

OPTIMUM SIGNAL DESIGN FOR COMMUNICATION THROUGH UNCERTAIN DISTORTING CHANNELS

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ABSTRACT

The problem of optimum finite-length discrete-time signal selection under a power constraint for communication through uncertain distorting channels embedded in additive colored noise is studied. Minimax matched filtering is employed in order to combat the uncertainty in the channel distortion, which is modeled by mean-square, maximum-absolute and mean-absolute deviation classes. The well-known solution, in the absence of uncertainties, i.e., the minimum-eigenvalue eigenspace of the noise covariance matrix, is generalized for the above uncertainty classes.

I. INTRODUCTION AND FORMULATION

The classical matched filtering problem solves for the linear system that gives the maximum output signal-to-noise ratio at some instant of time when the input is a deterministic signal embedded in additive zero-mean noise with given second-order statistics (e.g. power spectral density or covariance matrix). In the finite-length discrete-time case, it is easy to show that, for a given filter, the ratio between the output power due to the signal and the output power due to the noise at the $k-1$ th sampling instant is given by

$$\text{SNR}(h;s,\Sigma) = \frac{\sum_{i=0}^{k-1} s_i \tilde{h}_{k-1-i}}{\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \tilde{h}_{k-1-i} \tilde{h}_{k-1-j} (\Sigma)_{ij}} \quad (1.1)$$

where $\{s_i; i = 0, \dots, k-1\}$ is the deterministic signal, Σ is the noise covariance matrix (assumed to be positive definite), and $\{\tilde{h}_i, i = 0, \dots, k-1\}$ is the impulse response of the discrete-time linear filter. For convenience of notation we will deal with h , which denotes the vector of time reversed impulse response values, i.e., $h_i = \tilde{h}_{k-1-i}$. As is well known, the matched filter for s and Σ , $h^*(s,\Sigma)$, (i.e., the vector h maximizing (1.1)) is given via the Schwarz inequality by

$$h^*(s,\Sigma) = \Sigma^{-1} s \quad (1.2)$$

In some instances in which Σ and/or s are not known exactly by the designer, a useful alternative to the classical adaptive solution (see [3] and [5] for example) is the minimax robust approach ([4],[7] and [8]). The minimax robust matched filter is the fixed linear system that provides the best worst-case performance when the uncertain signal and noise covariance are known to belong to some specified classes \mathcal{S} and \mathcal{N} , respectively, i.e.

$$h_L = \arg \max_{h \in \mathbb{R}^k} \inf_{(s,\Sigma) \in \mathcal{S} \times \mathcal{N}} \text{SNR}(h;s,\Sigma) \quad (1.3)$$

In [8] this problem is solved for several types of sets \mathcal{S} and \mathcal{N} and the robustness of the resulting filters is illustrated by way of several examples. It turns out that the solution to (1.3) is the matched filter for the least-favorable signal and noise covariance, $(s_L, \Sigma_L) \in \mathcal{S} \times \mathcal{N}$; i.e. those elements with the worst optimal performance. Therefore, we have that for every pair of signal and noise uncertainty classes, the minimax approach to the design of robust matched filters guarantees a minimum level of signal-to-noise ratio, given by

$$\overline{\text{SNR}}(\mathcal{S}, \mathcal{N}) = \max_{h \in \mathbb{R}^k} \inf_{(s,\Sigma) \in \mathcal{S} \times \mathcal{N}} \text{SNR}(h;s,\Sigma) \quad (1.4)$$

In this paper we deal with the case in which the noise covariance set \mathcal{N} has a maximal element, Σ_L (which is the least favorable regardless of the received signal [8]), and in which the signal uncertainty is originated by some class of channel distortion (fading, jitter, ISI, nonlinearities, etc.) which is modeled by a waveform δ added to s_0 (the originally transmitted signal) and known only to belong to a class \mathcal{D} . Supposing that in order to combat these uncertainties in the received signal, and in the noise covariance a minimax robust filter

is used, then an interesting question arises as to how one can choose the transmitted signal, s_0 , in a way so as to achieve the best possible minimax performance for a given level of distortion. That is, the goal is to seek solutions to the problem

$$s^*(\mathcal{B}) = \arg \max_{s_0 \in \mathcal{B}(P)} \overline{\text{SNR}}(s_0 + \mathcal{B}, \mathcal{N}) \quad (1.5)$$

where $\mathcal{B}(P)$ is the k -dimensional hypersphere containing all vectors with Euclidean norm equal to $P^{1/2}$ (i.e., all signals with a given power P). This problem is discussed in the following section for three types of distortion classes \mathcal{B} , namely, mean-square, maximum-absolute and mean-absolute distortion.

II. SIGNAL SELECTION

Because the lower bound for the signal-to-noise ratio guaranteed by the minimax robust design coincides with the best SNR achievable at the least-favorable pair of signal and noise covariance [8], it is straightforward to show that, within the above assumptions,

$$\overline{\text{SNR}}(s_0 + \mathcal{B}, \mathcal{N}) = \langle s_0 + \delta_L, \Sigma_L^{-1}(s_0 + \delta_L) \rangle \quad (2.1)$$

where $s_0 + \delta_L$ denotes the least-favorable signal in $s_0 + \mathcal{B}$ and $\langle a, b \rangle$ denotes the scalar product $a^T b$.

In the classical case in which no distortion is allowed (i.e., $\delta_L = 0$), it is easy to see from (1.5) and (2.1) that the optimal transmitted signal is any vector of power P belonging to the eigenspace of the minimum eigenvalue of the noise covariance matrix [2]. This result will be generalized in the next subsections by allowing the presence of an unknown signal distortion modeled by three different types of uncertainty classes. The minimax robust matched filter solution for each of these classes is derived in [8].

A. Mean-Square Distortion

Let \mathcal{B}_2 be the power (ℓ_2 norm) constrained distortion class defined by

$$\mathcal{B}_2 = \{ \delta \in \mathbb{R}^k; \sum_{i=0}^{k-1} |\delta_i|^2 \leq \Delta^2 \} \quad (2.2)$$

It is shown in [8] that the least-favorable distortion, δ_L , for this case is given by

$$\delta_L = -\sigma_s^2 h_L \quad (2.3)$$

with σ_s^2 defined implicitly by

$$\sigma_s^2 \|h_L\| = \Delta \quad (2.4)$$

where $\|\cdot\|$ denotes Euclidean norm. Equation (2.3) implies that the robust filter for (2.2) is the matched filter for the nominal signal s_0 and Σ_L with an added component of white noise; i.e.,

$$h_L = (\Sigma_L + \sigma_s^2 I)^{-1} s_0 \quad (2.5)$$

It can be shown (see Prop. 8 [8]) that the above solution implies an interesting certainty-equivalence property, namely that, if s_0 is an eigenvector of the (least-favorable) noise covariance Σ_L , then the minimax robust matched filter coincides with the nominal (zero-distortion) matched filter. Moreover, as the following result states, the optimal transmitted signals for matched filtering with and without distortion also coincide for this distortion model.

Proposition 1

The optimum transmitted signal s_0 for robust matched filtering when the signal uncertainty is described by a mean-square distortion class is any minimum-eigenvalue eigenvector (with power P) of the least-favorable noise covariance matrix, Σ_L .

Proof

Denoting by $\lambda_M[A]$, $\lambda_L[A]$ the maximum and minimum eigenvalues, respectively, of the square matrix A , we define the (increasing function

$$\Lambda(\epsilon) = \lambda_M[\epsilon(\Sigma_L + \epsilon I)^{-1}] \quad (2.6)$$

for $\epsilon > 0$. Through (2.4-2.5), the dependence of the term σ_s^2 on the transmitted signal s_0 is made explicit by defining a function $\sigma^2: \mathbb{R} \rightarrow \mathbb{R}_+$ from the equation

$$\sigma^2(s) \cdot \|(\Sigma_L + \sigma^2(s)I)^{-1} s\| = \Delta, \quad (2.7)$$

the solution of which is guaranteed to exist if $\Delta \leq \|s\|$. Assuming that $2\Delta \leq P^{1/2}$ (see [9] for a proof of the general case $\Delta < P^{1/2}$) (2.6 - 2.7) imply that for every $s \in \mathcal{B}(P)$ we have

$$\sigma_0^2 \triangleq \Lambda^{-1} (\Delta/P^k) \leq \sigma^2(s) \quad (2.8)$$

with equality if and only if s is a minimum-eigenvector of Σ_L .

Now, substituting (2.3) in (2.1) we obtain

$$\begin{aligned} \overline{\text{SNR}}(s_0 + \mathcal{B}, \mathcal{N}) &= \\ &\langle s_0, (\Sigma_L + \sigma^2(s_0)I)^{-1} \Sigma_L (\Sigma_L + \sigma^2(s_0)I)^{-1} s_0 \rangle \\ &\leq \max_{s \in \mathcal{B}(P)} \langle s_0, (\Sigma_L + \sigma^2(s)I)^{-1} \Sigma_L (\Sigma_L + \sigma^2(s)I)^{-1} s_0 \rangle \end{aligned} \quad (2.9)$$

for every $s_0 \in \mathcal{B}(P)$. It is straightforward to see that the argument that maximizes the right side of (2.9) is the same that minimizes $\sigma^2(\cdot)$, i.e., a minimum-eigenvalue eigenvector of Σ_L . Moreover, it is easy to prove that \mathcal{X} is an eigenvector of Σ_L with eigenvalue λ if and only if \mathcal{X} is an eigenvector of $(\Sigma_L + \sigma_0^2 I)^{-1} (\Sigma_L + \sigma_0^2 I)^{-1}$ with eigenvalue $\lambda/(\lambda + \sigma_0^2)^2$. But since $\sigma_0^2 \leq \lambda_L(\Sigma_L)$ we have that, for every $s_0 \in \mathcal{B}(P)$,

$$\begin{aligned} &\langle s_0, (\Sigma_L + \sigma_0^2 I)^{-1} \Sigma_L (\Sigma_L + \sigma_0^2 I)^{-1} s_0 \rangle \\ &\leq P \lambda_L(\Sigma_L) / (\lambda_L(\Sigma_L) + \sigma_0^2)^2 \end{aligned} \quad (2.10)$$

Therefore, uniting (2.9) and (2.10) we have that

$$\overline{\text{SNR}}(s_0 + \mathcal{B}, \mathcal{N}) \leq P \lambda_L(\Sigma_L) / (\lambda_L(\Sigma_L) + \Lambda^{-1} (\Delta/P^k))^2 \quad (2.11)$$

for every $s_0 \in \mathcal{B}(P)$, with equality if and only if s_0 is a minimum eigenvalue eigenvector of Σ_L . \square

Proposition 1 is indeed a very appealing result. Not only does the minimum-eigenvalue eigenvector signal maximize the signal-to-noise ratio achieved under the nominal conditions, and, as we mentioned before, the corresponding robust matched filter exhibits a certainty-equivalence property, but this eigenvector also maximizes the minimax performance for any degree of mean-square signal distortion.

B. Maximum Absolute Distortion

In this and in the following subsection we assume that the least-favorable noise is uncorrelated, i.e., $\Sigma_L = \text{diag}(\lambda_0, \dots, \lambda_{k-1})$, and without loss of generality, that $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{k-1}$. Note that in our case, uncorrelatedness is a non-trivial restriction because the signal-distortion classes of subsections II.B-C are not independent

of the basis chosen for the k -dimensional Euclidean space.

The maximum absolute distortion class is defined by the (ℓ_∞ norm) class

$$\mathcal{B}_\infty = \{\delta \in \mathbb{R}^k; |\delta_i| \leq \Delta, i = 0, \dots, k-1\}. \quad (2.12)$$

The least-favorable distortion for (2.12) is given by (see [8])

$$\delta_{Li} = \begin{cases} -\Delta & \Delta < s_{oi} \\ -s_{oi} & -\Delta \leq s_{oi} \leq \Delta \\ \Delta & s_{oi} < -\Delta \end{cases}, \quad (2.13)$$

and the optimum transmitted signal is given by the following proposition.

Proposition 2

The optimum transmitted signal for robust matched filtering when the signal uncertainty is described by a maximum absolute distortion class and the least favorable noise is uncorrelated, concentrates all of its power in one minimum-eigenvalue sample.

Proof

Let s_0 denote a particular nominal signal. It is easy to see that, if for some sample i we have $|s_{oi}| \leq \Delta$, then (2.13) implies that $s_{Li} = 0$, and therefore such a nominal signal cannot be optimum, since the power in that sample is wasted. Alternately, suppose that for some $i < j$ we have $|s_{oi}| > \Delta$ and $|s_{oj}| > \Delta$. Again it can be proved that this is not optimum, since the nominal signal \tilde{s}_0 which is equal to s_0 except that $\tilde{s}_{oi}^2 = s_{oi}^2 + s_{oj}^2$ and $\tilde{s}_{oj} = 0$ gives better performance with the same power. To see this, consider

$$\begin{aligned} &\langle s_L(s_0), \Sigma_L^{-1} s_L(s_0) \rangle - \langle s_L(\tilde{s}_0), \Sigma_L^{-1} s_L(\tilde{s}_0) \rangle = \\ &= \frac{1}{\lambda_i} [(|s_{oi}| - \Delta)^2 - (|\tilde{s}_{oi}| - \Delta)^2] + \frac{1}{\lambda_j} (|s_{oj}| - \Delta)^2 \\ &\leq 2\Delta (|\tilde{s}_{oi}| - |s_{oi}| - |s_{oj}| + \Delta/2) / \lambda_i \\ &\leq 2\Delta [\Delta(2\Delta - |s_{oi}| - |s_{oj}|) - 2(|s_{oi}| - \Delta)(|s_{oj}| - \Delta)] \\ &\quad / (\gamma \lambda_i) \\ &\leq 0 \end{aligned} \quad (2.14)$$

where $\gamma = |s_{oi}| + |s_{oj}| + |s_{oj}| - \Delta/2$. Thus, only one sample can be non-zero in the optimum signal, and the proposition follows. \square

Proposition 2 shows an example in which the set of optimum signals for robust matched filtering no longer coincides with, but is included in, the set of optimal signals for the classical problem. (Of course, if the minimum-eigenvalue eigenspace has dimension one then both solutions are identical.) The following signal selection problem provides an example in which an optimal signal for robust matched filtering need not be optimal for the classical problem and vice versa.

C. Mean Absolute Distortion

Finally, we study the signal design problem for the (ℓ_1 norm) mean absolute distortion class defined by

$$\mathcal{D}_1 = \{ \delta \in \mathbb{R}^k; \sum_{i=0}^{k-1} |\delta_i| \leq \Delta \} \quad (2.15)$$

When the noise is uncorrelated the minimax robust filter is given by ([8])

$$h_{Li} = \begin{cases} h_{oi} & |h_{oi}| \leq C \\ C \operatorname{sgn}(h_{oi}) & |h_{oi}| > C \end{cases} \quad (2.16)$$

where $\{h_{oi}\}$ represent the samples of the nominal matched filter and C is a positive constant that satisfies the equation

$$\sum_{i=0}^{k-1} \lambda_i (|h_{oi}| - C)^+ = \Delta \quad (2.17)$$

Therefore, the robust filter has the interesting property of being a clipped version of the matched filter for the nominal problem (without distortion). This in turn means that the least favorable signal, $s_L = s_o + \delta_L$, is the result of a "stairlike" clipping of s_o , in which the clipping level is proportional to the corresponding eigenvalue and is scaled so as to introduce the maximum distortion allowed by the uncertainty class. From (2.16) it can be checked that the least favorable distortion satisfies

$$\delta_{Li} = \begin{cases} 0 & |s_{oi}| \leq \lambda_i C \\ [\lambda_i C - |s_{oi}|] \operatorname{sgn}(h_{oi}) & |s_{oi}| > \lambda_i C \end{cases} \quad (2.18)$$

The solution to the signal selection problem for mean absolute distortion is given by

Proposition 3

The optimum transmitted signal for robust

matched filtering when the signal uncertainty is described by a mean absolute distortion class and the least favorable noise is uncorrelated, is constant in absolute value for $i = 0, 1, \dots, q-1$ and zero elsewhere, where $q \in \{1, \dots, k\}$ maximizes

$$\Gamma(q) = \frac{[\sqrt{q} - \Delta/P_c]^{2q}}{\sum_{i=0}^{q-1} \lambda_i} \quad (2.19)$$

Proof

We first show that, in an optimum transmitted signal, the clipped samples must be constant in absolute value. To this end, note that the value of the filter impulse response clipping level is given through Eqns (2.17) and (2.18) by (denoting the summation over the clipped samples by Σ^*)

$$C = [(\Sigma^* |s_{oi}|) - \Delta] / \Sigma^* \lambda_i \quad (2.20)$$

and hence the contribution of the clipped samples to the signal-to-noise ratio in (2.1) is

$$\begin{aligned} \sum_i \lambda_i^{-1} s_{Li}^2 &= \sum_i \lambda_i^{-1} (\lambda_i C)^2 \\ &= [(\Sigma^* |s_{oi}|) - \Delta]^2 / \Sigma^* \lambda_i \\ &\leq [(qP_c) - \Delta]^2 / \sum_i \lambda_i \end{aligned} \quad (2.21)$$

where q and P_c are the number and power, respectively, of the clipped samples. Note that we have equality in (2.21) if and only if the absolute values of all the clipped samples are equal.

Now, we prove that if a nominal signal contains nonzero unclipped samples, then it is not optimum. Suppose that we distribute a fixed amount of power, P_b , between the clipped part of the signal and an unclipped sample s_{oj} , and furthermore, assume that the allocation of the power in the clipped part is optimum. Then the corresponding contribution to the signal-to-noise ratio is given by

$$g(s_{oj}^2) = ([(qP_b - s_{oj}^2) - \Delta]^2 / \sum_i \lambda_i) + \lambda_j^{-1} s_{oj}^2 \quad (2.22)$$

which is a convex function of s_{oj}^2 . Therefore, the optimal choice of $|s_{oj}|$ must lie on the boundary of the set of its possible values; i.e.

it must be either zero or clipped.

We have now shown that all nonzero samples of an optimum nominal signal have constant absolute value. Thus, we need only to specify the optimum number and locations of the nonzero samples. From (2.21) it is readily seen that the nonzero samples must correspond to the q lowest eigenvalues of the noise, and that the optimum q maximizes the right side of (2.21) with $P_c = \|s_0\|^2$ or equivalently, $\Gamma(q)$. \square

An interesting result which follows from Proposition 3 is that, if the first (lowest) m eigenvalues are equal, then we have that

$$\Gamma(n) = [1 - n^{-1/2} \Delta \|s_0\|]^2 / \lambda_0, \quad 1 \leq n \leq m, \quad (2.23)$$

and therefore $m \leq q$, i.e., the first m samples are assured to be nonzero. Applying this fact when the least favorable noise is white, we deduce that the optimum nominal signal is constant in absolute value for all its samples. Also, if $\Delta = 0$ it is easy to check that Proposition 3 results in the classical minimum-eigenvalue eigenvector solution. An important aspect in which Proposition 3 differs from Propositions 1 and 2 is that it gives an optimum signal that is dependent on the degree of distortion (through the ratio $r = \Delta/P^2$). Note from (2.19) that the solutions of Propositions 1, 2 and 3 coincide when $\lambda_1/\lambda_0 > H(r)$, where $H(r)$ is an increasing function defined on $[0,1]$ by

$$H(r) = (\sqrt{2} - r)^2 / (1 - r)^2 - 1$$

III. CONCLUDING REMARKS

The classical solution to the problem of optimum signal selection for matched filtering has been generalized in this paper to admit the existence of uncertainties in the received signal and in the noise covariance matrix. Following the minimax approach to the design of finite-length discrete-time robust matched filters, the goal of the selection (under a power constraint) of the transmitted signal is the optimization of the lower bound of performance guaranteed by the robust matched filter design.

The discussion has emphasized the presence of signal uncertainties due to channel distortion and the noise covariance uncertainty class has been restricted to contain a maximal element, or, equivalently, a signal-independent least favorable matrix (see [8]). Three types of distortion uncertainty models that cover a wide area of practical application have been studied and different results for the signal selection problem have been shown to hold. By use of weighted ℓ_1 , ℓ_2 and ℓ_∞ norms, these uncertainty models can be further generalized to accommodate for different degrees of distortion in the directions of the signal space. In such case the results related to minimax matched filtering and optimum signal design can be extended straightforwardly.

With respect to the mean-square distortion model, a threefold justification for the classical signal design using the minimum-eigenvalue eigenvector of the covariance matrix has been found: it optimizes the signal-to-noise ratio when the received signal coincides with the transmitted one, its associated matched filter is minimax robust for any degree of mean-square distortion and it optimizes the worst-case signal-to-noise ratio. However, for the other types of distortion considered here, the set of optimum transmitted signal under distortion no longer coincides with the minimum-eigenvalue eigenspace. The maximum and mean absolute distortion models lend themselves to an analytical solution of the signal design problem under a mean-square power constraint in the case of uncorrelated (not necessarily stationary) least-favorable noise. For these models the corresponding results indicate the advisability of avoiding comparatively small nominal signal samples and of allocating, in some cases, signal power to non-minimum-eigenvalue samples. Note finally that, for a given covariance matrix with a one dimensional minimum-eigenvalue eigenspace, and with a sufficiently large allowable power (relative to the degree of

distortion), the optimum signals for the three types of distortion classes coincide.

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REFERENCES

- [1] F. R. Gantmacher, The Theory of Matrices, vol. 1, New York: Chelsea, 1960.
- [2] T. L. Grettenberg, "Signal selection in communication and radar systems," IEEE Trans. Information Theory, vol. IT-9, no. 4, pp. 265-275, Oct. 1963.
- [3] U. Padan, "Adaptive digital matched filters," IEEE Trans. Information Theory, vol. IT-28, no. 6, pp. 890-904, Nov. 1982.
- [4] H. V. Poor, "Robust matched filtering," IEEE Trans. Information Theory, to appear.
- [5] H. J. Scudder, III, "Probability of error of some adaptive pattern recognition machines," IEEE Trans. Information Theory, vol. IT-11, pp. 363-371, July 1965.
- [6] G. L. Turin, "An introduction to digital matched filtering," Proc. of IEEE, vol. 64, no. 7, pp. 1092-1112, July 1976.
- [7] S. Verdú and H. V. Poor, "Least favorable signals and noise for discrete-time robust matched filtering," Rec. 1981 IEEE National Telecommunications Conference, pp. D.8.1-D.8.5, Nov. 1981.
- [8] S. Verdú and H. V. Poor, "Minimax robust discrete-time matched filters," IEEE Trans. on Communications, vol. COM-31, no. 2, Feb. 1983.
- [9] S. Verdú and H. V. Poor, "Signal selection for robust matched filtering," IEEE Trans. on Communications, vol. COM-31, to appear.

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