

# Optimal Signal Design for Band-Limited PAM Synchronous Multiple-Access Channels

Roger S. Cheng and Sergio Verdú  
Department of Electrical Engineering  
Princeton University  
Princeton, NJ 08544

## Abstract

The optimal signal design problem for a band-limited PAM symbol-synchronous Gaussian two-user multiple-access channel is investigated. Using the root-mean-square and the fractional out-of-band energy bandwidth definitions, we find the capacity region of the channel and the signature waveforms to achieve each point inside the capacity region. The optimal pair of signature waveforms are mirror images of each other, and are obtained by minimizing their cross-correlation subject to a fixed finite duration and the bandwidth constraint. The two-user capacity region, in the rms case, is found to contain the capacity region of the two-user strictly band-limited Gaussian channel. This demonstrates the fact that by relaxing constraints in the frequency domain, we can introduce structure (PAM) in the time domain and obtain a larger capacity region.

## 1. Introduction

The capacity region of the two-user discrete-time Gaussian multiple-access channel

$$y_i = x_{1i} + x_{2i} + n_i$$

where  $n_i$  is an i.i.d. Gaussian sequence with variance equal to  $\sigma^2$  and the energy of each codeword is constrained to satisfy

$$\frac{1}{n} \sum_{i=1}^n x_{ki}^2 \leq w_k \quad k = 1, 2$$

is equal to the Cover-Wyner pentagon [1], [2]:

$$C_D = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq \frac{1}{2} \log[1 + \frac{w_1}{\sigma^2}] \\ 0 \leq R_2 \leq \frac{1}{2} \log[1 + \frac{w_2}{\sigma^2}] \\ R_1 + R_2 \leq \frac{1}{2} \log[1 + \frac{w_1 + w_2}{\sigma^2}] \end{array} \right\} \quad (1)$$

in information units per channel use. Analogously, the capacity region of the continuous-time band-limited channel with noise power spectral density, bandwidth and  $k^{\text{th}}$  user signal power equal to  $\sigma^2$ ,  $B$ , and  $S_k$  respectively, is given by [2], as (in units per second)

$$C_C = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq B \log[1 + \frac{S_1}{2\sigma^2 B}] \\ 0 \leq R_2 \leq B \log[1 + \frac{S_2}{2\sigma^2 B}] \\ R_1 + R_2 \leq B \log[1 + \frac{S_1 + S_2}{2\sigma^2 B}] \end{array} \right\} \quad (2)$$

This capacity region is achieved by approximately band-limited and approximately time-limited waveforms which have no particular structure. In order to deal with modulation and demodulation schemes with manageable complexity, it is customary in digital communications to introduce structure on the transmitted waveforms by slotting the time domain into intervals of length  $T$  and sending a symbol in each slot by means of a digital modulation format such as PAM, PSK,

FSK, etc. In the case of PAM (Pulse Amplitude Modulation), the  $k^{\text{th}}$  user is assigned a fixed deterministic waveform,  $s_k(t)$ , which is time-limited to  $[0, T]$  and is modulated by the information stream. Then, assuming that the transmitters are symbol-synchronous, the PAM two-user multiple-access channel becomes

$$y(t) = \sum_{i=1}^n b_1(i)s_1(t - iT) + b_2(i)s_2(t - iT) + n(t) \quad (3)$$

where  $n(t)$  is white Gaussian noise with spectral density  $\sigma^2$  and  $\{b_k(i)\}$  is the symbol stream transmitted by the  $k^{\text{th}}$  user. Assuming that, without loss of generality, the signature waveforms have unit energy, the energy constraints on the transmitted waveforms become

$$\frac{1}{n} \sum_{i=1}^n b_k^2(i) \leq w_k \triangleq TS_k \quad k = 1, 2 \quad (4)$$

It is easy to show that if  $s_1(t) = s_2(t)$ , then the capacity of (3) under constraints (4) is equal to the Cover-Wyner pentagon (1) (this result remains true even if the users are completely asynchronous [3].) If the signature waveforms are not necessarily identical, then the Cover-Wyner pentagon generalizes to [4]

$$C_V = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq \frac{1}{2} \log[1 + \frac{w_1}{\sigma^2}] \\ 0 \leq R_2 \leq \frac{1}{2} \log[1 + \frac{w_2}{\sigma^2}] \\ R_1 + R_2 \leq \frac{1}{2} \log[1 + \frac{w_1 + w_2}{\sigma^2} + \frac{w_1 w_2}{\sigma^4} (1 - \rho^2)] \end{array} \right\} \quad (5a)$$

in information units per channel use or

$$C_V = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq \frac{1}{2T} \log[1 + \frac{TS_1}{\sigma^2}] \\ 0 \leq R_2 \leq \frac{1}{2T} \log[1 + \frac{TS_2}{\sigma^2}] \\ R_1 + R_2 \leq \frac{1}{2T} \log[1 + \frac{TS_1 + TS_2}{\sigma^2} + \frac{T^2 S_1 S_2}{\sigma^4} (1 - \rho^2)] \end{array} \right\} \quad (5b)$$

in information units per second, where  $\rho = \int_0^T s_1(t)s_2(t)dt$  is the cross-correlation between the signature waveforms.

A natural question to address is the choice of the unit-energy waveforms  $s_1(t)$  and  $s_2(t)$  to maximize the capacity region  $C_V$ . It is clear that the unconstrained solution is to choose orthogonal signature waveforms. Then,  $\rho = 0$ , and the multiple-access channel is decoupled into independent single-user channels, and each transmitter can transmit at single-user capacity. However, in practice, there are constraints on the choice of the signals (e.g. in Spread Spectrum CDMA systems, the waveforms may be constrained to be Pseudo Noise shift register sequences of given period,) and it is not always possible to assign orthogonal waveforms for all users. In this paper, we will address the optimization of the signature waveforms and their duration  $T$  under bandwidth constraints. Since the signature waveforms are strictly time-limited, they cannot be strictly band-limited, and the need arises to quantify the bandwidth of these signals. There are several established ways to accomplish this [5]. In this paper, we will consider the two bandwidth measures of baseband signals that have received most attention from the information theoretic community: the

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root mean square (rms) bandwidth and the fractional out-of-band energy (fobe) bandwidth.

The rms bandwidth was popularized by Gabor [6] (it is sometimes referred to as Gabor bandwidth) and studied subsequently in [5], [7], and [8]. A finite-energy signal  $s(t)$  has rms bandwidth  $B$  if its Fourier transform  $S(f)$  satisfies

$$\frac{\int_{-\infty}^{\infty} f^2 |S(f)|^2 df}{\int_{-\infty}^{\infty} |S(f)|^2 df} = B^2 \quad (6)$$

i.e. the rms bandwidth is the square root of the "second moment" of the energy spectral density ( $|S(f)|^2$ ) of the normalized signal or, proportional to the square root of the energy of its derivative,

$$\frac{1}{(2\pi)^2} \frac{\int_{-\infty}^{\infty} [\frac{d}{dt}s(t)]^2 dt}{\int_{-\infty}^{\infty} s^2(t) dt} = B^2 \quad (7)$$

The fobe bandwidth has been used in e.g.[5], [8] and is defined as the bandwidth necessary to encompass a given fraction (say  $\alpha$ ) of the signal energy, i.e. the  $\alpha$ -fobe bandwidth is  $B$  if

$$\int_{-B}^B |S(f)|^2 df = \alpha \int_{-\infty}^{\infty} |S(f)|^2 df \quad (8)$$

Notice that the bandwidth constraints imposed on the signature waveforms will be inherited by the transmitted signals because, as is well known [9], the power spectral density of  $\sum_i b_k(i)s_k(t - iT - \tau)$  where  $\tau$  is uniformly distributed in  $[0, T]$  and  $\{b_k(i)\}$  is an i.i.d. sequence, is a scaled version of the energy spectral density  $|S_k(f)|^2$ .

## 2. Single-user Channel

Before solving for the capacity region of the PAM multiple-access channel under bandwidth constraints, it is enlightening to examine the PAM single-user channel with constrained rms bandwidth. This channel differs from the classical band-limited Gaussian channel in that the allowable transmitted signals 1) have much more structure (PAM) and 2) are rms band-limited but not strictly band-limited. It turns out that the effect of the laxer bandwidth measure cancels the effect of the additional structure imposed on the transmitted signals in the time domain, and the capacity of the channel is given by the celebrated Shannon formula [10].

### Theorem 2.1.

The capacity of the single-user PAM white Gaussian channel with noise power spectral density, rms bandwidth and signal power equal to  $\sigma^2$ ,  $B$ , and  $S$  respectively is given by (in units per second)

$$C_S = B \log\left[1 + \frac{S}{\sigma^2 B}\right] \quad (9)$$

### Proof.

The single-user PAM white Gaussian channel is a special case of (3):

$$y(t) = \sum_{i=1}^n b(i)s(t - iT) + n(t) \quad (10)$$

Assuming that, without loss of generality,  $s(t)$  has unit energy, the power constraint becomes

$$\frac{1}{n} \sum_{i=1}^n b^2(i) \leq TS \quad (11)$$

and the  $T$ -shifts of  $s(t)$ ,  $\{s(t - iT)\}_{i=1}^n$ , form an orthonormal set. The projections of  $y(t)$  on this orthonormal set are equal to

$$y(i) = \int_{iT}^{(i+1)T} y(t)s(t - iT)dt \quad i = 1, \dots, n \quad (12)$$

or, substituting  $y(t)$  from (10),

$$y(i) = b(i) + n(i) \quad (13)$$

where  $\{n(i)\}$  is an i.i.d. Gaussian sequence with variance equal to  $\sigma^2$ .

The important point to note is that  $\{y(i)\}_{i=1}^n$  are sufficient statistics for the transmitted messages; therefore, the capacity of the PAM channel (10) for a fix  $T$  coincides with the capacity of the discrete time memoryless channel (13) with constraint (11), which is given by (e.g. [11]) (in units per second)

$$C_S(T) = \frac{1}{2T} \log\left[1 + \frac{ST}{\sigma^2}\right] \quad (14)$$

Since  $C_S(T)$  is monotonically decreasing in  $T$ , the capacity is maximized by minimizing  $T$ . However, due to the rms bandwidth constraint, the value of  $T$  cannot be arbitrarily small. Using the fact that the set  $\{\sqrt{\frac{2}{T}} \sin(\frac{i\pi t}{T})\}_{i=1}^{\infty}$  is a complete orthonormal set in the space of all rms band-limited signals in  $[0, T]$  [7], we can express  $s(t)$ , as

$$s(t) = \sum_{i=1}^{\infty} d_i \sqrt{\frac{2}{T}} \sin\left(\frac{i\pi t}{T}\right) \quad (15)$$

Then, the unit energy assumption and the constraint in the rms bandwidth (7) translate into

$$\sum_{i=1}^{\infty} d_i^2 = 1 \quad (16)$$

and

$$\sum_{i=1}^{\infty} i^2 d_i^2 \leq (2BT)^2 \quad (17)$$

respectively.

The minimum  $T$  consistent with (16) and (17) is chosen by taking equality in (17) and minimizing the left hand side of (17) subject to (16). Since

$$1 = \sum_{i=1}^{\infty} d_i^2 \leq \sum_{i=1}^{\infty} i^2 d_i^2 \quad (18)$$

with equality if and only if  $d_1 = 1$  and  $d_i = 0$  if  $1 < i$ , it follows that the optimum  $T$  is equal to  $\frac{1}{2B}$  which upon substitution in (14) results in the desired result. ■

## 3. Two-user Channel

We turn our attention to the main results of the paper, namely the optimization of the capacity region of the synchronous PAM channel (5b) with respect to the choice of the signature waveforms, including their duration  $T$ . In both the rms and the fobe bandwidth constrained problems, we will solve the problem in two stages:

1. Fix  $T$ , and find  $\rho^*(TB)$ , the minimum absolute cross-correlation,  $|\rho|$ , achievable under the time-bandwidth constraint (and the optimal waveforms which achieve that  $\rho$ .) Then, the capacity region for fixed  $T$  is given by  $C_V$  in (5b) evaluated at  $\rho = \rho^*(TB)$ . This is because  $C_V$  depends on the signature waveforms only through the rate-sum constraint which is monotonic decreasing in  $\rho$ .
2. Take the union of the capacity regions found in the first stage over all  $T$ . Note that there is a minimum value of  $T$  below which the time-bandwidth product is so small that no waveform can be found to satisfy the bandwidth constraint and therefore, the capacity region is an empty set. Also, there is a maximum value of  $T$  above which the allowed time-bandwidth product is so large that orthogonal signals can be assigned to both users, and therefore the capacity region decreases with  $T$  beyond that maximum value of  $T$ .

**Theorem 3.1.**

If  $TB \geq 0.5$ , then the minimum cross-correlation,  $\rho_G^*(TB)$ , between any two unit-energy signals of duration  $T$  and rms bandwidth less than or equal to  $B$  is

$$\rho_G^*(TB) = \max\{0, \frac{1}{3}[5 - 8(TB)^2]\}$$

and is achieved by the signature waveforms

$$s_1(t) = \sqrt{\frac{1 + \rho_G^*(TB)}{T}} \sin \frac{\pi t}{T} + \sqrt{\frac{1 - \rho_G^*(TB)}{T}} \sin \frac{2\pi t}{T}$$

$$s_2(t) = \sqrt{\frac{1 + \rho_G^*(TB)}{T}} \sin \frac{\pi t}{T} - \sqrt{\frac{1 - \rho_G^*(TB)}{T}} \sin \frac{2\pi t}{T}$$

If  $TB < 0.5$ , then there exists no signal of duration  $T$  and rms bandwidth less than or equal to  $B$ .

**Proof.**

If  $TB < 0.5$ , we have seen in the proof of Theorem 2.1, that there is no signal of duration  $T$  and rms bandwidth less than or equal to  $B$ .

If  $TB = 0.5$ , we have seen that there is only one signal of duration  $T$  and rms bandwidth  $B$  and is  $\sqrt{\frac{2}{T}} \sin \frac{\pi t}{T}$ ,  $t \in [0, T]$ . Therefore, the theorem follows immediately when  $TB = 0.5$ .

If  $TB > 0.5$ , let  $s_1(t)$ ,  $s_2(t)$  be any two unit-energy signals with duration  $T$  and rms bandwidth  $B$ . Using the same complete orthonormal set in the last theorem, we denote the vector  $\mathbf{M}(t) = [\sqrt{\frac{2}{T}} \sin(\frac{\pi t}{T}), \sqrt{\frac{2}{T}} \sin(\frac{2\pi t}{T}), \dots]^T$ ,  $t \in [0, T]$ , and express  $s_1(t)$  and  $s_2(t)$  as

$$s_k(t) = \mathbf{a}_k^T \mathbf{M}(t) \quad k = 1, 2 \quad (19)$$

Then, the rms bandwidth constraint can be expressed, via (7), as

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left(\frac{d}{dt} s(t)\right)^2 dt = \frac{1}{(2T)^2} \mathbf{a}_k^T \mathbf{\Pi} \mathbf{a}_k \leq B^2 \quad k = 1, 2 \quad (20)$$

where  $\mathbf{\Pi} = \text{diag}[1^2, 2^2, 3^2, \dots]$ . Denoting  $\rho$  as the cross-correlation, we can assume that, without loss of generality,  $0 \leq \rho$ . From the unit energy assumption, we have the cross-correlation matrix,  $\mathbf{H}$ , as

$$\mathbf{H} = \mathbf{A} \mathbf{A}^T \triangleq \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \end{bmatrix} [\mathbf{a}_1 \mathbf{a}_2] = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \quad (21)$$

Since the mapping between  $s_k(t)$  and  $\mathbf{a}_k$  is an one-to-one mapping, the problem is equivalent to finding the minimum  $\rho$  such that there exists  $\mathbf{A}$  satisfying (20) and (21).

We solve this problem by first giving a lower bound on the cross-correlation and then showing that the lower bound is achievable. Let  $B_a$  be the minimum of the sum of the rms bandwidth of  $M$  equal energy signals of duration  $T$  and correlation matrix,  $\mathbf{H}$ .  $B_a$  is found by Nuttall [7], as

$$B_a^2 = \frac{1}{(2T)^2} \frac{1}{M} \sum_{l=1}^r \mu_l^2 \quad (22)$$

where each  $\mu_l$  is the positive eigenvalue of  $\mathbf{H}$  with  $\mu_i \leq \mu_j$  for  $j \leq i$ , and  $r$  is the rank of  $\mathbf{H}$ .

Applying this result with  $M = 2$ ,  $r = 2$  (since  $s_1(t) \neq s_2(t)$  implies  $\rho \neq 1$ ) and the correlation matrix  $\mathbf{H}$  in (21), we get from (20) and (22) that

$$\frac{1}{2(2T)^2} [(1 + \rho) + 4(1 - \rho)] \leq B^2 \quad (23)$$

where it can be easily verified that  $1 + \rho$  and  $1 - \rho$  are eigenvalues of  $\mathbf{H}$  in (21).

After rearrangement, (23) becomes

$$\frac{1}{3} [5 - 8(TB)^2] \leq \rho \quad (24)$$

Since  $s_1(t)$  and  $s_2(t)$  are arbitrarily chosen, and  $\rho$  belongs to  $[0, 1]$ , we have the lower bound,

$$\max\left\{0, \frac{1}{3} [5 - 8(TB)^2]\right\} \leq \rho_G^*(TB)$$

We now show a signal pair that achieves this lower bound. Stimulated by the fact that the functions  $f(t)$  and  $f(T - t)$  have the same magnitude spectrum, we consider signature waveforms which are mirror images of each other about  $T/2$ . Also, we note that  $\sin \frac{\pi t}{T}$  is even about  $T/2$  while  $\sin \frac{2\pi t}{T}$  is odd about  $T/2$ . Therefore, we assume that the matrix  $\mathbf{A}$  has the form

$$\mathbf{A} = \begin{bmatrix} \alpha & \sqrt{1 - \alpha^2} & 0 & \dots \\ \alpha & -\sqrt{1 - \alpha^2} & 0 & \dots \end{bmatrix} \quad (25)$$

for some  $0 \leq \alpha \leq 1$ .

From (20), the rms bandwidth constraint becomes  $\sqrt{\frac{4 - 4(TB)^2}{3}} \leq \alpha$ . If we let  $\alpha = \sqrt{\frac{4 - 4(TB)^2}{3}}$  and substitute (25) into (21), we have  $\rho = 2\alpha^2 - 1 = \frac{5 - 8(TB)^2}{3}$ . If  $\frac{5 - 8(TB)^2}{3} < 0$ ,  $\sqrt{\frac{4 - 4(TB)^2}{3}} < \frac{1}{4}$  and we can let  $\alpha = \frac{1}{4}$  which gives  $\rho = 2\alpha^2 - 1 = 0$ . Therefore, we have shown that the lower bound is achievable by signature waveforms characterized by the matrix  $\mathbf{A}$  in (25), with  $\alpha = \sqrt{\frac{1 + \rho^*(TB)}{2}}$ . Then, (19) results in the optimal signature waveforms stated in the theorem. ■

**Theorem 3.2.**

The capacity region of the two-user PAM white Gaussian multiple-access channel with noise power spectral density, rms bandwidth and signal powers equal to  $\sigma^2$ ,  $B$ ,  $S_1$  and  $S_2$ , re-

spectively, is given by

$$C_G = \bigcup_{1 \leq \gamma \leq \sqrt{\frac{5}{2}}} (R_1, R_2) : \begin{cases} 0 \leq R_1 \leq \frac{B}{\gamma} \log[1 + \frac{S_1 \gamma}{2\sigma^2 B}] \\ 0 \leq R_2 \leq \frac{B}{\gamma} \log[1 + \frac{S_2 \gamma}{2\sigma^2 B}] \\ R_1 + R_2 \leq \frac{B}{\gamma} \log[1 + \frac{(S_1 + S_2) \gamma}{2\sigma^2 B}] + \\ \frac{S_1 S_2 \gamma^2}{4\sigma^4 B^2} (1 - \frac{4}{3}(\frac{5}{2} - \gamma^2)^2) \end{cases} \quad (26)$$

**Proof.**

Recall that the capacity region,  $C_G$ , is the union of  $C_V$  in (5b) evaluated at  $\rho^*(TB)$  over  $T$ . We proceed to find the range of  $T$  of interest. From the last theorem, if  $TB < 0.5$ , no signature waveforms can be found to satisfy the constraints and the capacity region is an empty set. Also, if  $TB \geq \sqrt{\frac{5}{8}}$ ,  $\rho^*(TB) = 0$ , and the capacity region for fixed  $T$  is a pentagon which is monotonic decreasing in  $T$ . Therefore, the range of  $T$  in interest is the interval  $[\frac{1}{2B}, \frac{1}{2B}\sqrt{\frac{5}{2}}]$ . Denoting  $2TB$  by  $\gamma$ , and substituting  $\gamma$  into  $C_V$  in (5b), we have, after taking the union,  $C_G$  in the theorem. ■

At a first glance, it seems that there is a conflict with Theorem 2.1 since the total capacity of  $C_G$  is larger than the single-user capacity of an rms band-limited channel with power constraint  $S_1 + S_2$ . However, the signal transmitted over the channel in the two-user case is a sum of two PAM signals and, in general, it is no longer a PAM signal since the signals in different time slots need not have the same shape.

Figure 1 shows the capacity region of the rms band-limited PAM two-user channel,  $C_G$  and the strictly band-limited two-user channel,  $C_C$ . In contrast to the single-user case where they coincide,  $C_C$  is a subset of  $C_G$ . It can also be seen from (26) and (2) that  $C_C$  is the pentagon inside the union in (26) when  $\gamma = 1$ . However, by increasing  $\gamma$ , we trade off the decrease in the single-user rate by the increase in the rate sum, such that the union gives a larger capacity region,  $C_G$ . This indicates that, in the two-user case, the laxer bandwidth constraint more than offsets the additional structure (PAM) in the time domain.

Figure 2 and 3 show the signature waveforms which achieve the boundary points of the capacity region for two different time-bandwidth products. The signature waveforms are mirror images of each other and as  $\gamma$  increases, they become more asymmetric so as to decrease the cross-correlation while maintaining the same rms bandwidth.

Finally, although the union in Theorem 3.2 is taken over  $\gamma$  in the interval  $[1, \sqrt{\frac{5}{2}}]$ , not every  $\gamma$  in that interval achieves some boundary points of  $C_G$ . The set of values of  $\gamma$  that achieves boundary points of  $C_G$  is a function of the signal-to-noise ratios,  $\frac{S_k}{2\sigma^2 B}$ ,  $k = 1, 2$ . According to Figure 1, the boundary points in the segments AB and EF are achieved by  $\gamma = 1$ , while those in the segment CD are achieved by some  $\gamma_{\max}$  in  $[1, \sqrt{\frac{5}{2}}]$  depending on the signal-to-noise ratios. The boundary points in BC and DE are achieved by  $1 \leq \gamma \leq \gamma_{\max}$ .

We now proceed to the optimal signal design problem under  $\alpha$ -fobe bandwidth constraint. Denote the prolate spheroidal wave functions ([12], [13], and [14]) as  $\psi_i(TB, t)$  and the associated eigenvalues as  $\lambda_i(TB)$ , i.e.

$$\lambda_i(TB) \psi_i(TB, t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_i(TB, \tau) \frac{\sin[2\pi TB(t - \tau)]}{\pi(t - \tau)} d\tau$$

for  $i = 0, 1, 2, \dots$  and  $\lambda_0(TB) > \lambda_1(TB) > \lambda_2(TB) >$

$\dots$ . It is known [8] that  $\psi_0(TB, \frac{t}{T} - \frac{1}{2})$  and  $\psi_1(TB, \frac{t}{T} - \frac{1}{2})$  are even and odd about  $\frac{T}{2}$  respectively and the set  $\{\frac{1}{\sqrt{\lambda_i(TB)}} \psi_i(TB, \frac{t}{T} - \frac{1}{2})\}$  forms a complete orthonormal set in  $[0, T]$ . Also,  $\lambda_0(TB)$  and  $\lambda_0(TB) + \lambda_1(TB)$  are continuous and monotonic increasing in  $TB$  (Figure 4).

**Theorem 3.3.**

For any  $0 < \alpha < 1$ ,

If  $TB \geq \lambda_0^{-1}(\alpha)$ , then the minimum cross-correlation,  $\rho_F^*(TB)$ , between any two unit-energy signals of duration  $T$ , and  $\alpha$ -fobe bandwidth less than or equal to  $B$  is

$$\rho_F^*(TB) = \max \left\{ 0, \frac{2\alpha - \lambda_0(TB) - \lambda_1(TB)}{\lambda_0(TB) - \lambda_1(TB)} \right\}$$

and is achieved by the signature waveforms

$$s_1(t) = \sqrt{\frac{1 + \rho_F^*(TB)}{2\lambda_0(TB)}} \psi_0(TB, \frac{t}{T} - \frac{1}{2}) + \sqrt{\frac{1 - \rho_F^*(TB)}{2\lambda_1(TB)}} \psi_1(TB, \frac{t}{T} - \frac{1}{2})$$

$$s_2(t) = \sqrt{\frac{1 + \rho_F^*(TB)}{2\lambda_0(TB)}} \psi_0(TB, \frac{t}{T} - \frac{1}{2}) - \sqrt{\frac{1 - \rho_F^*(TB)}{2\lambda_1(TB)}} \psi_1(TB, \frac{t}{T} - \frac{1}{2})$$

If  $TB < \lambda_0^{-1}(\alpha)$ , then there exists no signal of duration  $T$  and  $\alpha$ -fobe bandwidth less than or equal to  $B$ .

**Proof.**

As in Theorem 3.1, we would like to find a suitable complete orthonormal set in  $[0, T]$ . To that end, we rewrite the definition of  $\alpha$ -fobe bandwidth as

$$\begin{aligned} \alpha &\leq \int_{-B}^B |S(f)|^2 df \\ &= \int_0^T \int_0^T s(t)s(\tau) \int_{-B}^B e^{j2\pi f(t-\tau)} df dt d\tau \\ &= \int_0^1 \int_0^1 s\left(\frac{t}{T}\right) \frac{\sin(2\pi TB(t-\tau))}{\pi(t-\tau)} s\left(\frac{\tau}{T}\right) dt d\tau \quad (27) \end{aligned}$$

Since the prolate spheroidal wave functions are eigenfunctions of the kernel  $2TB \text{sinc}(t - \tau)$ , a good choice for the complete orthonormal set will be the set of all prolate spheroidal wave functions.

For notational convenience, we will drop the explicit dependence on  $TB$  of the eigenvalues of the prolate spheroidal wave functions. If  $TB \geq \lambda_0^{-1}(\alpha)$ , we can express any  $s_1(t)$  and  $s_2(t)$  in terms of  $\Psi(t) = [\frac{1}{\sqrt{\lambda_0}} \psi_0(TB, \frac{t}{T} - \frac{1}{2}), \frac{1}{\sqrt{\lambda_1}} \psi_1(TB, \frac{t}{T} - \frac{1}{2}), \dots]$ ,  $t \in [0, T]$ , as

$$s_k(t) = \mathbf{a}_k^T \Psi(t) \quad k = 1, 2 \quad (28)$$

Using (27) and (28), we have

$$\alpha \leq \int_{-B}^B |S(f)|^2 df = \mathbf{a}_k^T \mathbf{\Lambda} \mathbf{a}_k = \text{tr}(\mathbf{\Lambda} \mathbf{a}_k \mathbf{a}_k^T) \quad k = 1, 2 \quad (29)$$

where  $\mathbf{\Lambda} = \text{diag}[\lambda_0, \lambda_1, \lambda_2, \dots]$ . Also, the cross-correlation matrix,  $\mathbf{H}$ , is

$$\mathbf{H} = \mathbf{A} \mathbf{A}^T = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \end{bmatrix} [\mathbf{a}_1 \mathbf{a}_2] = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \quad (30)$$

Similar to the rms case, we find the lower bound by maximizing the average over  $k = 1, 2$  of the right hand side of (29). Rewriting the average, we have

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^2 \mathbf{a}_k^T \Lambda^T \mathbf{a}_k &= \frac{1}{2} \text{tr}(\Lambda \Lambda^T \mathbf{A}) \\ &= \frac{1}{2} \text{tr}(\Lambda \mathbf{P}_A \Xi_A \mathbf{P}_A^T) \\ &= \frac{1}{2} \text{tr}(\Xi_A \mathbf{P}_A^T \Lambda \mathbf{P}_A) \end{aligned} \quad (31)$$

where  $\Lambda^T \mathbf{A}$  is diagonalized by the orthonormal matrix,  $\mathbf{P}_A$ , and  $\Xi_A = \text{diag}[\xi_1 \ \xi_2 \ 0 \ \dots]$ . Since the eigenvalues of  $\Lambda \Lambda^T$  and  $\Lambda^T \mathbf{A}$  are the same, we have  $\xi_1 = 1 + \rho$  and  $\xi_2 = 1 - \rho$ .

Now, let's denote  $\mathbf{P}$  as the  $2 \times \infty$  matrix formed by taking only the first two rows of  $\mathbf{P}_A^T$ , and  $\Xi$  as  $\text{diag}[\xi_1 \ \xi_2]$ . Then, the maximum of the average is

$$\max_{\mathbf{P} \mathbf{P}^T = \mathbf{I}_{2 \times 2}} \text{tr}(\Xi \mathbf{P} \Lambda \mathbf{P}^T) \quad (32)$$

We will solve the maximization problem using the Lagrange multiplier method. We form the Lagrangian,

$$\sum_{k=1}^2 \xi_k \mathbf{P}_k^T \Lambda \mathbf{P}_k + \sum_{k=1}^2 \sum_{n=1}^k z_{nk} (\mathbf{P}_k^T \mathbf{P}_n - \delta_{nk}) \quad (33)$$

where  $\mathbf{P}_k^T$  is the  $k^{\text{th}}$  row of  $\mathbf{P}$ . Taking derivative with respect to  $\mathbf{P}_k$ , we have

$$\xi_1 \Lambda \mathbf{P}_1 + z_{11} \mathbf{P}_1 + 2z_{12} \mathbf{P}_2 = 0 \quad (34)$$

and

$$\xi_2 \Lambda \mathbf{P}_2 + z_{22} \mathbf{P}_2 + 2z_{12} \mathbf{P}_1 = 0 \quad (35)$$

If we pre-multiply (34) by  $\mathbf{P}_2^T$  and (35) by  $\mathbf{P}_1^T$ , we have  $z_{12} = 0$  since  $\xi_1 \neq \xi_2$ ; therefore, from (34) and (35),  $\mathbf{P}_1$  and  $\mathbf{P}_2$  are eigenvectors of  $\Lambda$ . Since  $\Lambda$  is diagonal and the diagonal elements are distinct and decreasing down the diagonal axis, we have

$$\mathbf{P} = [\mathbf{I}_{2 \times 2} \ 0] \quad (36)$$

Substituting back into (31), we show that the maximum value of (31) is  $\lambda_0 \frac{1+\rho}{2} + \lambda_1 \frac{1-\rho}{2}$ . Comparing to (29), we have

$$\alpha \leq \frac{1}{2} [(\lambda_0 + \lambda_1) + \rho(\lambda_0 - \lambda_1)]$$

or, together with  $0 \leq \rho \leq 1$ ,

$$\rho \geq \max \left\{ 0, \frac{2\alpha - \lambda_0 - \lambda_1}{\lambda_0 - \lambda_1} \right\}$$

The achievability of the lower bound can be verified, as in the rms case, by letting

$$\mathbf{A} = \begin{bmatrix} \sqrt{\frac{1+\rho}{2}} & \sqrt{\frac{1-\rho}{2}} & 0 & \dots \\ \sqrt{\frac{1+\rho}{2}} & -\sqrt{\frac{1-\rho}{2}} & 0 & \dots \end{bmatrix} \quad (37)$$

which corresponds to the optimal set of signals stated in the theorem.

The proof of the second part of the theorem ( $TB < \lambda_0^{-1}(\alpha)$ ) can be found in [13, p.54]. ■

### Theorem 3.4.

The capacity region of the two-user PAM white Gaussian multiple-access channel with noise power spectral density,  $\alpha$ -lobe bandwidth and signal powers equal to  $\sigma^2$ ,  $B$ ,  $S_1$ , and  $S_2$ , respectively, is given by

$$C_F = \bigcup_{\gamma_{\min} \leq \gamma \leq \gamma_{\max}} \left\{ (R_1, R_2) : \begin{cases} 0 \leq R_1 \leq \frac{B}{\gamma} \log \left[ 1 + \frac{S_1 \gamma}{2\sigma^2 B} \right] \\ 0 \leq R_2 \leq \frac{B}{\gamma} \log \left[ 1 + \frac{S_2 \gamma}{2\sigma^2 B} \right] \\ R_1 + R_2 \leq \frac{B}{\gamma} \log \left[ 1 + \frac{(S_1 + S_2) \gamma}{2\sigma^2 B} + \frac{S_1 S_2 \gamma^2}{4\sigma^4 B^2} \left[ 1 - \left( \frac{2\alpha - \lambda_0(\frac{\gamma}{2}) - \lambda_1(\frac{\gamma}{2})}{\lambda_0(\frac{\gamma}{2}) - \lambda_1(\frac{\gamma}{2})} \right)^2 \right] \right] \end{cases} \right\}$$

where  $\alpha = \lambda_0(\frac{\gamma_{\min}}{2}) = \frac{1}{2} [\lambda_0(\frac{\gamma_{\max}}{2}) + \lambda_1(\frac{\gamma_{\max}}{2})]$ .

### Proof.

The proof is very similar to that in Theorem 3.2 where  $\gamma = 2TB$ . The lower limit of  $\gamma$  is carried over from Theorem 3.3, while  $\gamma_{\max}$  is the smallest  $\gamma$  such that  $\rho_F(\frac{\gamma}{2}) = 0$ . ■

Notice that the range of  $\gamma$  in taking the union is only a function of  $\alpha$ . In Figure 4, we show  $\lambda_0(TB)$  and  $\frac{1}{2}[\lambda_0(TB) + \lambda_1(TB)]$  vs the time-bandwidth product, and  $\gamma_{\min}$  and  $\gamma_{\max}$  can be obtained directly from the figure. Also, Figure 5 shows the capacity region,  $C_F$ , with the capacity region of the strictly band-limited channel,  $C_C$ . Similar comments to those we made in the rms case apply to the values of  $\gamma$  that achieve the boundary points in the capacity region. However, we see that for sufficiently high  $\alpha$ ,  $C_F$  does not contain  $C_C$  in contrast to the rms case.

Finally, in Figure 6, we show the signature waveforms which are, as expected, mirror images of each other. However, in contrast to the rms case where the signature waveforms must be zero at the end points to have finite rms bandwidth, the transmitted signal waveform in the  $\alpha$ -lobe case may have jumps at  $t = iT$ .

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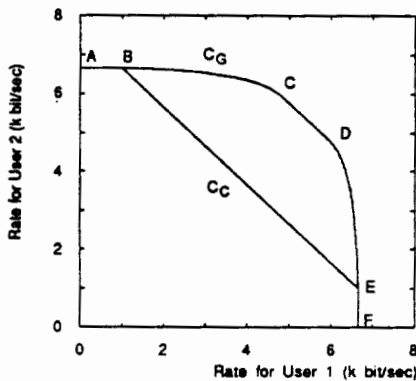


Figure 1. Capacity Regions in the rms case for SNR1=SNR2=20db, B=1kHz

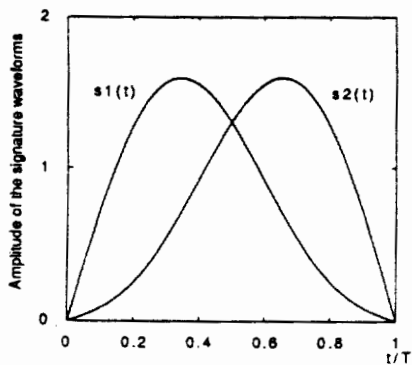


Figure 2. Signature waveforms for two-user rms band-limited channel, TB=0.6

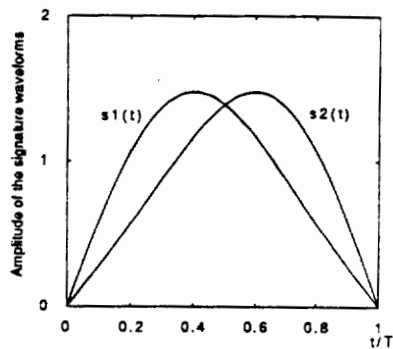


Figure 3. Signature waveforms for two-user rms band-limited channel, TB=0.525

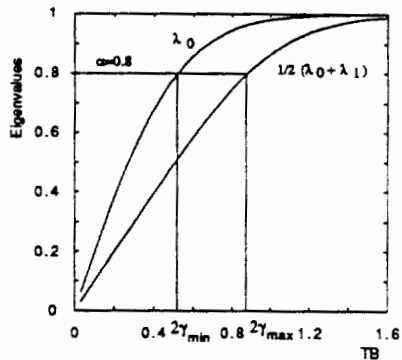


Figure 4. Eigenvalues vs. time-bandwidth product

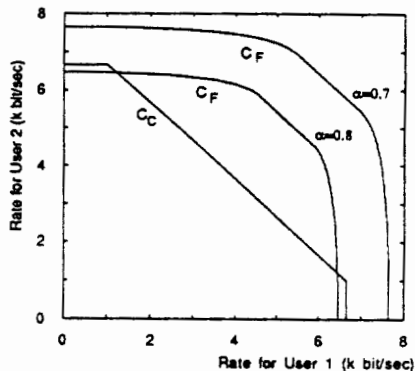


Figure 5. Capacity Regions in the lobe case for SNR1=SNR2=20db, B=1kHz

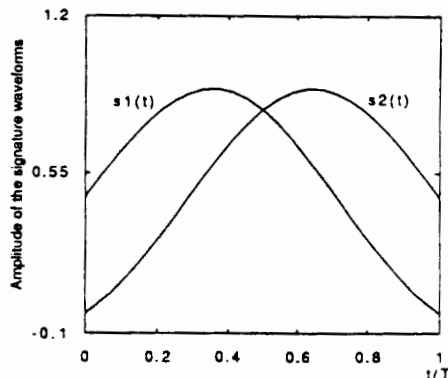


Figure 6. Signature waveforms for two-user 0.8-lobe band-limited channel, TB=0.955