Optimal Signal Design for Band-Limited PAM Synchronous Multiple-Access Channels

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Abstract

The optimal signal design problem for a band-limited PAM symbol-synchronous Gaussian two-user multiple-access channel is investigated. Using the root-mean-square and the fractional out-of-band energy bandwidth definitions, we find the capacity region of the channel and the signature waveforms that achieve each point in the capacity region. The optimal pair of signature waveforms are mirror images of each other, and are obtained by minimizing their cross-correlation subject to a fixed duration constraint and an in-band constraint. The two-user capacity region, in the rms case, is found to contain the capacity region of the two-user strictly band-limited Gaussian channel. This demonstrates the fact that by relaxing constraints in the frequency domain, we can introduce structure (PAM) in the time domain and obtain a larger capacity region.

1. Introduction

The capacity region of the two-user discrete-time Gaussian multiple-access channel

\[ y_i = x_1 + x_2 + n_i \]

where \( n_i \) is an i.i.d. Gaussian sequence with variance equal to \( \sigma^2 \) and the energy of each codeword is constrained to satisfy

\[ \frac{1}{N} \sum_{i=1}^{N} x_i^2 \leq W_0 \]

is equal to the Cover-Wyner pentagon [1,2]:

\[ C_D = \left\{ (R_1, R_2) : 0 \leq R_1 \leq \frac{1}{2} \log(1 + \frac{W_0}{\sigma^2}) \right\} \]

(1)

in information units per channel use. Analogously, the capacity region of the continuous-time band-limited channel with noise power spectral density, bandwidth, and 4th user signal power equal to \( w^2 \), \( B \), and \( X_4 \) respectively, is given by [3], as (in units per second)

\[ C_C = \left\{ (R_1, R_2) : 0 \leq R_1 \leq B \log(1 + \frac{W_0}{\sigma^2}) \right\} \]

(2)

This capacity region is achieved by approximately band-limited and approximately time-limited waveforms which have no particular structure. In order to deal with modulation and demodulation schemes with manageable complexity, it is customary in digital communications to introduce structure on the transmitted waveforms by slotting the time domain into intervals of length \( T \) and sending a symbol in each slot by means of a digital modulation format such as PAM. PAM...

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PAM, etc. In the case of PAM (Pulse Amplitude Modulation), the \( k \)th user is assigned a fixed deterministic waveform, \( a_k(t) \), which is time-limited to \( [0, T] \) and is modulated by the information stream. The fact that the two transmitters are symbol-synchronous, the PAM two-user multiple-access channel becomes

\[ y(t) = \sum_{k=1}^{K} a_k(t)(1 - \delta(t - T_k)) + n(t) \]

(3)

where \( n(t) \) is white Gaussian noise with spectral density \( \sigma^2 \) and \( \{a_k(t)\} \) is the symbol stream transmitted by the \( k \)th user. Assuming that, without loss of generality, the signature waveforms have unit energy, the energy constraints on the transmitted waveforms become

\[ \sum_{k=1}^{K} \| a_k(t) \|_2^2 = W_0 \]

(4)

It is easy to show that if \( a_1(t) = a_2(t) \), then the capacity of (2) under constraints (4) is equal to the Cover-Wyner pentagon (1) (this result remains true even if the users are completely asynchronous [3]). If the signature waveforms are not necessarily identical, then the Cover-Wyner pentagon generalizes to

\[ C_Y = \left\{ (R_1, R_2) : 0 \leq R_1 \leq \frac{1}{2} \log(1 + \frac{W_0}{\sigma^2}) \right\} \]

(24)

in information units per channel use or

\[ C_Y = \left\{ (R_1, R_2) : 0 \leq R_1 \leq B \log(1 + \frac{W_0}{\sigma^2}) \right\} \]

(25)

in information units per second, where \( \rho = \frac{1}{2} \| a_1(2T) \|_2^2 \) is the cross-correlation between the signature waveforms. A natural question to address is the choice of the unit-energy waveforms \( a_1(t) \) and \( a_2(t) \) to maximize the capacity region \( C_Y \). It is clear that the unconstrained solution is to choose orthogonal signature waveforms. Then, \( \rho = 0 \), and the multiple-access channel is decoupled into independent single-user channels, and each transmitter can transmit at single-user capacity. However, in practice, there are constraints on the choice of the signals (e.g., in Spread Spectrum CDMA systems, the waveform may be constrained to be Pseudo Noise shift register sequences of given period, and it is not always possible to assign orthogonal waveforms for all users). In this paper, we will address the optimization of the signature waveforms and their duration \( T \) under bandwidth constraints. Since the signature waveforms are strictly time-limited, they cannot be strictly band-limited, and the need arises to quantify the bandwidth of these waveforms. There are several established ways to accomplish this [5]. In this paper, we will consider the two bandwidth measures of baseband signals that have received most attention from the information theoretic community: the
root mean square (rms) bandwidth and the fractional out-of-band energy (fobe) bandwidth. The fobe bandwidth was popularized by Gabor [6] (it is sometimes referred to as Gabor bandwidth) and studied subsequently in [5], [7], and [8]. A finite-energy signal \( x(t) \) has rms bandwidth \( B \) if the Fourier transform \( \hat{x}(f) \) satisfies

\[
\int_{-\infty}^{\infty} |\hat{x}(f)|^2 df \leq B^2 \tag{6}
\]

i.e., the rms bandwidth is the square root of the "second moment" of the energy spectral density \( |\hat{x}(f)|^2 \) of the normalized signal set, proportional to the square root of the energy of its derivative.

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{x}(f)|^2 df = B^2 \tag{7}
\]

The fobe bandwidth has been used in, say, [3] and is defined as the bandwidth necessary to encompass a given fraction (say \( \alpha \)) of the signal energy, i.e., the \( \alpha \) fobe bandwidth is \( B \) if

\[
\int_{-\infty}^{B} |\hat{x}(f)|^2 df = \alpha \int_{-\infty}^{\infty} |\hat{x}(f)|^2 df \tag{8}
\]

Notice that the bandwidth constraints imposed on the signature waveforms will be inherited by the transmitted signals because, as is well known [8], the power spectral density of \( \sum_k h_k(t-kT-t) \) where \( t \) is uniformly distributed in \([0, T]\) and \( h_k(t) \) is an i.i.d. sequence, is a scaled version of the energy spectral density \( |\hat{h}_k(f)|^2 \).

2. Single-user Channel

Before solving for the capacity of the PAM multiple-access channel under bandwidth constraints, it is enlightening to examine the Fourier transform of a channel with constrained rms bandwidth. This channel differs from the channel band-limited Gaussian channel in that the allowable transmitted signals \( 1 \) have much more structure (PAM and 2 D) and are rms band-limited but not strictly band-limited. It turns out that the effect of the latter bandwidth measure cancels the effect of the additional structure imposed on the transmitted signals in the time domain, and the capacity of the channel is given by the celebrated Shannon formula [6].

\[ C_S = B \log_2[1 + \frac{S}{2PB}] \tag{9} \]

Proof.

The single-user PAM white Gaussian channel is a special case of (3):

\[ y(t) = \sum_k h_k(t-kT) + n(t) \tag{10} \]

Assuming that, without loss of generality, \( n(t) \) has unit energy, the power constraint becomes

\[ \frac{1}{n} \sum_{i=1}^{n} y_i^2 \leq TS \tag{11} \]

and the T-shifts of \( n(t) \). (\( n(t-T) \)) form an orthogonal set. The projections of \( y(t) \) on this orthonormal set are equal to

\[ y_i = \int_{-T}^{T} y(t) n(t-iT)dt \quad i = 1, \ldots, n \tag{12} \]

or, substituting \( y(t) \) from (10),

\[ y_i = h_i + n_i \tag{12} \]

where \( h_i(t) \) is an i.i.d. Gaussian sequence with variance equal to \( n_i \).

The important point to note is that \( y_i(t) \) are sufficiently correlated for the transmitted messages; therefore, the capacity of the PAM channel (10) for a fixed \( T \) coincides with the capacity of the discrete time memoryless channel (13) with constraint (11), which is given by (e.g. [11]) in units per second

\[ C_T(T) = \frac{1}{2T} \log_2 [1 + \frac{S^2}{2T}] \tag{14} \]

Since \( C_T(T) \) is monotonically decreasing in \( T \), the capacity is maximized by minimizing \( T \). However, due to the rms bandwidth constraint, the value of \( T \) cannot be arbitrarily small. Using the fact that the set \( \{ \sqrt{\text{var}[n(t)]} \}_{i=1}^{n} \) is a complete orthonormal set in the space of all \( n(t) \) bounded signals in \([0, T]\), we can express \( y(t) \), as

\[ y(t) = \sum_{i=1}^{n} d_i \sqrt{\text{var}[n(t)]} \tag{15} \]

Then, the exit entropy assumption and the constraint in the rms bandwidth (7) translate into

\[ \sum_{i=1}^{n} d_i^2 = 1 \tag{16} \]

and

\[ \sum_{i=1}^{n} d_i^2 \leq (2BT)^2 \tag{17} \]

respectively. The minimum \( T \) consistent with (16) and (17) is chosen by taking equality in (17) and minimizing the left hand side of (17) subject to (16). Since

\[ \sum_{i=1}^{n} d_i^2 \leq \sum_{i=1}^{n} d_i^2 \tag{18} \]

with equality if and only if \( d_i = 1 \) and \( d_i = 0 \) if \( i \neq \) then it follows that the optimum \( T \) is equal to \( \frac{1}{\text{var}[n(t)]} \) which upon substitution in (14) results in the desired result.

3. Two-user Channel

We turn our attention to the main results of the paper, namely the optimization of the capacity region of the synchronous PAM channel (3b) with respect to the choice of the signature waveforms, including their duration \( T \). In both the rms and the fobe bandwidth constrained problems, we will solve the problem in two stages.
1. Fit $T$, and find $\rho(T)$, the minimum absolute cross-correlation, $\rho(T)$, achievable under the time-bandwidth constraint (and the optimal waveform) which achieve that $\rho$. Then, the capacity region for fixed $T$ is given by $C_T$ in (18) evaluated at $\rho = \rho(T)$. This is because $C_T$ depends on the signature waveform only through the max-min constraint which is non-convex decreasing in $\rho$.

2. Take the union of the capacity regions found in the first stage over all $T$. Note that there is a minimum value of $T$ below which there are no waveforms that can satisfy the time-bandwidth constraint and therefore the capacity region is an empty set. Also, there is a maximum value of $T$ above which the allowed time-bandwidth product is so large that orthogonal signals can be assigned to both users, and therefore the capacity region decreases with $T$ beyond that maximum value of $T$.

Theorem 3.1.

If $TB \geq 0.5$, then the minimum cross-correlation, $\rho_T(TB)$, between any two-unit energy signals of duration $T$ and rms bandwidth less than or equal to $B$ is

$$\rho_T(TB) = \max_0\left\{ \frac{1}{TB^2}, \frac{1}{TB} \right\}$$

and is achieved by the optimum waveform

$$x(t) = \frac{\text{sign}(t)}{\sqrt{T}}, \quad y(t) = \frac{\text{sign}(t)}{\sqrt{B}}$$

If $TB < 0.5$, then there exists no signal of duration $T$ and rms bandwidth less than or equal to $B$.

Proof.

If $TB < 0.5$, we have seen in the proof of Theorem 2.1, that there is no signal of duration $T$ and rms bandwidth less than or equal to $B$.

If $TB = 0.5$, we have seen that there is only one signal of duration $T$ and rms bandwidth $B$ is $\frac{\sqrt{2}}{\sqrt{T}}$, $\frac{\sqrt{2}}{\sqrt{B}}$, $t \in [0, T]$. Therefore, the theorem follows immediately when $TB = 0.5$.

If $TB > 0.5$, let $x(t), y(t)$ be any two unit-energy signals with duration $T$ and rms bandwidth $B$. Using the same complete orthonormal set in the last theorem, we denote the vector $M(t) = \{ \sqrt{T} \sin(T), \int_0^T \sin^2(t), \int_0^T \sin(t) \} t \in [0, T]$.

Then $x(t) = x(t) M(t)$ and $y(t) = y(t) M(t)$ (19)

Then, the rms bandwidth constraint can be expressed, via (7), as

$$\int_0^T |x(t)|^2 dt = B^2$$

Then $x(t)$ is identical.$^{2}, 3^*, 4^*, \ldots$. Denoting $\rho$ as the cross-correlation, we can assume that, without loss of generality, $0 \leq \rho$. From the unit energy assumption, we have the cross-correlation matrix, $H$ as

$$H = AA^H \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix} \begin{bmatrix} \rho & 1 \\ \rho^* & 1 \end{bmatrix}$$

Since the mapping between $x(t)$ and $y(t)$ is an one-to-one mapping, the problem is equivalent to finding the minimum $\rho$ such that the lower bound on the signal-to-noise ratio (SNR) is achievable. Let $\rho_{\text{min}}$ be the minimum of the sum of the rms bandwidth of all equal energy signals of duration $T$ and correlation matrix $H$. Hence, $\rho_{\text{min}}$ is found by solution (17), as

$$B^2 \rho = \frac{1}{(2\pi T)^2} \frac{1}{\int_0^T |x(t)|^2 dt}$$

where each $\rho_{\text{min}}$ is the positive eigenvalue of $H$ with $\rho_{\text{min}} \leq \rho_{\text{max}}$ for $\rho_{\text{max}} = \rho_{\text{max}}$. Applying this result with $M = 2, \tau = 2$ (since $x(t) = x(t)$ implies $\rho = 1$) and the correlation matrix $H$ in (21), we get from (20) and (22) that

$$\frac{1}{(2\pi T)^2} \left[ (1 + \frac{1}{\rho_{\text{max}}}) + (1 - \rho_{\text{max}}) \right] \leq B^2$$

where it can be easily verified that $1 + \rho_{\text{max}}$, $1 - \rho_{\text{max}}$ are eigenvalues of $H$ in (21).

After rearrangement, (23) becomes

$$\frac{1}{(2\pi T)^2} \left[ (1 + \frac{1}{\rho_{\text{max}}}) + (1 - \rho_{\text{max}}) \right] \leq B^2$$

Since $x(t)$ and $y(t)$ are arbitrarily chosen, and $\rho$ belongs to $[0, 1]$, we have the lower bound,

$$\rho_{\text{min}} \leq \rho$$

We now show a signal pair that achieves this lower bound. Stimulated by the fact that the functions $(t)$ and $\sqrt{T}$ are both in the same magnitude spectrum, we consider signaI waveforms which are mirror images of each other about $T/2$.

Also, we note that $t$ is even about $T/2$ while $t$ is odd about $T/2$. Therefore, we assume that the matrix $A$ has the form

$$A = \begin{bmatrix} a & \sqrt{a} & \sqrt{a} & \ldots \\ \sqrt{a} & \sqrt{a} & \sqrt{a} & \ldots \\ \sqrt{a} & \sqrt{a} & \sqrt{a} & \ldots \\ \end{bmatrix}$$

(25)

for some $0 \leq a \leq 1$.

From (20), the rms bandwidth constraint becomes

$$\frac{1}{(2\pi T)^2} \rho \leq B^2$$

If we let $a = \sqrt{\frac{1}{(2\pi T)^2} \rho}$ and substitute (25) into (21), we have $\rho = 2a^2 - 1 = \frac{1}{2} \sqrt{1 - \frac{1}{\rho}}$. If $\frac{1}{\rho} < 0$, then $\frac{1}{\rho} < 0$ and we can let $a = \frac{1}{2}$ which gives $\rho = 2a^2 - 1 = 0$. Therefore, we have shown that the lower bound on the signal-to-noise ratio is achievable by signature waveform characterized by the matrix $A$ in (25), with $a = \frac{1}{2} \sqrt{1 - \frac{1}{\rho}}$. Then, (23) results in the optimal signature waveform stated in the theorem.

Theorem 3.2.

The capacity region of the two-user $1/2$-user white Gaussian multiplex-access channel with noise power spectral density, rms bandwidth and signal power equal to $\sigma^2$, $B$, $\tau$, and $\Sigma$, respectively.
Repeal. Proof. Recall that the capacity region \( C_C \) is the union of \( C_Y \) in (26) evaluated at \( \phi(T) \) over \( T \). We proceed to find the range of \( T \) of interest. From the last theorem, if \( T < 0.5 \), then signum-waveforms can be found to satisfy the constraints and the capacity region is an empty set. Also, if \( T \geq \sqrt{4/\pi} \), then \( \cap \{ T \geq \sqrt{4/\pi} \} \) is a perfect square, which is monotonically decreasing in \( T \). Therefore, the range of \( T \) in interest is the interval \( \{ T : \sqrt{4/\pi} \leq T < 1 \} \). Denoting \( \beta(T) \) by \( \gamma \), and substituting \( \gamma \) into \( C_Y \) in (26), we now, after taking the union, the capacity region \( C_C \) in the theorem.

At first glance, it seems that there is a conflict with Theorem 2.1 since the total capacity of \( C_C \) is larger than the single-user capacity of \( \sqrt{1/2} \beta(T) \) over \( T \). We proceed to find the
range of \( T \) of interest. From the last theorem, if \( T < 0.5 \), then signum-waveforms can be found to satisfy the constraints and the capacity region is an empty set. Also, if \( T \geq \sqrt{4/\pi} \), then \( \cap \{ T \geq \sqrt{4/\pi} \} \) is a perfect square, which is monotonically decreasing in \( T \). Therefore, the range of \( T \) in interest is the interval \( \{ T : \sqrt{4/\pi} \leq T < 1 \} \). Denoting \( \beta(T) \) by \( \gamma \), and substituting \( \gamma \) into \( C_Y \) in (26), we now, after taking the union, the capacity region \( C_C \) in the theorem.

Theorem 3.3

For any \( 0 < a < 1 \),
\[ S_T \triangleright \lambda^a(\sigma) \]
the minimum cross-correlation, \( \rho^a(T) \) satisfies, among any two unit-energy signals of duration \( T \), and a fano bandwidth less than or equal to \( B \)
\[ \rho^a(T) = \max \left\{ 0, \frac{2a - 2\lambda^a(T) - \Lambda^a(T)}{\lambda^a(T) - \Lambda^a(T)} \right\} \]
and is achieved by the signature waveforms
\[ s(t) = \begin{cases} \frac{1}{\sqrt{2a}} s(t), & \text{for } |s(t)| = \frac{1}{\sqrt{2a}} \sqrt{2a} \beta(1-\gamma(T)) + \frac{1}{\sqrt{2a}} s(t) \beta(1-\gamma(T)) \end{cases} \]
\[ s(t) = \begin{cases} \lambda^a(T) - \lambda^a(T) s(t), & \text{for } |s(t)| = \frac{1}{\sqrt{2a}} \sqrt{2a} \beta(1-\gamma(T)) + \frac{1}{\sqrt{2a}} s(t) \beta(1-\gamma(T)) \end{cases} \]

If \( T < \lambda^a(\sigma) \), then there exists no signal of duration \( T \) and a fano bandwidth less than or equal to \( B \).

Proof.

As in Theorem 3.1, we would like to find a suitable complete orthogonal set in \([0, T]\). To that end, we rewrite the definition of fano bandwidth as

\[ \alpha \leq \int_0^T |s(t)|^2 d\tau \]

\[ \leq \frac{1}{2} \int_0^T \beta(1-\gamma(T)) s(t)^2 d\tau + \frac{1}{2} \int_0^T \beta(1-\gamma(T)) s(t)^2 d\tau \]

\[ \leq \frac{1}{2} \int_0^T \beta(1-\gamma(T)) d\tau \]

Since the prolate spherical wave functions are eigenfunctions of the kernel \( F_2B(\cdot, \cdot) \), a good choice for the complete orthogonal set will be the set of all prolate spherical wave functions. For notational convenience, we will drop the explicit dependence on \( T \) of the eigenvalues of the prolate spherical wave functions. \( \rho(t) \) is a good choice for the complete orthogonal set.

For notational convenience, we will drop the explicit dependence on \( T \) of the eigenvalues of the prolate spherical wave functions. \( \rho(t) \) is a good choice for the complete orthogonal set.

We now proceed to the optimal signal design problem under o-fano bandwidth constraints. Define the prolate spherical wave functions, \( W(i, \sigma, k) \) as \( \psi(t, \sigma, k) \) and the associated eigenvalues as \( \lambda^a(T) \).

\[ \lambda^a(T, \psi(t, \sigma, k)) = \int_0^T \psi(t, \sigma, k)(\alpha \beta(T) + \frac{1}{2} \int_0^T |s(t)|^2 d\tau) d\tau \]

for \( i = 0, 1, 2, \ldots \) and \( \lambda^a(T) > \lambda^a(T) > \lambda^a(T) > \ldots \)

In view of (8) \( \psi(t, \sigma, k) \) and \( \psi(t, \sigma, k) \) are even and odd about \( \frac{T}{2} \) respectively and the set \( \{ \lambda^a(T) \psi(t, \sigma, k) \} \) forms a complete orthonormal set in \([0, T]\). Also, \( \psi(t, \sigma, k) \) and \( \lambda^a(T) \) (and \( \psi(t, \sigma, k) \)) are continuous and monotonic increasing in \( T \) (Figure 4).

\[ \lambda^a(T) = \max \left\{ 0, \frac{2a - 2\lambda^a(T) - \Lambda^a(T)}{\lambda^a(T) - \Lambda^a(T)} \right\} \]

and is achieved by the signature waveforms

\[ s(t) = \begin{cases} \frac{1}{\sqrt{2a}} \sqrt{2a} \beta(1-\gamma(T)) + \frac{1}{\sqrt{2a}} s(t) \beta(1-\gamma(T)) \end{cases} \]

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Similar to the rms case, we find the lower bound by maximizing the average over $k = 1, 2$ of the right hand side of (29). Rewriting the average, we have

$$
\frac{2}{k} \sum_{l=1}^{2} \frac{1}{2} \text{tr}(AA^T A_l) = \frac{1}{2} \text{tr}(AA^T A)
$$

where $A_l$ is diagonalized by the orthonormal matrix $P_A$, and $S = \text{diag}(\xi_1, \xi_2, \ldots)$. Since the eigenvalues of $AA^T$ and $A^T A$ are the same, we have $\xi_1 = 1 + \sigma$ and $\xi_2 = 1 - \sigma$. Now, let's denote $P$ as the two by two matrix formed by taking only the first two rows of $P_A$, and $E = \text{diag} (\xi_1, \xi_2)$. Then, the maximum of the average is

$$
\max_{P \in \mathcal{P}^*} \frac{1}{2} \text{tr}(EE P P^T) = \max_{P \in \mathcal{P}} \frac{1}{2} \text{tr}(EE P P^T)
$$

We will solve the maximization problem using the Lagrange multiplier method. We form the Lagrangian,

$$
\sum_{k=1}^{2} \sum_{l=1}^{K} \xi_k \gamma_{k l}^2 + \sum_{k=1}^{2} \sum_{l=1}^{K} \gamma_{k l} \left( p_{k l} - \xi_{k l} \right)
$$

where $p_{k l}$ is the $k$th row of $P$. Taking derivative with respect to $p_{k l}$, we have

$$
\xi_k \gamma_{k l} = 2 \sum_{j \neq k} \frac{\xi_j}{\xi_j - \xi_k} p_{j l},
$$

and

$$
\sum_{l=1}^{K} \gamma_{k l} - \xi_k = 0.
$$

If we pre-multiply (34) by $p_{k l}^T$ and (35) by $p_{k l}$, we have $p_{k l}^T = 0$ since $\xi_1 \neq \xi_2$; therefore, from (34) and (35), $p_{k l}$ and $p_{k l}$ are eigenvectors of $A$. Since $A$ is diagonal and the diagonal entries are distinct and decreasing down the diagonal axis, we have

$$
P = \begin{bmatrix}
0 & 1
\end{bmatrix}.
$$

Substituting back into (32), we show that the maximum value of (31) is $\frac{1}{2} \lambda_1^2 + \frac{1}{2} \lambda_2^2$. Comparing to (29), we have

$$
\sigma \leq \frac{1}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right),
$$

or, together with $0 \leq \sigma \leq 1$,

$$
\sigma \geq \max \left( \frac{2 - \lambda_1}{\lambda_1}, \frac{2 - \lambda_2}{\lambda_2} \right).
$$

The achievability of the lower bound can be verified, as in the rms case, by letting

$$
A = \begin{bmatrix}
\sqrt{\frac{\lambda_1^2}{2}} & \sqrt{\frac{\lambda_2^2}{2}} & 0 & \ldots
\end{bmatrix},
$$

which corresponds to the optimal set of signals stated in the theorem. The proof of the second part of the theorem ($TB < \lambda_1(\alpha)$) can be found in [33, p. 44].

**Theorem 3.4.**

The capacity region of the two-user PAU white Gaussian multiple-access channel with noise power spectral density, additive white Gaussian noise, and input powers equal to $c_1$, $c_2$, and $c_3$, respectively, is given by

$$
C_F = \left\{ \begin{array}{ll}
0 \leq \alpha_1 \leq \log \left( 1 \frac{c_1}{c_1 + c_2} \right), & 0 \leq \alpha_2 \leq \log \left( 1 \frac{c_2}{c_1 + c_2} \right), \\
0 \leq \alpha_3 \leq \log \left( 1 \frac{c_3}{c_3 + c_2} \right), & \alpha_1 + \alpha_2 + \alpha_3 \leq \log \left( 1 \frac{c_1 + c_2 + c_3}{c_1 + c_2 + c_3} \right)
\end{array} \right.
$$

where $\alpha = \log(\alpha_3) = \frac{1}{2} \left( \log(\alpha_1) + \log(\alpha_2) \right)$.

**Proof.**

The proof is very similar to that in Theorem 3.2 where $\gamma = TB$. The lower limit of $\gamma$ is carried over from Theorem 3.3, while the upper limit is the smallest $\gamma$ such that $c_1(\gamma) = c_0$.

Notice that the range of $\gamma$ in taking the union is only a function of $\alpha$. In Figure 4, we show $\alpha(F)$ and $\frac{1}{2} \log(\phi(\gamma) + \lambda_2(\gamma))$ vs. the time-bandwidth product, and $\gamma$ and $\alpha$ can be obtained directly from the figure. Also, Figure 5 shows the capacity region $C_F$ with the capacity region of the strictly band-limited channel $C_C$. Similar comments to those made in the rms case apply to the values of $\gamma$ that achieve the boundary points in the capacity region. However, we see that for sufficiently high $\alpha$, $C_F$ does not contain $C_C$ in contrast to the rms case.

Finally, in Figure 6, we show the signature waveforms which are, as expected, mirror images of each other. However, in contrast to the rms case where the signature waveforms must be zero at the end points to have finite rms bandwidth, the transmitted signal waveform in the o-frequency case may have jumps at $t = 0$.

**References**

7. Nuttall, A. H., "Minimum rms bandwidth of M time-limited signals with specified code or correlation matrix,"