalized Kaplan's condition is verified, it is enough to conclude that (X_t, S_t) is nonergodic.

Several examples are gathered in Table 1 below.

	C_n	$\eta_0 = \lim_{n\to\infty} C_n$	$\eta_c = \sup_{A>0} e^{-A} \sum_{n=1}^{\infty} C_n \frac{A^n}{n!}$
conventional collision channel [1]	$1 \quad n = 1$ $0 \quad n > 1$	0	e^{-1}
q-frequency frequency hopping [4]	$n(1-\frac{1}{q})^{n-1}$	0	$q\, e^{-1}$
mobile users with pairwise transmission [4]	1	1	1
capture - power discrimination [4]	$1 \quad n = 1$ $\frac{1}{\beta^2} \quad n > 1$	$\frac{1}{\beta^2}$	$\frac{1}{\beta^2} + (1-\frac{1}{\beta^2}) \exp(-\frac{\beta^2}{\beta^2-1})$
capture - timing discrimination [2]	$1 \quad n = 1$ $(1-Q)^n \quad n > 1$	0	$\max_{A>0} \{ (AQ-1)\, e^{-A} + e^{-AQ} \}$

Table 1. Examples.

$(1-\frac{z_i}{i})^{i-j} \frac{C_j}{j!}$

$1-\frac{z_i}{i})^{i-j} - e^{-y} y^j \mid \frac{C_j}{j!}$

$+ e^{-y} \sum_{j=M+1}^{\infty} y^n \frac{C_n}{n!}$

n are smaller than $\sum_{j=M+1}^{\infty} Z^j \frac{C_j}{j!} < \epsilon$ for
n goes to zero. It is shown in Appendix A
vs from the results in [8] and [11].

to compute the exact value of $F^*(i)$, since
*, the maximum stable throughput of the
lgorithm of the form (1) can have a max-
y using a result of [15]. Consider the Mar-
t $\lambda>\eta_c$. Then

$(1-F(s))^{n-j} C_j \ge d_n(p_n^*) \ge \frac{\lambda-\eta_c}{2}$

positive outside a finite subset of the state
s condition is verified, it is enough to con-

$\eta_c = \sup_{A>0} e^{-A} \sum_{n=1}^{\infty} C_n \frac{A^n}{n!}$
e^{-1}
$q\ e^{-1}$
1
$\frac{1}{\beta^2} + (1-\frac{1}{\beta^2}) \exp(-\frac{\beta^2}{\beta^2-1})$
$\max_{A>0} \{ (AQ-1)\, e^{-A} + e^{-AQ} \}$

3. Optimal control for the multipacket channel with bounded simultaneous successes

In this section we assume that there exists an upper bound on the number of packets that can be demodulated simultaneously and successfully, i.e. $\epsilon_{ij}=0$ if $j>K$. Note that in the case of a multipoint-to-point channel, such a limitation on the receiver capabilities is likely to happen in practice. For instance, systems with capture (see [5], [10]) are included in the case $K=1$. We assume here that the feedback is binary. More precisely, if slot n is empty then $Z_n=0$ and $Z_n=\bar{0}$ otherwise. As suggested by the results of Section 2, we consider retransmission probabilities of the form $p_n = A/S_n$. To fully take advantadge of the feedback information, it seems quite natural to consider a linear recursion for the backlog estimate, that is $S_{t+1} = \max\{ A, S_t + \alpha\, I(Z_t=0) + \beta\, I(Z_t=\bar{0}) \}$, where $I(E)$ denotes the indicator function of the event E.

Theorem 2. Assume that the new packet arrivals $(A_k)_{k\geq 0}$ are exponential type [6] (e.g. Poisson) and that the sequence $(C_n)_{n\geq 1}$ is nonincreasing and nonconstant. Denote by A the solution of (4) and $\eta_c = e^{-A} \sum_{n=1}^{\infty} C_n \frac{A^n}{n!}$. If $\alpha<0$ and $\beta>0$ verify the following two conditions,

C1. $\beta > \lambda$

C2. $\beta(1-e^{-A}) + \eta_c - \lambda + \alpha e^{-A} = 0$

then the control algorithm $\begin{cases} p_t = \dfrac{A}{S_t} \\ S_{t+1} = \max\{ A,\, S_t + \alpha I(Z_t=0) + \beta I(Z_t=\bar{0}) \} \end{cases}$ has a maximum stable throughput η_c.

Proof. The proof is based on the method developed in [14]. The idea is to use the properties of (X_n, S_n) to build a Lyapunov function whose drift is negative in the entire plane. Clearly, $(X_n, S_n)_{n\geq 0}$ is a homogeneous two dimensional vector Markov chain. Denote by $c(n,s)$ the backlog drift, $c(n,s) = E[X_{t+1} - X_t \mid X_t=n,\ S_t=s]$, by $\tilde{X}_t = S_t - X_t$ the error in the backlog estimate, and by $d(n,s) = E[\tilde{X}_{t+1} - \tilde{X}_t \mid X_t=n,\ S_t=s]$ its drift. Equation (2) being still valid here, we get for all $s \geq A$

$$c(0,s) = \lambda \tag{10}$$

$$c(n,s) = \lambda - \sum_{j=1}^{n} \binom{n}{j} (\frac{A}{s})^j (1-\frac{A}{s})^{n-j} C_j \qquad (n\geq 1)$$

Let us now compute the drift of the backlog estimate:

$$E[S_{t+1} - S_t \mid X_t=n,\ S_t=s] = \beta + (\alpha-\beta)(1-\frac{A}{s})^n \qquad (s \geq A-\alpha) \tag{11}$$

$$E[S_{t+1} - S_t \mid X_t=n,\ S_t=s] = \beta + (A-s-\beta)(1-\frac{A}{s})^n \qquad (A \leq s \leq A-\alpha)$$

It then follows from (10)-(11) that if $s\geq A-\alpha$:

$$d(0,s) = \alpha - \lambda \tag{12}$$

$$d(n,s) = \beta - \lambda + (\alpha-\beta)(1-\frac{A}{s})^n + \sum_{j=1}^{n} \binom{n}{j} (\frac{A}{s})^j (1-\frac{A}{s})^{n-j} C_j \qquad (n\geq 1)$$

and if $A\leq s\leq A-\alpha$:

$$d(0,s) = A-s-\lambda \tag{13}$$

$$d(n,s) = \beta - \lambda + (A-s-\beta)(1-\frac{A}{s})^n + \sum_{j=1}^{n} \binom{n}{j} (\frac{A}{s})^j (1-\frac{A}{s})^{n-j} C_j \qquad (n\geq 1)$$

Most of the properties of the drifts that are needed to build the Lyapunov function depend on the ratio $x=n/s$. For instance, if x is close to 1, the backlog estimate is close to its ideal value, and we should have $c(n,s) < 0$. Also, a well-behaved estimate should be such that $d(n,s) < 0$ (resp. $d(n,s) > 0$) for $x<1$ (resp. $x>1$). For this reason, an approximation of $c(n,s)$ and $d(n,s)$ in terms of x is needed. If ν is the difference $\eta_c - \lambda = \nu>0$, we can fix L such that $\sum_{L+1}^{\infty} A^n \frac{C_n}{n!} < \frac{\nu}{2}$, and so

$$\lambda < e^{-A} \sum_{n=1}^{L} C_n \frac{A^n}{n!} - \frac{\nu}{2} \tag{14}$$

Lemma 2. There exists two positive functions, $\epsilon(n,s)$ that goes to zero when either n or s goes to infinity, and $\epsilon'(n)$ that goes to zero when n goes to infinity, such that the following statements hold if $s > A$

$$(i)\ c(n,s) \leq F(x) + L\epsilon(n,s) \quad (n > L) \text{ with } F(x) = \lambda - e^{-Ax} \sum_{j=1}^{L} (Ax)^j \frac{C_j}{j!}$$

$$(ii)\ d(n,s) \leq N(x) + \epsilon(n,s) + \epsilon'(n) \quad (n \geq 1,\ s \geq A-\alpha) \text{ with } N(x) = \beta + \eta_c - \lambda + (\alpha-\beta)e^{-Ax}$$

$$(iii)\ d(n,s) \geq P_N(x) - N\epsilon(n,s) \quad (n > N) \text{ with } P_N(x) = \beta - \lambda + (\alpha-\beta)e^{-Ax} + e^{-Ax} \sum_{j=1}^{N} (Ax)^j \frac{C_j}{j!}$$

Proof of Lemma 2. The proof follows from (10)-(13), Theorem 1 and Lemma 3 below, whose proof is omitted because of space limitations.

Lemma 3. Given an integer $L \geq 1$, we have for $n > L$, for $s > A$ and for all $j \in \{0, 1, \ldots, L\}$:

$$\left(\frac{An}{s}\right)^j \left| \frac{n(n-1)\cdots(n-j+1)}{n^j} \left(1 - \frac{A}{s}\right)^{n-j} - e^{-\frac{An}{s}} \right| < \epsilon(n,s)$$

where $\epsilon(n,s)$ is a positive function going to zero when either n or s goes to infinity.

Define the following two regions in the (n,s) plane:

$$C(\lambda_0,\lambda_1) = \{ (n,s) : n \geq 0,\ s \geq 0,\ 1+\lambda_0 \leq \frac{n}{s} \leq 1+\lambda_1 \}$$

$$U_M = \{ (n,s) : n \geq M \text{ or } s \geq M \}$$

where λ_0 and λ_1 are any two real numbers such that $-\infty \leq \lambda_0 \leq \lambda_1 \leq +\infty$. We first show some simple geometric properties of these regions.

Lemma 4. Consider $\gamma > 0$, $B > 0$, and $\gamma - 1 < \lambda_0 < \lambda_1 < +\infty$; and assume that $|n - n'| \leq B$, $|s - s'| \leq B$, and $Q \geq \frac{B}{\gamma}(1+|\lambda_1|)(\lambda_1+2+\gamma)$. Then

1) $(n,s) \in C(\lambda_0,\infty) \cap U_Q \Rightarrow (n',s') \in C(\lambda_0-\gamma,\infty) \cap U_{Q-B}$

2) $(n,s) \in C(-\infty,\lambda_1) \cap U_Q \Rightarrow (n',s') \in C(-\infty,\lambda_1+\gamma) \cap U_{Q-B}$

3) $(n,s) \in C(\lambda_0,\lambda_1) \cap U_Q \Rightarrow (n',s') \in C(\lambda_0-\gamma,\lambda_1+\gamma) \cap U_{Q-B}$

Proof of Lemma 4. We show that the first statement holds, the proof of the other two being similar. Since $Q > B$, either $n \geq Q > B$, or $s \geq Q$, in which case $n > (1+\lambda_0)\frac{B}{\gamma}(1+|\lambda_1|)(2+\lambda_1+\gamma)$ which is strictly larger than B. Therefore $n > B$. Then

$$\frac{n'}{s'} \geq \frac{(n-B)(1+\lambda_0)}{n+B(1+\lambda_0)} = 1 + \frac{\lambda_0 n - 2B(1+\lambda_0)}{n+B(1+\lambda_0)}$$

so that $n'/s' \geq 1+\lambda_0-\gamma$ iff

$$\gamma n \geq B(1+\lambda_0)(2+\lambda_0-\gamma) \tag{15}$$

If $n \geq Q$, it is easily seen that (15) holds, and if $s \geq Q$ we have $n \geq (1+\lambda_0)Q > (1+\lambda_0)\frac{B}{\gamma}(1+\lambda_1)(2+\lambda_1+\gamma)$, from which (15) follows

Let us now use the approximations of Lemma 2 to study some properties of the drifts in the plane.

Lemma 5. There exists constants $M > 0$, $\delta > 0$ and $\gamma \in]0,1/5[$ such that

(i) for all $(n,s) \in C(-5\gamma,5\gamma) \cap U_M$, $c(n,s) \leq -\delta$

(ii) for all $(n,s) \in C(-\infty,-\gamma) \cap U_M$, $d(n,s) \leq -\delta$

(iii) for all $(n,s) \in C(\gamma,+\infty) \cap U_M$, $d(n,s) \geq +\delta$.

Proof of Lemma 5.

(i) $F(x)$ is a continuous function and from (14) $F(1) < 0$, so we can fix $\gamma \in]0,1/5[$ and $\delta_1 > 0$ such that $F(x) \leq -2\delta_1$ for all $x \in [1-5\gamma, 1+5\gamma]$. Now if we choose $M > \max\{\frac{L}{1-5\gamma}, A(1+5\gamma)\}$ large enough so that $\epsilon(n,s) \leq \delta_1$ for $(n,s) \in U_M$ then we get from Lemma 2 (i): $c(n,s) \leq -\delta_1$ for all $(n,s) \in C(-5\gamma,5\gamma) \cap U_M$.

(ii) In $C(-\infty,-\gamma)\cap U_M$, $s\geq M$. Take $M\geq A-\alpha$. If $n=0$, $d(n,s)=\alpha-\lambda<0$. Fix N such that $\epsilon'(n)<|\frac{N(1-\gamma)}{3}|$ for $n>N$. For $1\leq n\leq N$, $d(n,s)\leq\beta-\lambda+(\alpha-\beta)(1-\frac{A}{s})^N+\frac{A}{s}N!\sum_{j=1}^{N}\frac{C_j}{j!}$, which goes to $\alpha-\lambda$ as s goes to infinity. Therefore we can take M large enough so that for $s\geq M$ and $1\leq n\leq N$, $d(n,s)\leq\frac{\alpha-\lambda}{2}$. $N(x)$ is a strictly increasing function and from C2 $N(1)=0$. So if $n>N$, we get from Lemma 2 (ii): $d(n,s)\leq\frac{2}{3}N(1-\gamma)+\epsilon(n,s)$ for $x<1-\gamma$.

(iii) If $s=A$ and $n\geq 1$, then $d(n,s)=\beta-\lambda+C_n>0$ from C1. If $s>A$ and $n>N$, Lemma 2 (iii) holds. Considering that

$$\frac{dP_N}{dx}/(Ae^{-A})=\beta-\alpha+C_1-(Ax)^N\frac{C_N}{N!}-\sum_{j=1}^{N-1}(Ax)^j\frac{C_j-C_{j+1}}{j!}$$

it can be seen that $\frac{dP_N}{dx}$ decreases from a positive value, $\beta-\alpha+C_1$, to $-\infty$. Therefore, $P_N(x)$ increases from $\alpha-\lambda<0$ to a positive maximum and then decreases towards $\beta-\lambda>0$. Denote by r_N the only zero of P_N. Clearly, $P_{N+1}\geq P_N$, so that (r_N) is a decreasing sequence; let r be its limit. It is easily shown that $\lim_{N\to\infty}P_N(r_N)=\beta-\lambda+(\alpha-\beta)e^{-Ar}+e^{-Ar}\sum_{j=1}^{\infty}(Ar)^j\frac{C_j}{j!}=0$, and that the equation $\beta-\lambda+(\alpha-\beta)e^{-Ax}+e^{-Ax}\sum_{j=1}^{\infty}(Ax)^j\frac{C_j}{j!}=0$ has only one solution. It then follows from C2 that $r=1$. Now fix N such that $r_N<1+\frac{\gamma}{2}$. Then $P_N(1+\gamma)>0$, so M can be chosen large enough so that (iii) holds.

⋮

From now on, M and γ are fixed so that Lemma 5 holds. Let us define the following Lyapunov function

$$V(n,s)=\max\{\,n,\ \frac{1+3\gamma}{3\gamma}(n-s),\ \frac{1-3\gamma}{3\gamma}(s-n)\,\}$$

$V(n,s)$ is equal to the first, second and third term inside the bracket when (n,s) is in $C(-3\gamma,3\gamma)$, $C(3\gamma,+\infty)$, and $C(-\infty,-3\gamma)$ respectively. We would like to have $E[V(X_{t+1},S_{t+1})-V(X_t,S_t)\mid X_t=n,S_t=s]\leq-\Delta$ for all $(n,s)\in U_M$, but this may fail if (X_{t+1},S_{t+1}) is not in the same region than (X_t,S_t).

Lemma 6. Given $\epsilon>0$, there exists some $\Delta>0$ and J_0 such that for $J>J_0$

$$E[V(X_{t+J},S_{t+J})-V(X_t,S_t)\mid X_t=n,S_t=s]\leq-\Delta J+\epsilon$$

for all $(n,s)\in U_{HJ}$, where H is a positive constant independent of J.

Proof of Lemma 6. Given any integer $J>0$, and any real number $\phi>0$, let us define the stopping time $\tau_{J,\phi}=\min\{s\geq 0\ \sum_{k=t}^{s+t}A_k\geq\phi J\}$. ϕ can be chosen such that the event $\{\tau_{J,\phi}\leq J\}$ is unlikely.

Lemma 7. There exists $\phi>0$ such that $\lim_{J\to\infty}J\,P[\tau_{J,\phi}\leq J]=0$.

Proof of Lemma 7. Since $(A_k)_{k\geq 0}$ are exponential type, their common characteristic function $\psi(t)=E[e^{tA_k}]$ exists on an interval $[0,d]$. Denoting by $S_J=\sum_{k=1}^{J}A_{t+k}$, we get $\sum_{k=0}^{\infty}P[S_J=k]e^{tk}=\psi(t)^J$. Therefore $P[S_J\geq N]\leq\psi(t)^Je^{-Nt}$, so that if $N>J\frac{\log\psi(t)}{t}$ for some $t\in[0,d]$, we have $\lim_{J\to\infty}JP[S_J\geq N]=0$.

From now on, ϕ is fixed such that Lemma 7 holds, and we just write τ_J. Consider the decomposition

$$E[V(X_{t+J},S_{t+J})-V(X_t,S_t)\mid X_t=n,\ S_t=s]=$$

$$\sum_{k=0}^{J-1}E\{\,E[V(X_{t+k+1},S_{t+k+1})-V(X_{t+k},S_{t+k})\mid X_{t+k},\ S_{t+k}]I(\tau_J\leq J)\mid X_t=n,\ S_t=s\}$$
$$+\sum_{k=0}^{J-1}E\{\,E[V(X_{t+k+1},S_{t+k+1})-V(X_{t+k},S_{t+k})\mid X_{t+k},\ S_{t+k}]I(\tau_J>J)\mid X_t=n,\ S_t=s\} \qquad (16)$$

Denote by T_1 and T_2 the two sums on the right hand side of (16). Since $|X_{t+1}-X_t|\leq K+A_t$ and $|S_{t+1}-S_t|\leq|\alpha|+\beta$, we have

$$|V(X_{t+1},S_{t+1})-V(X_t,S_t)|\leq\max\{1,\ \frac{1+3\gamma}{3\gamma},\ \frac{1-3\gamma}{3\gamma}\}(K+|\alpha|+\beta+A_t)\leq R(1+A_t)$$

where R is some constant independent of J. Therefore

$$T_1 \leq R(1+\lambda)JP[\tau_J \leq J] \tag{17}$$

so that from Lemma 7, T_1 goes to zero when J goes to infinity.

Lemma 8. If $\tau_J > J$, there exists $\mu > 0$ independent of J such that for $0 \leq k \leq J-1$ and $(n,s) \in U_{Q(J)}$ $E[V(X_{t+k+1},S_{t+k+1}) - V(X_{t+k},S_{t+k}) \mid X_{t+k},\ S_{t+k}] \leq -\mu$, where $Q(J)$ is a positive constant depending on J

Proof of Lemma 8. To simplify notations, let us define $\Delta V_k = E[V(X_{t+k+1},S_{t+k+1}) - V(X_{t+k},S_{t+k}) \mid X_{t+k},\ S_{t+k}]$, for $0 \leq k \leq J-1$. Since $\tau_J > J$, we have $\sum_{k=0}^{J-1} |X_{t+k+1} - X_{t+k}| \leq J(\phi+K)$ and $\sum_{k=0}^{J-1} |S_{t+k+1} - S_{t+k}| \leq J(|\alpha|+\beta)$. Set $B = J\max\{\phi+K,\ |\alpha|+\beta\}$, and define by $Q(J)$ any constant such that $Q \geq \max\{B+M,\ \frac{B}{\gamma}(1+4\gamma)(2+5\gamma)\}$.

Case 1: $(X_t,S_t) \in C(-2\gamma,2\gamma) \cap U_{Q(J)}$

From Lemma 4, $(X_{t+k},S_{t+k}) \in C(-3\gamma,3\gamma) \cap U_M \subseteq C(-5\gamma,5\gamma) \cap U_M$ for $0 \leq k \leq J-1$. Thus from Lemma 5, $\Delta V_k = c(X_{t+k},S_{t+k}) \leq -\delta$.

Case 2: $(X_t,S_t) \in C(4\gamma,\infty) \cap U_{Q(J)}$

Then from Lemma 4, $(X_{t+k},S_{t+k}) \in C(3\gamma,\infty) \cap U_M \subseteq C(\gamma,\infty) \cap U_M$, so from Lemma 5 $\Delta V_k = -\frac{1+3\gamma}{3\gamma}\, d(X_{t+k},S_{t+k}) \leq -\delta$.

Case 3: $(X_t,S_t) \in C(-\infty,-4\gamma) \cap U_{Q(J)}$

Since from Lemma 4, $(X_{t+k},S_{t+k}) \in C(-\infty,-3\gamma) \cap U_M \subseteq C(-\infty,-\gamma) \cap U_M$, we get $\Delta V_k = \frac{1-3\gamma}{3\gamma}\, d(X_{t+k},S_{t+k}) \leq -\frac{1-3\gamma}{3\gamma}\delta$

Case 4: $(X_t,S_t) \in C(2\gamma,4\gamma) \cap U_{Q(J)}$

Then from Lemma 4, we have $(X_{t+k},S_{t+k}) \in C(\gamma,5\gamma) \cap U_M$, so from the definition of V, we get for $0 \leq k \leq J-1$:

$$V(X_{t+k+1},S_{t+k+1}) = \max\ \{X_{t+k+1},\ \frac{1+3\gamma}{3\gamma}(X_{t+k+1}-S_{t+k+1})\}$$

$$V(X_{t+k},S_{t+k}) = \max\ \{X_{t+k},\ \frac{1+3\gamma}{3\gamma}(X_{t+k}-S_{t+k})\}$$

so that another 4 cases have to be distinguished depending on the value of the two maxima above. Since both (X_{t+k+1},S_{t+k+1}) and (X_{t+k},S_{t+k}) belong to $C(-5\gamma,5\gamma) \cap C(\gamma,\infty) \cap U_M$, we have in any case from Lemma 5 $\Delta V_k \leq \max\ \{-\delta,\ -\frac{1+3\gamma}{3\gamma}\delta\} \leq -\delta$. For instance, if $X_{t+k+1} > \frac{1+3\gamma}{3\gamma}(X_{t+k+1}-S_{t+k+1})$ and $X_{t+k} < \frac{1+3\gamma}{3\gamma}(X_{t+k}-S_{t+k})$, we get

$$\Delta V_k = E[X_{t+k+1} - \frac{1+3\gamma}{3\gamma}(X_{t+k}-S_{t+k}) \mid X_{t+k},S_{t+k}] \leq c(X_{t+k},S_{t+k}) \leq -\delta$$

Case 5: $(X_t,S_t) \in C(-4\gamma,-2\gamma) \cap U_{Q(J)}$ is analogous to case 4.

We can now conclude the proof of Lemma 6. Define $H = \frac{(1+4\gamma)(2+5\gamma)}{\gamma}\max\{\phi+K,\ |\alpha|+\beta\}$. Choose J large enough so that $Q(J) = HJ$ and $P[\tau_J > J] > 1/2$. Then from (16) and Lemma 8, we get for $(n,s) \in U_{HJ}$: $T_2 \leq -\mu JP[\tau_J > J] < -\mu\frac{J}{2}$, which together with Lemma 7 and (17), ends the proof of Lemma 6.

Define the following

$$W_t = V(X_{tJ},S_{tJ})$$
$$F_t = \sigma\ \{\ A_{s-1},\ X_s, 0 \leq s \leq t\ \}$$
$$a = HJ\ \max\{1,\frac{1+3\gamma}{3\gamma},\frac{1-3\gamma}{3\gamma}\}$$

Lemma 9 (Hajek [6]). Let $\{W_t\}$ be a sequence of random variables adapted to an increasing family of σ-fields $\{F_t\}$. Suppose that W_0 is deterministic, that $\{W_t,F_t\}$ is exponential type and that for some $\epsilon > 0$ and $a > 0$ we have $E[(W_{t+1} - W_t + \epsilon)\, I(W_t > a) \mid F_t] \leq 0$ for all $t \geq 0$. Then for each value of W_0 the stopping time $\tau = \min\ \{\ t \geq 0;\ W_t \leq a\}$ is exponential type.

If $V(X_t,S_t) > a$, then $(X_t,S_t) \in U_{HJ}$, so we can apply Lemma 9 to our system to conclude that $\tau = \min\{\ t \geq 0,\ V(X_{Jt},S_{Jt}) \leq a\ \}$ is exponential type for any initial state. Since $V(X_t,S_t) \leq a$ implies that $X_t \leq a$ and $S_t \leq a/(1-\gamma)$, it follows that $\tau' = \min\{\ t \geq 0,\ X_t \leq a$ and $S_t \leq a/(1-\gamma)\ \}$ is also exponential type for any initial state, and from [9] that (X_t,S_t) is geometrically ergodic.

Note that Theorem 2 can be applied to the usual collision channel. To our knowledge, this is the only decentralized control algorithm yielding the optimal throughput e^{-1} with only a binary feedback. It is conjectured that Theorem 2 can be extended to the general reception matrix case. The drifts remain the same and Lemma 5 is still verified, but the proof of Lemma 6 seems rather difficult to extend since the number of successes per slot is no longer bounded.

Appendix A: Kaplan's condition

Consider a Markov chain with denumerable state space D, and one-step transition probability matrix $(P_{xy})_{(x,y)\in D}$. Let $V(x)$ be a Lyapunov function on D. Then the generalized Kaplan's condition holds if there exists a positive constant B such that for all $z \in [0,1]$ and all $x \in D$

$$z^{V(x)} - \sum_{y\in D} P_{xy}\, z^{V(y)} \geq -B(1-z)$$

1) One dimensional Kaplan's condition. Consider the model of Section 2 with a control scheme of the form $p_n = F(X_n)$, and the Lyapunov function $V(x)=x$. To check the Kaplan's condition, it is enough from [12] to show that the downward part of the drift, $D(i)$, is bounded below. For $i\geq 1$ and $1\leq k\leq i$ we have

$$P_{i,i-k} = \sum_{n=0}^{i-k} \lambda_n \sum_{j=k+n}^{i} \binom{i}{j} F(i)^j \, (1-F(i))^{i-j}\, \epsilon_{j,k+n}$$

after a change of variable, it follows that

$$D(i) = -\sum_{j=1}^{i} \binom{i}{j} F(i)^j (1-F(i))^{i-j} \sum_{n=0}^{j-1} \lambda_n \sum_{k=n+1}^{j} (k-n)\, \epsilon_{j,k} \tag{A-1}$$

a) If $p_n = \dfrac{A}{X_n}$, then Kaplan's condition holds. (A-1) yields in this case for $i \geq \max\{1, A\}$.

$$D(i) = -\sum_{j=1}^{i} \frac{i(i-1)\cdots(i-j+1)}{i^j}\, \frac{A^j}{j!} \left(1-\frac{A}{i}\right)^{i-j} \sum_{n=0}^{j-1} \lambda_n \sum_{k=n+1}^{j} (k-n)\, \epsilon_{jk} \geq -Ae^A$$

b) If $(C_n)_{n\geq 1}$ is bounded, then Kaplan's condition holds independently of the retransmission policy. Denoting by U an upper bound for (C_n), (A-1) becomes

$$D(i) \geq -\sum_{j=1}^{i} \binom{i}{j} F(i)^j (1-F(i))^{i-j} \sum_{n=0}^{j-1} \lambda_n\, C_j \geq -\sum_{j=1}^{i} \binom{i}{j} F(i)^j (1-F(i))^{i-j}\, C_j \geq -U$$

2) Two dimensional Kaplan's condition. Consider now the general multipacket Aloha channel with a control algorithm of the form (1). Then (X_n, S_n) is the Markov chain of interest, and the relevant Lyapunov function is $V(n,s) = n$. We prove again that provided that $(C_n)_{n\geq 1}$ is bounded, Kaplan's condition holds. From [12], it is enough to show that the downward part $T(x)$ of the generalized drift is bounded below, with $T(x) = \sum_{y/V(y)<V(x)} P_{xy}\, (V(y)-V(x))$. Given a state $x=(i,s)$, we have

$$T(x) = -\sum_{r=1}^{i} r \sum_{k} P[X_{n+1}=i-r,\ S_{n+1}=k \mid X_n=i,\ S_n=s] = -\sum_{r=1}^{i} r\ P[X_{n+1}=i-r \mid X_n=i,\ S_n=s]$$

which is, in the same way as before

$$T(x) = -\sum_{r=1}^{i} r \sum_{n=0}^{i-r} \lambda_n \sum_{j=r+n}^{i} \binom{i}{j} (F(s))^j (1-F(s))^{i-j}\, \epsilon_{j,r+n} = -\sum_{j=1}^{i} \binom{i}{j} F(s)^j (1-F(s))^{i-j} \sum_{n=0}^{j-1} \lambda_n \sum_{r=n+1}^{j} (r-n)\, \epsilon_{jr}$$

this expression is similar to (A-1), and the end of the proof is the same.

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