Single-User Detectors for Multiuser Channels

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Abstract—**Optimum decentralized demodulation for asynchronous Gaussian multiple-access channels is considered. It is assumed that the receiver is the destination of the information transmitted by only one active user, and single-user detectors that take into account the existence of the other active users in the channel are obtained. This approach is in contrast to both conventional demodulation, which is fully decentralized but neglects the presence of multiple-access interference, and globally optimum demodulation, which requires centralized sequence detection. The problem considered is one of signal detection in additive colored non-Gaussian noise, and attention is focused on one-shot structures where detection of each symbol is based only on the received process during its corresponding interval. Particular emphasis is placed on asymptotically optimum detectors for each of the following situations: 1) weak interferers, 2) CDMA signature waveforms with long spreading codes, and 3) low background Gaussian noise level.**

I. Introduction

THE conventional approach to the demodulation of code-division multiplexed multiuser digital communications is to demodulate each user as if it were the only user in the channel. The multiple-access capability of such systems is thus achieved by using complex signal constellations that exhibit favorable cross-correlation properties. (See, for example, [1] for a description of conventional multiple-access demodulation techniques.) However, recent work by Verdu [2] has shown that substantial performance gains can be achieved in coherent multiuser systems by using a receiver that takes advantage of the structure of the multiple-access interference. For example, this approach can be used to alleviate such limitations as the near/far problem in the direct-sequence spread-spectrum multiple-access (DS/SSMA) format. The performance gains realized by the receiver proposed in [2] are achieved by the use of simultaneous sequence detection of all users in the channel, a task that requires a centralized implementation and a high degree of software complexity (for example, the decision algorithm required is a dynamic program (DP) whose complexity is $O(2^K)$ where K is the number of users in the channel). Since the implementation costs of such fully centralized detection algorithms may be unacceptably high for many applications, and since network security restrictions may not permit the distribution of all user's signaling waveforms to all demodulating terminals, it is of interest to consider demodulators that lie between these two philosophies of conventional demodulation, in which other users' signaling waveforms to all demodulating terminals, it is optimum demodulation, in which all users in the channel are tracked and demodulated simultaneously. The performance results obtained in [2]–[4] indicate that an attractive compromise in practice is to use optimum multiuser demodulators for only a subset of the active users and simply neglect the presence of all other users. In order to take advantage of the superior performance achievable by multiuser detectors, the subset of active users to be taken into account at the receiver should contain all the users whose power is sufficient regardless of whether their messages are destined to that particular location.

In this paper, we consider the case of full decentralization where the receiver is constrained to track and lock to the signal of only one user, but unlike the conventional single-user detector, is optimized to take into account the structure of multiple-access interference in making decisions. We consider several design approaches that can be used to optimize these structures depending on the amount of information that one is able to assume to be known about the signature waveforms assigned to the interfering users.

This paper is organized as follows. In Section II, we will first discuss the general structure of optimum decentralized demodulators that simultaneously track and demodulate a group of D users from a total population of K users sharing a common communications channel where $D \leq K$. We then consider the structure of optimum *single-user* detectors ($D = 1$), and particularize to the case $K = 2$ to illustrate this structure. The results of Section II are for general antipodal signaling formats. In Section III, we turn to the development of specific results for single-user demodulation of DS/SSMA transmissions. In this modulation technique, which among coherent signaling formats is of particular practical interest, each signature waveform consists of a sequence of chip waveforms whose polarities are determined by a binary word assigned to each user. The specific structure of the direct-sequence format allows for the development of useful approximations to optimum single-user detection which are asymptotically exact as either the length of the spreading codes or the signal-to-background-noise ratio (SBNR) increases without bound. We also show that (with $K = 2$) even in the absence of any prior knowledge about either the spreading codes or the timing of the multiuser interference, the optimum single-user detector is *not* multiple-access noise-limited as the background thermal noise level vanishes. This is in contrast to the conventional detector, which can incur an irreducible error probability even in the absence of background noise. All of these results for DS/SSMA require only that the chip waveform (which is usually common to all users in a given network) be known. Thus, these techniques can be applied in secure networks where the distribution of one user's spreading code to other users is not desirable.

In Section IV, we return to general coherent signaling formats to consider the problem of optimum single-user detection in the presence of weak unlocked interfering users. We model this problem by assuming that the multiple-access interference is multiplied by a small amplitude factor ϵ. We then derive an expression for the likelihood ratio statistic for optimum symbol decisions on the locked user that is of the form of the conventional correlation statistic, modified by an ϵ^2 term involving signal cross-correlation functions, and then having higher order terms of order ϵ^4. The resulting locally optimum detector correlates the observation with a replica of the waveform of the user of interest, suitably smoothed to take into account the presence of multiple-access interference.

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II. Optimum Decentralized Detection for Multiuser Channels

Throughout this paper, we consider a received signal model of the form

$$r_t = S_t(b) + n_t, \qquad -\infty < t < \infty \tag{2.1}$$

where $\{n_t;\ -\infty < t < \infty\}$ represents white Gaussian noise with spectral height $N_0/2$, and where the received signal $S_t(b)$ is the superposition of transmissions received from K separate asynchronous users, i.e.,

$$S_t(b) = \sum_{k=1}^{K} \sum_{i=-M}^{M} b_k(i) s_k(t - iT - \tau_k) \tag{2.2}$$

where T is the symbol interval, $b_k(i)$ is the ith symbol of the kth user, τ_k is the relative delay (modulo T) with which the kth user's transmission is received, and $s_k(t)$ is the signature waveform assigned to the kth user. (It is assumed that $s_k(t)$ is zero for $t \notin [0, T]$.) Note that $(2M + 1)$ is the number of symbols per user in the given transmission, and b denotes the $K \times (2M + 1)$ matrix whose (k, i) entry is $b_k(i)$.

Suppose that we wish to demodulate some group of D users from the total population of K users where $D \leq K$. For simplicity of notation, we assume that these D users of interest are labeled 1-D. Thus, we know $s_1, \cdots, s_D$ and $\tau_1, \cdots, \tau_D$, and the *maximum likelihood* demodulator chooses a symbol matrix $b_D \equiv \{b_k(i);\ k = 1, \cdots, D\}_{i=-M}^{M}$ to maximize the log-likelihood function

$$\frac{2}{N_0}\int_{-\infty}^{\infty} r_t S_t^D(b_D)\,dt - \frac{1}{N_0}\int_{-\infty}^{\infty} [S_t^D(b_D)]^2\,dt + \log E\left\{\exp\left[\frac{2}{N_0}\int_{-\infty}^{\infty} [r_t - S_t^D(b_D)]S_t^{MA}\,dt - \frac{1}{N_0}\int_{-\infty}^{\infty} [S_t^{MA}]^2\,dt\right]\right\} \tag{2.3}$$

where

$$S_t^D(b_D) = \sum_{k=1}^{D} \sum_{k=-M}^{M} b_k(i) s_k(t - iT - \tau_k), \qquad -\infty < t < \infty \tag{2.4}$$

$$S_t^{MA} = S_t(b) - S_t^D(b_D) \tag{2.5}$$

and where the expectation is over the ensemble of all unknown quantities in S_t^{MA}, including delays, symbols, and (possibly) waveforms.

Note that, even if we ignore the complexity of computing the expectation in (2.3), the time-complexity-per-demodulated-bit (TCB) of brute-force maximization of the log-likelihood function is $O(|A|^{D(2M+1)}/DM|A|)$ where $|A|$ is the size of the symbol alphabet. Thus, unless some simpler algorithm can be found, simultaneous maximum likelihood sequence detection of D users is out of the question from a practical point of view. For the particular case of fully centralized detection ($K = D$), it turns out that a much simpler algorithm can indeed be found. In particular, for this case, the expectation term in (2.3) disappears (since $S_t^{MA} \equiv 0$), and the remaining terms can be decomposed in a way that allows maximization of (2.3) with a dynamic programming algorithm yielding a TCB of $O(|A|)^K$. (See [2], [3] for details of this analysis.)

Unfortunately, for $D < K$, the decomposition of (2.3) necessary for a dynamic programming solution is not possible because of the coupling among symbols in the expectation term. This means that maximum likelihood sequence detection is not generally computationally feasible (its TCB is exponential in the number of symbols per user) if all users' signaling waveforms are not known and locked. Thus, in considering decentralized demodulators in multiuser channels, we will restrict our attention to algorithms which demodulate only a single symbol at a time, i.e., we consider *one-shot detectors.* We also will restrict attention to the binary signaling case, in which $b_k(i) \in \{-1, +1\}$ for all i, k. Extensions to general alphabets are, in most cases, straightforward.

In the sequel, we will consider the case of full decentralization, i.e., single-user detection, which can be modeled by the binary hypothesis-testing problem

$$H_0:\ r_t = s_1(t) + S_t^{MA} + n_t, \qquad 0 \leq t \leq T$$
$$H_1:\ r_t = -s_1(t) + S_t^{MA} + n_t, \qquad 0 \leq t \leq T \tag{2.6}$$

where $\{n_t;\ -\infty < t < \infty\}$ is the white Gaussian noise and where

$$S_t^{MA} = \sum_{k=2}^{K} [b_k^L s_k(t - \tau_k + T) + b_k^R s_k(t - \tau_k)], \qquad 0 \leq t \leq T \tag{2.7}$$

with b_k^L and b_k^R denoting the kth user's bits in the intervals $[-T + \tau_k, \tau_k)$ and $[\tau_k, T + \tau_k)$, respectively. We also assume that the receiver is coherent with user 1 so that $\{s_1(t);\ t \in [0, T]\}$ is a deterministic waveform, and that each user's signaling waveform is of the form

$$s_k(t) = (2w_k)^{1/2} a_k(t) \cos(\omega_c t + \theta_k) \tag{2.8}$$

where ω_c is known. We assume that $(\omega_c T/2\pi)$ is an integer large enough so that integrals of $2\omega_c$ components can be neglected.

Optimum (maximum likelihood/minimum error probability) decisions for (2.6) are based on comparing the likelihood ratio to a threshold. With this in mind, we give the following result.

Proposition 2.1: Suppose that the phase vector of the interfering users $\theta = (\theta_2, \cdots, \theta_K)$ is uniformly distributed on $[0, 2\pi)^{K-1}$ and is independent of $b_k = (b_k^L, b_k^R)$, $k = 2, \cdots, K$, $\tau = (\tau_2, \cdots, \tau_K)$ and $(a_k(t), t \in [0, T], k = 2, \cdots, K)$. If the dependence of $\|S^{MA}\|$ on θ can be neglected,[1] then the likelihood ratio for (2.6) can be written in the following form:

$$\exp\left[\frac{4w_1^{1/2}}{N_0}\int_0^T r_p(t) a_1(t)\,dt\right] \frac{E\left[\prod_{k=2}^{K} I_0((\rho_k^2(b_k, \tau_k, 1) + \psi_k^2(b_k, \tau_k))^{1/2}) \exp\left(-\sum_{j=2}^{k-1} \Gamma_{kj}(b_k, b_j, \tau_k, \tau_j)\right)\right]}{E\left[\prod_{k=2}^{K} I_0((\rho_k^2(b_k, \tau_k, -1) + \psi_k^2(b_k, \tau_k))^{1/2}) \exp\left(-\sum_{j=2}^{k-1} \Gamma_{kj}(b_k, b_j, \tau_k, \tau_j)\right)\right]} \tag{2.9}$$

where the expectation is over the ensemble of bits, delays, and possibly waveforms of the interfering users[2] and we use the notation

$$\rho_k(b_k, \tau_k, e) = \frac{2(w_k)^{1/2}}{N_0}\int_0^T \alpha_k(b_k^L, b_k^R, t - \tau_k) \cdot [r_p(t) - e w_1^{1/2} a_1(t)]\,dt \tag{2.10}$$

[1] For a waveform $x = \{x(t);\ 0 \leq t \leq T\}$, $\|x\|^2$ denotes $\int_0^T x^2(t)\,dt$.

[2] $I_0(\cdot)$ is the modified Bessel function of the first kind of order 0, i.e., $I_0(x) = 1/2\pi \int_0^{2\pi} \exp(x \cos\theta)\,d\theta$.

$$\psi_k(b_k, \tau_k) = \frac{2(w_k)^{1/2}}{N_0} \int_0^T \alpha_k(b_k^L, b_k^R, t-\tau_k) r_q(t)\, dt \quad (2.11)$$

$$\Gamma_{kj}(b_k, b_j, \tau_k, \tau_j) = \frac{2(w_k w_j)^{1/2}}{N_0} \int_0^T \alpha_k(b_k^L, b_k^R, t-\tau_k) \cdot \alpha_j(b_j^L, b_j^R, t-\tau_j)\, dt \quad (2.12)$$

$$r_p(t) = \sqrt{2} r_t \cos(\omega_c t + \theta_1) \quad (2.13)$$

$$r_q(t) = \sqrt{2} r_t \sin(\omega_c t + \theta_1) \quad (2.14)$$

$$\alpha_k(b, c, \lambda) = b a_k(\lambda + T) + c a_k(\lambda). \quad (2.15)$$

Proof: The likelihood ratio for (2.6) is equal to the ratio of expected values of conditionally Gaussian *a priori* densities where the expected value is taken with respect to all random quantities in S^{MA}; this is given straightforwardly by

$$LR = \frac{\exp\left[-\frac{1}{N_0}\|r - s_1\|^2\right]}{\exp\left[-\frac{1}{N_0}\|r + s_1\|^2\right]} \cdot \frac{E\left\{\exp\left[-\frac{1}{N_0}\|S^{MA}\|^2 + \frac{2}{N_0}\langle r - s_1, S^{MA}\rangle\right]\right\}}{E\left\{\exp\left[-\frac{1}{N_0}\|S^{MA}\|^2 + \frac{2}{N_0}\langle r + s_1, S^{MA}\rangle\right]\right\}} \quad (2.16)$$

where, for functions x and y on $[0, T]$, the notation $\langle x, y\rangle$ denotes $\int_0^T x(t)y(t)\, dt$. The first ratio in the above expression is readily shown to be equal to $\exp\left[(4(w_1)^{1/2}/N_0)^T \int_0^T r_p(t)a_1(t)\, dt\right]$. Now, neglecting the dependence of $\|S^{MA}\|$ on θ, we have for every $\boldsymbol{b}$ and τ

$$\frac{1}{N_0}\|S^{MA}\|^2 = \left[\sum_{k=2}^{K}\left(\frac{w_k}{N_0} + \sum_{j=2}^{k-1}\Gamma_{kj}(b_k, b_j, \tau_k, \tau_j)\right)\right]. \quad (2.17)$$

So, it remains to show that for all $(\alpha_k, b_k, \tau_k, k = 2, \cdots, K)$, we have

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \exp\left[\frac{2}{N_0}\langle r - ex_1, S^{MA}\rangle\right] \frac{d\theta_2 \cdots d\theta_K}{(2\pi)^{K-1}} = \prod_{k=2}^{K} I_0\left((\rho_k^2(b_k, \tau_k, e) + \psi_k^2(b_k, \tau_k))^{1/2}\right). \quad (2.18)$$

To this end, we note that the following sequence of equalities holds:

$$\frac{2}{N_0}\langle r - es_1, S^{MA}\rangle = \frac{2\sqrt{2}}{N_0}\sum_{k=2}^{K}\int_0^T (r(t) - es_1(t)) w_k^{1/2}\alpha_k(b_k^L, b_k^R, t-\tau_k) \cdot \cos(\omega_c t - \omega_c\tau_k + \theta_k)\, dt$$

$$= \frac{2}{N_0}\sum_{k=2}^{K} w_k^{1/2}\int_0^T [(r_p(t) - 2ew_1^{1/2}a_1(t) \cdot \cos^2(\omega_c t + \theta_1))\alpha_k(b_k^L, b_k^R, t-\tau_k) \cdot \cos(\theta_k - \omega_c\tau_k - \theta_1) - (r_q(t) - ew_1^{1/2}a_1(t) \cdot \sin(2\omega_c t + \theta_1))\alpha_k(b_k^L, b_k^R, t-\tau_k)) \cdot \sin(\theta_k - \omega_c\tau_k - \theta_1)]\, dt$$

$$= \sum_{k=2}^{K} \rho_k(b_k, \tau_k, e)\cos(\theta_k - \omega_c\tau_k - \theta_1) - \psi_k(b_k, \tau_k)\sin(\theta_k - \omega_c\tau_k - \theta_1) \quad (2.19)$$

where the last equality follows by neglecting the integrals of the $2\omega_c$ terms. Equation (2.18) is immediate from (2.19) and the result follows. □

Note that the structure (2.9) consists of the single-user ($K = 1$) correlation statistic

$$\frac{2w_1^{1/2}}{N_0}\int_0^T r_p(t)a_1(t)\, dt \quad (2.20)$$

used by conventional single-user receivers, modified by an additive correction term which accounts for the other users in the channel. Note that the received waveform enters this correction term through the sliding correlation statistics of (2.10) and (2.11).

The simplifying approximation in Proposition 2.1, which states that the energy of the multiple-access interference process is independent of the carrier phases, is certainly accurate when $\omega_c T$ is sufficiently large and the normalized (i.e, unit energy) cross correlations between the interfering users are low. We assume throughout this section and the following one that this independence is valid. If such an approximation is not assumed, then it can be shown straightforwardly that the multiplicative correction term in the likelihood ratio (2.16) is equal to

$$\frac{E\int_0^{2\pi}\cdots\int_0^{2\pi}\exp\left[\sum_{k=2}^{K}\rho_k(b_k, \tau_k, 1)\cos\alpha_k - \psi_k(b_k, \tau_k)\sin\alpha_k - \sum_{j=2}^{k-1}\Gamma_{kj}\cos(\alpha_k - \alpha_j)\right] d\alpha_2\cdots d\alpha_K}{E\int_0^{2\pi}\cdots\int_0^{2\pi}\exp\left[\sum_{k=2}^{K}\rho_k(b_k, \tau_k, -1)\cos\alpha_k - \psi_k(b_k, \tau_k)\sin\alpha_k - \sum_{j=2}^{k-1}\Gamma_{kj}\cos(\alpha_k - \alpha_j)\right] d\alpha_2\cdots d\alpha_K}. \quad (2.21)$$

Several variants of the general structure of (2.9) are of interest and will be considered here. One such variant is that in which the *modulation* waveforms of the interfering users $\{a_k(t);\ 2 \le k \le K\}$ are known, and the remaining unknown quantities in $\{s_k(t);\ 2 \le k \le K\}$ are all independent with the data bits and delays uniformly distributed in their ranges. In this case, the expectations in (2.9) reduce to

$$E\{(\cdot)\} = \frac{1}{(4T)^{K-1}}\sum_{b' \in 4^{K-1}} \cdot \int_{[0,T]^{K-1}} E\{(\cdot)|b', \tau\}\, d\tau_2\cdots d\tau_K \quad (2.22)$$

where $\boldsymbol{b}' = (b_2, \cdots, b_K)$ and where the inner expectation is over the amplitudes. Thus, the computation of this likelihood ratio is of exponential complexity in K. Moreover, there will be a further substantial computational burden in computing the $(K-1)$-dimensional integral corresponding to averaging over the relative delays $\tau_2, \cdots, \tau_K$.

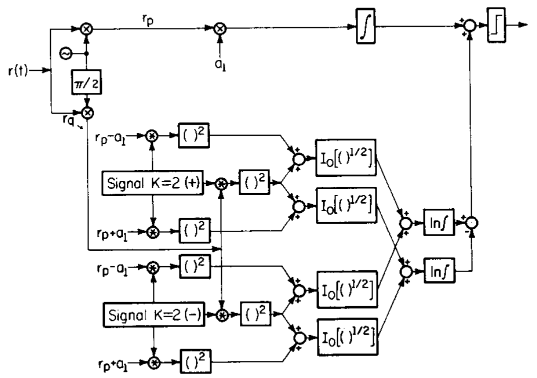

Fig. 1. Optimum single-user detector ($K = 2$). Replicas of $\alpha_2(+, +, -t)$ and $\alpha_2(+, -, -t)$ are generated by the blocks corresponding to the signal of the second user.

Fig. 1 illustrates the particularization of the demodulator derived in Proposition 2.1 to the two-user case. Note that the quadrature component of the input is used and that convolutions [required to generate (2.10) and (2.11)] and nonlinear memoryless operations are also needed.

For $K > 2$, the delay integrals do not appear to be obtainable in closed form. However, even if they could be, the exponential (in K) complexity of $\Sigma_{b'}$ shows that optimum one-shot single-user detection in a K-user channel is at least as computationally burdensome as centralized simultaneous sequence detection of fully locked users. However, one-shot single-user detection does not require tracking phases, delays, and amplitudes of all users, and thus may be preferred if these quantities are not stable for relatively long periods of time. Moreover, and perhaps more importantly, Proposition 2.1 also applies to situations in which the modulating *waveforms* of the interferers $\{a_k(t);\ k = 2, \cdots, K\}$ are not known. This situation is the norm for the noncentral nodes in many practical radio networks, and thus the centralized detection algorithm of [2] cannot be applied to such cases unless the receiver estimates the unknown signal cross correlations. Furthermore, as it is shown in Section III, an important reduction in the complexity of computing (2.9) results from the modeling of the modulation waveforms of the interfering users as being signature sequences.

III. Single-User Detectors for DS/SSMA Channels

In practice, one of the most important types of code-division multiple-access systems is direct-sequence spread spectrum. This corresponds to the particular case of the model (2.1), (2.2), and (2.8), in which the kth user's signature waveform is of the form

$$a_k(t)=\sum_{i=0}^{N-1} c_{ki}\psi(t-iT_c), \qquad 0\le t\le T \tag{3.1}$$

where $\{c_{ki}\}_{i=0}^{N-1}$ is a signature sequence of binary (± 1) digits, the chip waveform ψ is nonzero only on $[0, T_c]$, and the chip duration T_c is given by $T_c = T/N$. In many DS/SSMA multipoint-to-multipoint channels, it is frequently reasonable to assume that user 1 knows the chip waveforms of users 2-K, but not the specific signature sequences they employ. Since these sequences are usually chosen to be pseudonoise sequences, it is reasonable to model them (from the viewpoint of user 1) as independent sequences of independent, equiprobable binary digits. In this section, we apply this model for the interfering users in the likelihood ratio formula of Proposition 2.1. As we will see below, this affords a much more manageable form for the likelihood ratio in the limiting cases of practical interest, namely, when the number of chips is large and when the white Gaussian noise level is low.

A. Optimum Single-User Detection for Long Spreading Sequences

To study the large-N behavior of the likelihood ratio of Proposition 1, we first define the following functions:

$$\xi_e(\lambda)=\int_0^T [r_p(t)-ew_1^{1/2}a_1(t)]\psi(t-\lambda)\,dt \tag{3.2}$$

$$\phi(\lambda)=\int_0^T r_q(t)\psi(t-\lambda)\,dt \tag{3.3}$$

and

$$g_e(\lambda,\theta)=\xi_e(\lambda)\cos\theta-\phi(\lambda)\sin\theta \tag{3.4}$$

where the parameter e takes on the values $+1$ and -1 and g_e is abbreviated as g_+ and g_-, respectively, in the remainder of the section. Now fix $\tau \in (0, T)$ and suppose that $n \in \{1, \cdots, N\}$ is such that $(n-1)T_c < \tau \le nT_c$. Then define

$$d_{kj}=\begin{cases} b_k^R & c_{kj-n} \quad j-n\ge 0 \\ b_k^L & c_{kj-n+N} \quad j-n<0 \end{cases} \qquad j=0,\cdots,N \tag{3.5}$$

and notice that $g_e(\tau - T + iT_c, \theta) = 0$ for $i \le N - n - 1$ and $g_e(\tau + iT_c, \theta) = 0$ for $i \ge N - n + 1$ because $\psi(t) = 0$ for $t \notin [0, T_c]$. Then it follows that

$$\begin{aligned}
&\frac{N_0}{2w_k^{1/2}}[\rho_k(b_k,\tau,e)\cos\theta-\psi_k(b_k,\tau)\sin\theta]\\
&=\sum_{i=0}^{N-1} c_{ki}[b_k^L g_e(\tau-T+iT_c,\theta)+b_k^R g_e(\tau+iT_c,\theta)]\\
&=c_{kN-n}b_k^L g_e(\tau-nT_c,\theta)+c_{kN-n}b_k^R g_e(\tau-nT_c+T,\theta)\\
&\quad+\sum_{i=1}^{N-1} d_{ki}g_e(\tau+(i-n)T_c,\theta)\\
&=\sum_{i=0}^{N} d_{ki}g_e(\tau+(i-n)T_c,\theta)
\end{aligned} \tag{3.6}$$

and thus, the distribution of $I_0((\rho_k^2(b_k, \tau_k, e) + \psi_k^2(b_k, \tau_k))^{1/2})$ is the same modulo T_c when τ_k is uniformly distributed.

Let us now consider the particular case of a single interferer $K = 2$ which may also be used to approximate the situation in which we have a single *dominant* interferer. In this case, the correction term of the likelihood ratio is equal to

$$\frac{\displaystyle\int_0^{T_c}\int_0^{2\pi} E\exp\left(\frac{2w_2^{1/2}}{N_0}\sum_{i=0}^{N} d_{2i}g_+(iT_c-\lambda,\theta)\right)d\theta\,d\lambda}{\displaystyle\int_0^{T_c}\int_0^{2\pi} E\exp\left(\frac{2w_2^{1/2}}{N_0}\sum_{i=0}^{N} d_{2i}g_-(iT_c-\lambda,\theta)\right)d\theta\,d\lambda} \tag{3.7}$$

where the expectation is over the independent and equiprobable sequence $d_{2i} \in \{-1, 1\}$, $i = 0, \cdots, N$. The integrands in the numerator and denominator of (3.7) are products of hyperbolic cosines which do not lend themselves to further simplification. However, if N (the number of chips) is large, the distribution of the discrete random variable $\Sigma_{i=0}^{N} d_{2i}g_e(iT_c - \lambda, \theta)$ approximates the normal curve, and further simplification of (3.7) is possible. To justify this approximation, we show that for each θ, λ and each realization of

$$x(t)=[r_p(t)-ew_1^{1/2}a_1(t)]\cos\theta-r_q(t)\sin\theta$$

such that $\sup\{|x(t)| \mid t \in (0, T)\} < \infty$, the following triangular array of random variables[3]

$$\zeta_{ni} = d_{2i} \int_{(i-1)T/n}^{iT/n} x(t-\lambda)\, dt \qquad i = 1, \cdots, n \tag{3.8}$$

satisfies the Lindeberg–Feller condition (e.g., [5])

$$\lim_{n\to\infty} \sum_{i=1}^{n} E[(\zeta_{ni}/e_n)^2 I\{|\zeta_{ni}| > \delta e_n\}] = 0 \qquad \text{for every } \delta > 0 \tag{3.9}$$

where $e_n^2 = E[\Sigma_{i=1}^n \zeta_{ni}^2]$. To check (3.9), first note that

$$\lim_{n\to\infty} \frac{n}{T} \sum_{i=1}^{n} \zeta_{ni}^2 = \lim_{n\to\infty} \frac{n}{T} \sum_{i=1}^{n} x^2(t_i^n) T^2/n^2 = \|x\|^2 \tag{3.10}$$

where $t_i^n \in ((i - 1/n)T, (iT/n))$ and the first equation in (3.10) uses the mean-value theorem on the integral of (3.8). Therefore, for every $\epsilon > 0$, there exists n_0 such that for all $1 \le i \le n$ and $n > n_0$,

$$I\{|\zeta_{ni}| > \delta e_n\} < I\left\{\frac{n}{T}\zeta_{ni}^2 > \delta^2(\|x\|^2 - \epsilon)\right\}. \tag{3.11}$$

But for each $\mu > 0$, we can find n_1 such that for $n > n_1$, we have

$$I\left\{\frac{n}{T}|\zeta_{ni}|^2 > \mu\right\} \le I\left\{\frac{T}{n}\sup_t{}^2|x(t)| > \mu\right\} = 0. \tag{3.12}$$

Hence, only a finite number of terms on the left-hand side of (3.9) are nonzero, and since $\lim_{n\to\infty} \zeta_{ni}^2/e_n^2 = 0$, (3.9) follows.

If $\rho \cos\theta - \psi \sin\theta$ is a Gaussian random variable, then it is straightforward to check that

$$EI_0(\sqrt{\rho^2+\psi^2}) = \exp\left(\frac{1}{4}(E[\rho^2]+E[\psi^2])\right) I_0\left(\sqrt{\frac{1}{16}(E[\rho^2]-E[\psi^2])^2 + \frac{1}{4}E^2[\rho\psi]}\right). \tag{3.13}$$

Hence, using the Gaussian approximation[4] to the distribution of $\Sigma_{i=0}^N d_{2i} g_e(iT_c - \lambda, \theta)$, the correction term in (3.7) reduces to

$$\frac{\int_0^{T_c} \exp\left(\frac{w_2}{N_0^2}(\Xi_+(\lambda)+\Phi(\lambda))\right) I_0\left(\frac{w_2}{N_0^2}\sqrt{(\Xi_+(\lambda)-\Phi(\lambda))^2+\Theta_+^2(\lambda)}\right) d\lambda}{\int_0^{T_c} \exp\left(\frac{w_2}{N_0^2}(\Xi_-(\lambda)+\Phi(\lambda))\right) I_0\left(\frac{w_2}{N_0^2}\sqrt{(\Xi_-(\lambda)-\Phi(\lambda))^2+\Theta_-^2(\lambda)}\right) d\lambda} \tag{3.14}$$

where

$$\Xi_e(\lambda) = \sum_{i=0}^{N} \xi_e^2(iT_c - \lambda) \tag{3.15}$$

$$\Phi(\lambda) = \sum_{i=0}^{N} \phi^2(iT_c - \lambda) \tag{3.16}$$

and

$$\Theta_e(\lambda) = 2\sum_{i=0}^{N} \xi_e(iT_c-\lambda)\phi(iT_c-\lambda). \tag{3.17}$$

This structure is illustrated in Fig. 2. Note that there is considerable simplification in this structure over that of Fig. 1. In particular, each of the "+" and "−" channels involves chip matched filtering of the in-phase and quadrature components followed by chip-rate sampling, quadratic accumulation, memoryless nonlinear transformation, and integration over the offset λ. This latter operation can be implemented in parallel form by decimating an M/T_c-rate sampler (rather than a $1/T_c$-rate sampler) where M is the number of points taken in the numerical computation of the integral.

Further simplification of the correction term (2.9) is also possible in the case $K > 2$ by using the Gaussian approximation. We can obtain an expression similar to (3.14) where the integration is now over the hypercube $[0, T_c]^{K-1}$. Analogously to the case $K = 2$, for each $\boldsymbol{\theta} = (\theta_2, \cdots, \theta_K)$ and $\boldsymbol{\tau} = (\tau_2, \cdots, \tau_K)$, the distribution of

$$\sum_{k=2}^{K} \rho_k(b_k, \tau_k, e)\cos\theta_k - \psi_k(b_k, \tau_k)\sin\theta_k - \sum_{j=2}^{k-1} \Gamma_{kj}(b_k, b_j, \tau_k, \tau_j)$$

is approximately Gaussian, and since Γ_{kj} is uncorrelated with ρ_k, ψ_k, ρ_j, and ψ_j, both the numerator and the denominator of the correction term in (2.9) are approximated by

$$\int_{[0,T_c]}^{K-1} \cdots \int d\tau_2 \cdots d\tau_k E \cdot \exp\left(-\sum_{k=2}^{K}\sum_{j=2}^{k-1} \Gamma_{kj}(b_k, b_j, \tau_k, \tau_j)\right) \cdot \sum_{k=2}^{K} EI_0\left(\sqrt{\rho_k^2(b_k, \tau_k, e) - \psi_k^2(b_k, \tau_k)}\right).$$

Now, $E\exp(-\Sigma_{k=2}^K \Sigma_{j=2}^{k-1} \Gamma_{kj}(b_k, b_j, \tau_k, \tau_j))$ depends only on $\boldsymbol{\tau}$ and on the chip waveform, and since if $X \sim N(0, \sigma)$, then $E\exp X = \exp \sigma^2/2$, we have

$$E\exp\left(-\sum_{k=2}^{K}\sum_{j=2}^{k-1}\Gamma_{kj}(b_k, b_j, \tau_k, \tau_j)\right) \approx \exp\left(\frac{1}{2}E\left[\left(\sum_{k=2}^{K}\sum_{j=2}^{k-1}\Gamma_{kj}(b_k, b_j, \tau_k, \tau_j)\right)^2\right]\right)$$

$$= \exp\left(\frac{1}{2}\sum_{k=2}^{K}\sum_{j=2}^{k-1} E\Gamma_{kj}^2(b_k, b_j, \tau_k, \tau_j)\right)$$

$$\approx \prod_{k=2}^{K}\prod_{j=2}^{k-1} \exp\left[\frac{w_k w_j}{N_0^2}\left(\frac{|\tau_k-\tau_j|^2 + (T_c - |\tau_k-\tau_j|)^2}{T_c}\right)\right] \tag{3.18}$$

[3] For the sake of notational simplicity, here we consider the case of a rectangular chip waveform. In this case, $\zeta_{Ni} = d_{2i} g_e(iT_c - \lambda, \theta)$.

[4] It should be noted that the use of a central-limit theorem here is quite different from the Gaussian approximations used in many previous analyses of conventional single-user receivers. Here, we do not claim that the multiple-access interference is asymptotically a white Gaussian process; however, we do show via the Lindeberg–Feller condition (3.9) that the decision statistics in (3.7) are conditionally Gaussian random variables as the number of chips per symbol goes to infinity.

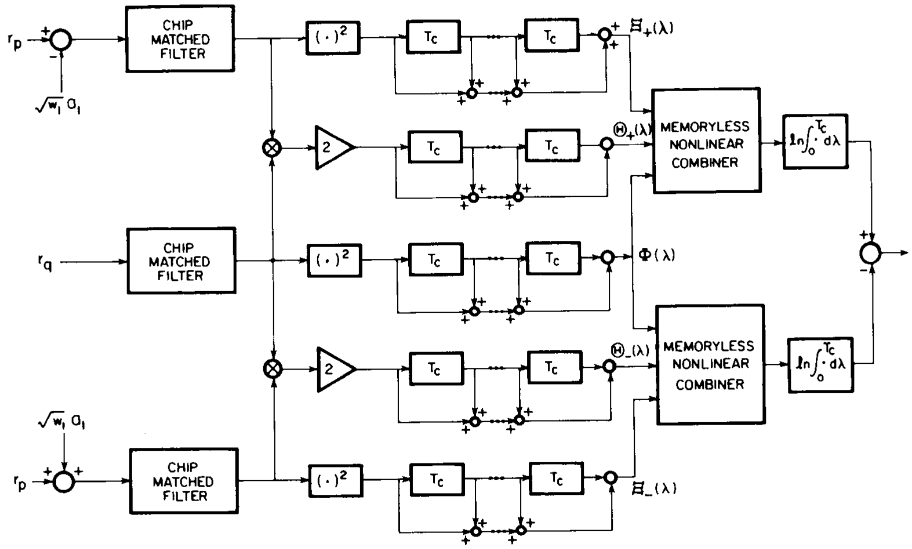

Fig. 2. Correction statistic for single-user detector with long spreading sequences.

where the last approximation follows by assuming that $\psi(t) = 1/T^{1/2}$ for $t \in [0, T_c]$ and by neglecting an $O(1/N^2)$ term in the exponent. Hence, the overall correction term is approximately equal to

$$\frac{\displaystyle\int_{[0,T_c]} \cdots \int_{[0,T_c]} \prod_{k=2}^{K} \exp\left(\frac{w_k}{N_0^2}(\Xi_+(\tau_k)+\Phi(\tau_k))\right) I_0\left(\frac{w_k}{N_0^2}\sqrt{(\Xi_+(\tau_k)-\Phi(\tau_k))^2+\Theta_+^2(\tau_k)}\right)}{\displaystyle\int_{[0,T_c]} \cdots \int_{[0,T_c]} \prod_{k=2}^{K} \exp\left(\frac{w_k}{N_0^2}(\Xi_-(\tau_k)+\Phi(\tau_k))\right) I_0\left(\frac{w_k}{N_0^2}\sqrt{(\Xi_-(\tau_k)-\Phi(\tau_k))^2+\Theta_-^2(\tau_k)}\right)}$$
$$\cdot \frac{\displaystyle\prod_{j=2}^{k-1} \exp\left[\frac{2w_k w_j}{NN_0^2}\left(\frac{|\tau_k-\tau_j|^2+(T_c-|\tau_k-\tau_j|)^2}{T_c}\right)\right] d\tau_2 \cdots d\tau_K}{\displaystyle\prod_{j=2}^{k-1} \exp\left[\frac{2w_k w_j}{NN_0^2}\left(\frac{|\tau_k-\tau_j|^2+(T_c-|\tau_k-\tau_j|)^2}{T_c}\right)\right] d\tau_2 \cdots d\tau_K}. \tag{3.19}$$

Notice that the term that couples the integrals in (3.19) is asymptotically independent of τ as $N \to \infty$. Hence, (3.19) approaches the product of $K - 1$ (3.14)-like terms (substituting w_2 by w_k). Thus, in this limiting case, implementation of the multiuser correction term in the likelihood ratio involves the implementation of only one chip-matched-filter/quadratic-accumulator section followed by multiple averaging channels, one for each different value of w_k. Fig. 2 shows an implementation of the correction statistic to be added to the output of the single-user matched filter in the case of a single interferer. The general structure is the same, except that the memoryless nonlinearities output a process for each interferer which is then passed through a separate logarithmic integrator.

B. Optimum Single-User Detection for High SBNR

We now turn to another limiting case of the single-user detector for which a simplified form of the likelihood ratio exists, namely, the case when the power spectral density of the additive Gaussian noise goes to zero. In the above case, we saw that when the rest of the parameters are fixed, we can use a Gaussian approximation as $N \to \infty$. However, for fixed N, the error between the expected values of the exponentials, according to the true and Gaussian distributions, diverges as $N_0 \to 0$.[5] Hence, rather than using (3.19), we must take the limit as $N_0 \to 0$ of the original likelihood ratio (2.9). As in the previous analysis, we will first focus attention on the case of a single interferer ($K = 2$).

Since the spreading codes of the interfering users are modeled by the single-user receiver as equiprobable and independent binary sequences, the correction term of the likelihood ratio is given by (3.7) and the log-likelihood ratio is (except for a positive multiplicative constant) equal to

$$2\int_0^T r_p(t)a_1(t)\,dt + \frac{N_0}{2w_1^{1/2}} \cdot \log \int_0^{T_c}\int_0^{2\pi} E\exp\left(\frac{2w_2^{1/2}}{N_0} d_{2i} g_+(iT_c-\lambda,\theta)\right) d\theta\, d\lambda$$
$$-\frac{N_0}{2w_1^{1/2}} \log \int_0^{T_c}\int_0^{2\pi} \cdot E\exp\left(\frac{2w_2^{1/2}}{N_0}\sum_{i=0}^{N} d_{2i} g_-(iT_c-\lambda,\theta)\right) d\theta\, d\lambda \tag{3.20}$$

[5] This is due to the fact that as the variance goes to infinity, the error between the distributions accumulates on the tails (the true random variable is bounded) on which the expected value of the exponential largely depends.

where the expectation is over the independent and equiprobable sequence $d_{2i} \in \{-1, 1\}$, $i = 0, \cdots, N$. On taking the limit of the correction terms in (3.20), we obtain

$$\lim_{N_0 \to 0} \frac{N_0}{2w_1^{1/2}} \log \int_0^{T_c} \int_0^{2\pi} \cdot E \exp\left(\frac{2w_2^{1/2}}{N_0} \sum_{i=0}^{N} d_{2i} g_e(iT_c - \lambda, \theta)\right) d\theta\, d\lambda$$

$$= \lim_{N_0 \to 0} \frac{1}{w_1^{1/2}} \log \left[\int_0^{T_c} \int_0^{2\pi} \cdot \left[\exp\left(w_2^{1/2} \sum_{i=0}^{N} |g_e(iT_c - \lambda, \theta)|\right)\right]^{2/N_0} d\theta\, d\lambda\right]^{N_0/2}$$

$$- \lim_{N_0 \to 0} \frac{N_0}{2w_1^{1/2}} (N+1) \log 2$$

$$= \frac{1}{w_1^{1/2}} \log \sup_{\lambda, \theta} \left\{\exp\left(w_2^{1/2} \sum_{i=0}^{N} |g_e(iT_c - \lambda, \theta)|\right)\right\}$$

$$= (w_2/w_1)^{1/2} \sup_{\substack{\lambda \in [0, T_c] \\ \theta \in [0, 2\pi]}} \sum_{i=0}^{N} |g_e(iT_c - \lambda, \theta)|. \tag{3.21}$$

Therefore, in the limit as $N_0 \to 0$, the optimum single-user detector for $K = 2$ in the case of unknown interfering codes compares the test statistic

$$2 \int_0^T r_p(t) a_1(t)\, dt + (w_2/w_1)^{1/2} \sup_{\substack{\lambda \in [0, T_c] \\ \theta \in [0, 2\pi]}} \sum_{i=0}^{N} |g_+(iT_c - \lambda, \theta)|$$

$$- (w_2/w_1)^{1/2} \sup_{\substack{\lambda \in [0, T_c] \\ \theta \in [0, 2\pi]}} \sum_{i=0}^{N} |g_-(iT_c - \lambda, \theta)| \tag{3.22}$$

to a zero threshold. Note that as might be expected, (3.22) is also the limiting form of the generalized likelihood ratio test or maximum likelihood detector (see Helstrom [6, p. 291], for example).

We now investigate the error probability of the test in (3.22) when $N_0 = 0$. It was shown in [4] that when the delays, phases, and waveforms of all users are known, the fully centralized optimum detector achieves perfect demodulation with probability 1 in the absence of background noise. This is a nontrivial result, as is illustrated by the behavior of the *conventional* single-user detector which becomes multiple-access limited, i.e., the limit of its error probability as $N_0 \to 0$ is nonzero for sufficiently powerful interfering users. However, as in the present case, the conventional detector does not have access to the delays, phases, or signature sequences of the interfering users. So, the question arises as to whether an optimum single-user detector can achieve error-free performance regardless of the energies of the interfering users without knowledge of those parameters. The answer, in the two-user case, is given in the affirmative by the following result which does not put any restrictions on the signature sequences.

Proposition 3.1: Suppose $K = 2$ and $w_1 > 0$. If $r(t) = bs_1(t) + S^{MA}(t)$, $b \in \{-1, 1\}$, then

$$\text{sgn}\left[2 \int_0^T r_p(t) a_1(t)\, dt + (w_2/w_1)^{1/2} \cdot \sup_{\substack{\lambda \in [0, T_c] \\ \theta \in [0, 2\pi]}} \sum_{i=0}^{N} |g_+(iT_c - \lambda, \theta)| - (w_2/w_1)^{1/2} \cdot \sup_{\substack{\lambda \in [0, T_c] \\ \theta \in [0, 2\pi]}} \sum_{i=0}^{N} |g_-(iT_c - \lambda, \theta)|\right] = b \tag{3.23}$$

with probability 1.

Proof: See Appendix.

In the general case of $K > 2$ users, the log-likelihood ratio is proportional to [cf. (2.16)]

$$2w_1^{1/2} \int_0^T r_p(t) a_1(t)\, dt + \frac{N_0}{2} \log \int_{[0, T_c]} \cdots \int_{K-1} \int_{[0, 2\pi]} \cdots \int_{K-1} \cdot E \exp\left[-\frac{1}{N_0} \|S^{MA}\|^2 + \frac{2}{N_0} \langle r - s_1, S^{MA}\rangle\right] d\theta\, d\tau$$

$$- \frac{N_0}{2} \log \int_{[0, T_c]}^{K-1} \cdots \int \int_{[0, 2\pi]}^{K-1} \cdots \int \cdot E \exp\left[-\frac{1}{N_0} \|S^{MA}\|^2 + \frac{2}{N_0} \langle r + s_1, S^{MA}\rangle\right] d\theta\, d\tau \tag{3.24}$$

where the expectation is over the independent sequences $d = \{d_{ki} \in \{-1, 1\};\ i = 0, \cdots, N,\ k = 2, \cdots, K\}$. As in (3.21), this expectation is dominated as $N_0 \to 0$ by the atom corresponding to the largest integrand, i.e.,

$$d^* \in \arg_d \max_d \Omega_e(d, \tau, \theta) \tag{3.25}$$

where

$$\Omega_e(d, \tau, \theta) = \langle r - es_1, S^{MA}(d)\rangle - \frac{1}{2} \|S^{MA}(d)\|^2 \tag{3.26}$$

and

$$S^{MA}(t, d) = \sum_{i=0}^{N} \sum_{k=2}^{K} d_{ki} (2w_k)^{1/2} \psi(t - (i-1)T_c - \tau_k) \cdot \cos(\omega_c t + \theta_k - \omega_c \tau_k). \tag{3.27}$$

Since there are $2^{K(N+1)}$ possible sequences, it is necessary to find an efficient way to carry out the maximization in (3.25). But (3.26) and (3.27) have the same structure as (2.3) and (2.4), respectively, so we can apply the results of [2] to carry out the maximization of (3.25) with linear complexity in N. On taking the limit of (3.24) as $N_0 \to 0$, we obtain the test statistic

$$2w_1^{1/2} \int_0^T r_p(t) a_1(t)\, dt + \sup_{\tau, \theta} \Omega_+^*(\tau, \theta) - \sup_{\tau, \theta} \Omega_-^*(\tau, \theta) \tag{3.28}$$

where $\Omega_e^*(\tau, \theta) = \Omega_e(d^*, \tau, \theta)$. Even if these quantities are obtained through efficient dynamic programming recursions as in [2], the main computational burden of (3.28) is the maximization over $[0, T_c]^{K-1}$ and $[0, 2\pi]^{K-1}$, which imposes severe limitations on its feasibility for even a moderate number of users. However, note that, in performing the maximization, the receiver is essentially acquiring the chip timing and carrier phases of the interfering users. Thus, in practice, it would normally be unnecessary to undergo a full search for the maximizing τ and θ in each symbol interval since these quantities will change little from symbol interval to symbol interval. For this reason, (3.28) might be reasonably efficient to implement in approximate form.

IV. Locally Optimum Single-User Detectors with Weak Interferers

We have seen in the preceding sections that in multiple-access environments with many users, the complexity of optimum detection is increased considerably (over centralized reception) when the unwanted users are unlocked. This is true even without sequence detection and regardless of whether the interfering waveforms are known. However, one of the main incentives for the study of optimum decentralized detectors is the situation in which all or some of the interfering users are comparatively weak, so that it may be impractical to provide reliable synchronization for them. The objective of this section is to derive locally optimum (up to a third-order approximation) decentralized detectors for reception in the presence of weak unlocked users. As we shall see, such detectors can be viewed as versions of the detector that would be optimum without the weak interferers, modified to be robust against small deviations from the nominal white Gaussian noise statistics caused by weak multiple-access interference. As in the preceding sections, we consider both the case in which the waveforms of the interfering users are known, and the case in which they are coded with binary signature sequences unknown to the receiver. We will see here that the *locally optimum* version takes care only of the nonwhiteness of the multiple-access noise.

The approach we follow to derive locally optimum decentralized demodulators is to obtain an asymptotic form of the log-likelihood ratio for signal detection in contaminated white Gaussian noise given by the following result.

Lemma 4.1: Consider the following pair of statistical hypotheses:

$$H_0:\ r_t = s_t^0 + \epsilon\tilde{n}_t + n_t \qquad t \in [t_p, t_f]$$
$$H_1:\ r_t = s_t^1 + \epsilon\tilde{n}_t + n_t \qquad t \in [t_p, t_f] \tag{4.1}$$

where s^1 and s^0 are deterministic finite-energy signals, $\{n_t\}$ is white Gaussian noise with spectral height σ^2, and $\{\tilde{n}_t, t \in [t_0, t_f]\}$ in a symmetric random process such that $\|\tilde{n}\| < B$ (a.s.) for some constant B, and whose correlation function is denoted by $C_{t,\lambda} = E[\tilde{n}_t\tilde{n}_\lambda]$, $(t, \lambda) \in [t_0, t_f]^2$. Then the log-likelihood ratio for (4.1) admits in the following expression:

$$\log LR(\epsilon) = \frac{1}{\sigma^2}\int_{t_0}^{t_f}\left[(s_t^1 - s_t^0) - \left(\frac{\epsilon}{\sigma}\right)^2 \cdot \int_{t_0}^{t_f} C_{t,\lambda}(s_\lambda^1 - s_\lambda^0)\, d\lambda\right]\left(r_t - \frac{1}{2}s_t^1 - \frac{1}{2}s_t^0\right) dt + O(\epsilon^4). \tag{4.2}$$

Proof: Using the Cameron–Martin likelihood ratio formula, we obtain

$$\log LR(\epsilon) = \log\frac{D_1(\epsilon)}{D_0(\epsilon)} \tag{4.3}$$

where

$$D_i(\epsilon) = E\left[\exp\left(-\frac{1}{2\sigma^2}\left(\|s^i + \epsilon\tilde{n}\|^2 - 2\int_{t_p}^{t_f}(s_t^i + \epsilon\tilde{n}_t)\, dr_t\right)\right)\right] \qquad i = 0, 1 \tag{4.4}$$

where the expectation is over the ensemble of sample functions of $\{\tilde{n}_t, t \in [t_p, t_f]\}$.

In order to derive (4.2), we take the Taylor series expansion of (4.3) around the origin. Since $\tilde{n}_t$ is a symmetric random variable, it follows that $D_i(-\epsilon) = D_i(\epsilon)$, and hence the odd terms in the Taylor expansion of $D_i(\epsilon)|_{\epsilon=0}$ and $\log D_i(\epsilon)|_{\epsilon=0}$ are equal to zero.

Using the fact that $\|\tilde{n}\| < B$ a.s. and the Schwarz inequality, it follows that the expectation of every coefficient in the series expansion of the exponential in (4.4) exists, and we can write

$$D_i(\epsilon) = D_i(0)\left[1 + \frac{\epsilon^2}{2}E\left[\left(\frac{1}{\sigma^2}\int_{t_p}^{t_f}\tilde{n}_t(r_t - s_t^i)\, dt\right)^2 - \frac{1}{\sigma^2}\|\tilde{n}\|^2\right] + O(\epsilon^4)\right]. \tag{4.5}$$

Now, since $\log(1 + x) = x + O(x^2)$, we obtain

$$\log\frac{D_1(\epsilon)}{D_0(\epsilon)} = \log\frac{D_1(0)}{D_0(0)} + \frac{\epsilon^2}{2}E\left[\left(\frac{1}{\sigma^2}\int_{t_p}^{t_f}\tilde{n}_t(r_t - s_t^1)\, dt\right)^2 - \left(\frac{1}{\sigma^2}\int_{t_p}^{t_f}\tilde{n}_t(r_t - s_t^0)\, dt\right)^2\right] + O(\epsilon^4) \tag{4.6}$$

and (4.2) follows straightforwardly. □

Notice that the stringent condition $\|\tilde{n}\| < B$ (a.s.) allows a straightforward proof of Lemma 4.1 and is satisfied in the case in which we are interested, namely,

$$\tilde{n}_t = \sum_{i=-M}^{M}\sum_{k=D+1}^{K} b_k(i)s_k(t - iT - \tau_k);\ b_k(i) \in \{-1, 1\}. \tag{4.7}$$

If the waveforms $\{a_k(t), k = D + 1, \cdots, K\}$ are known by the receiver, then the autocorrelation function of $\tilde{n}$ with support in $\mathbb{R}^2$ (for $M = \infty$) is equal to

$$C_{t,\lambda}^{MA} = \frac{1}{T}\cos(\omega_c(t - \lambda))\sum_{k=D+1}^{K} w_k R_k(t - \lambda) \tag{4.8}$$

where

$$R_k(t) = \int_0^T a_k(s - t)a_k(s)\, ds. \tag{4.9}$$

If the waveforms of the interfering users have the form in (3.1) and the code of each user is unknown by the receiver and assumed to be equiprobably distributed among all $\{-1, 1\}$ sequences of length N, then the autocorrelation is

$$C_{t,\lambda}^{MA} = \frac{1}{T}\cos(\omega_c(t - \lambda))\sum_{k=D+1}^{K} w_k\Psi(t - \lambda) \tag{4.10}$$

where the autocorrelation of the chip waveform is denoted by $\Psi(t) = \int_0^{T_c}\psi(s)\psi(s - t)\, dt$.

The one-shot single-user detector can be obtained readily from the result of Lemma 4.1. Since the signal of user 1 is antipodally modulated, we have

$$s_t^1 - s_t^0 = 2\sqrt{2w_1}\, a_1(t) \cos(\omega_c t + \theta_1)$$

and (4.2) becomes

$$\log LR = \frac{4\sqrt{w_1}}{N_0} \int_0^T r_p(t) \cdot \left[a_1(t) - \frac{1}{N_0 T} \sum_{k=2}^{K} w_k \int_0^T a_1(\lambda) R_k(t-\lambda)\, d\lambda \right] dt + O(\max_{k>1} w_k^2). \tag{4.11}$$

Hence, the locally optimum one-shot single-user detector is a conventional correlation receiver in which $a_1(t)$ is replaced by $a_1(t) - (1/N_0 T) \sum_{k=2}^K w_k \int_0^T a_1(\lambda) R_k(t-\lambda)\, d\lambda$, $t \in [0, T]$, i.e., the pseudosignal is the output in $[T, 2T]$ of a causal linear filter, driven by $a_1(t)$, and whose impulse response is equal to $\delta(t-T) - (1/N_0 T) \sum_{k=2}^K w_k R_k(t-T)$. If the signature sequences are unknown, the impulse response is $\delta(t - T) - (1/N_0 T) \sum_{k=2}^K w_k \Psi(t - T)$, which amounts to a mild smoothing of the signal replica of the user of interest.

The locally optimum detector that locks to D of K users is, in fact, a generalization of this conclusion. Using Lemma 4.1, it can be shown (see [3, ch. 5] for details) that the locally optimum D-user detector is a centralized detector whose correlators use replicas of the unmodified waveforms of the users of interest. However, the input is processed by a causal filter that whitens the interference due to unlocked users, and whose impulse response depends on the autocorrelation function and signal-to-noise ratio of each interfering signal. This requires a modification of the DP algorithm to account for the intersymbol interference introduced by the prefilter, and results in a complexity of $O(2^{2D})$ as opposed to $O(2^D)$ for the corresponding algorithm that neglects the additional $K - D$ interferers.

V. Summary

In this paper, we have obtained decentralized single-user detectors which take into account the presence of interfering users. The general decentralized demodulation problem is one of sequence detection in additive colored non-Gaussian noise, and results in nonlinear detectors whose decision algorithms do not admit recursive forms and hence are more complex than their centralized counterparts. Important reductions in complexity occur when attention is focused on one-shot single-user detectors.

The general form of the single-user likelihood ratio obtained in Proposition 2.1 is equal to the single-user likelihood ratio affected by a correction term which depends on both the in-phase and quadrature components of the input. Both the case where the baseband interfering waveforms are known and the case where they are coded by an unknown signature sequence have been studied.

Under the assumption that the assigned waveforms are signature sequences with N chips per bit, we have obtained limiting forms of the correction term for $N \gg 1$ and for $N_0 = 0$. In the first case, the correction term depends on the received waveform only through the functions $\Xi_\pm(\lambda)$, $\Phi(\lambda)$, and $\Theta_\pm(\lambda)$ which represent the l_2 norms and inner product, respectively, of the subintegrals of an N partition (with offset $\lambda \in [0, T_c]$) of the in-phase and quadrature components of the received noise process under both hypotheses. The correction term when $N_0 = 0$ is best illustrated in the single-interferer case where it is obtained through the maximization over the relative phase and delay of the l_1 norm of the above subintegrals. It has been shown that this detector (which assumes knowledge of only the chip waveform and energy of the interfering user) achieves perfect demodulation in the absence of Gaussian noise regardless of the energy of the interference, thus avoiding the multiple-access limitation that plagues the conventional receiver. Using dynamic programming, the single-user detector can be implemented in linear time in N; however, its main computational burden is the maximization over $[0, T_c]^{K-1}$ and $[0, 2\pi]^{K-1}$ needed in the correction term.

Using an asymptotic form of the log-likelihood ratio for signal detection in contaminated white Gaussian noise, we have derived locally optimum detectors up to a third-order approximation in the amplitude of the interfering users. The locally optimum one-shot detector has been shown to be a single-user correlation receiver which uses a smooth replica of the signal of interest. It has been shown in [3] that this approach can be generalized to the case of partial decentralization ($D > 1$), resulting in robustified versions of the centralized D-user receiver, which may offer substantial computational savings over the optimum K-user receiver.

Appendix

Proof of Proposition 3.1

We assume that the bit transmitted by user 1 is $b = 1$, the proof being identical in the antipodal case. For notational convenience and without lost of generality, we suppose that the relative delay of the interfering user is $0 < \tau_2 \leq T_c$; then it follows that

$$\alpha_2(b_2^L, b_2^R, t-\tau_2) = \sum_{i=0}^{N} d_i \psi(t - iT_c + \lambda_2) \tag{A.1}$$

where $\lambda_2 = T_c - \tau_2$, $d_0 = c_{2N-1} b_2^L$, and $d_{i+1} = c_{2i} b_2^R$ for $i = 0, N - 1$. Let $\beta = \theta_1 + \omega_c \tau_2 - \theta_2$; then it is easy to show that

$$\int_0^T r_p(t) a_1(t) = w_1^{1/2} + w_2^{1/2} \cos\beta \cdot \int_0^T a_1(t) \alpha_2(b_2^L, b_2^R, t-\tau_2)\, dt \tag{A.2}$$

$$g_+(iT_c - \lambda, \theta) = w_2^{1/2} \cos(\theta + \beta) \cdot \int_0^T \alpha_2(b_2^L, b_2^R, t-\tau_2) \psi(t - iT_c + \lambda)\, dt \tag{A.3}$$

and

$$g_-(iT_c - \lambda, \theta) = 2w_1^{1/2} \cos\theta \cdot \int_0^T a_1(t) \psi(t - iT_c + \lambda)\, dt + g_+(iT_c - \lambda, \theta). \tag{A.4}$$

We show now that

$$\sup_{\substack{\lambda \in [0, T_c] \\ \theta \in [0, 2\pi]}} \sum_{i=0}^{N} |g_+(iT_c - \lambda, \theta)| = w_2^{1/2}. \tag{A.5}$$

To that end, using (A.1) and (A.3), we obtain for every $\lambda \in [0, T_c]$

$$\sup_{\theta \in [0, 2\pi]} \sum_{i=0}^{N} |g_+(iT_c - \lambda, \theta)| = w_2^{1/2} \sum_{i=0}^{N} \left| \int_0^T \alpha_2(b_2^L, b_2^R, t-\tau_2) \psi(t - iT_c + \lambda)\, dt \right|$$

$$= w_2^{1/2} \sum_{i=0}^{N} \left| \int_0^T \sum_{j=0}^{N} d_j \psi(t - jT_c + \lambda_2) \psi(t - iT_c + \lambda) \, dt \right|$$

$$\le w_2^{1/2} \int_0^T \sum_{i=0}^{N} \sum_{j=0}^{N} |\psi(t - jT_c + \lambda_2)| |\psi(t - iT_c + \lambda)| \, dt$$

$$\le w_2^{1/2} \left(\int_0^T \sum_{j=0}^{N} \psi^2(t - jT_c + \lambda_2) \, dt \right)^{1/2}$$

$$\cdot \left(\int_0^T \sum_{i=0}^{N} \psi^2(t - iT_c + \lambda) \, dt \right)^{1/2} = w_2^{1/2} \qquad \text{(A.6)}$$

where the last two equations follow from the Schwarz inequality and from the relationship $\int_0^{T_c} (\psi^2(t + s) + \psi^2(t - T_c + s)) \, dt = \int_0^{T_c} \psi^2(t) = 1/N$, $0 \le s \le T_c$, respectively. But the right-hand side of (A.6) is achieved when $\lambda = \lambda_2$; hence, (A.5) follows. Consequently, in order to show that the sign of the log-likelihood ratio is positive, one has to prove that

$$2w_1 + w_2 + w_1^{1/2} w_2^{1/2} \int_0^T 2a_1(t) \alpha_2(b_2^L, b_2^R, t - \tau_2)$$

$$\cdot \cos \beta \, dt - w_2^{1/2} \sum_{i=0}^{N} |g_-(iT_c - \lambda, \theta)| > 0 \quad \text{(A.7)}$$

for all $\lambda \in [0, T_c]$ and $\theta \in [0, 2\pi]$. Using (A.3) and (A.4), we obtain

$$2w_1 + w_2 + w_1^{1/2} w_2^{1/2} \int_0^T 2a_1(t) \alpha_2(b_2^L, b_2^R, t - \tau_2) \cos \beta \, dt$$

$$- w_2^{1/2} \sum_{i=0}^{N} |g_-(iT_c - \lambda, \theta)|$$

$$= 2w_1 + w_2 + w_1^{1/2} w_2^{1/2}$$

$$\cdot \int_0^T 2a_1(t) \alpha_2(b_2^L, b_2^R, t - \tau_2) \cos \beta \, dt$$

$$- w_2^{1/2} \sum_{i=0}^{N} \left| \int_0^T (2w_1^{1/2} a_1(t) \cos \theta + w_2^{1/2} \right.$$

$$\left. \cdot \alpha_2(b_2^L, b_2^R, t - \tau_2) \cos(\theta + \beta)) \psi(t - iT_c + \lambda) \, dt \right|. \qquad \text{(A.8)}$$

The last term on the right-hand side of the above equation can be bounded as follows:

$$\sum_{i=0}^{N} \left| \int_0^T (2w_1^{1/2} a_1(t) \cos \theta + w_2^{1/2} \alpha_2(b_2^L, b_2^R, t - \tau_2) \right.$$

$$\left. \cdot \cos(\theta + \beta)) \psi(t - iT_c + \lambda) \, dt \right|$$

$$\le \sum_{i=0}^{N} \int_0^T |2w_1^{1/2} a_1(t) \cos \theta + w_2^{1/2} \alpha_2(b_2^L, b_2^R, t - \tau_2)$$

$$\cdot \cos(\theta + \beta)| \cdot |\psi(t - iT_c + \lambda)| \, dt$$

$$= \int_0^T |2w_1^{1/2} a_1(t) \cos \theta + w_2^{1/2} \alpha_2(b_2^L, b_2^R, t - \tau_2)$$

$$\cdot \cos(\theta + \beta)| \sum_{i=0}^{N} |\psi(t - iT_c + \lambda)| \, dt$$

$$\le \left[\int_0^T (2w_1^{1/2} a_1(t) \cos \theta + w_2^{1/2} \alpha_2(b_2^L, b_2^R, t - \tau_2) \right.$$

$$\left. \cdot \cos(\theta + \beta))^2 \, dt \right]^{1/2} \left[\int_0^T \sum_{i=0}^{N} \psi^2(t - iT_c + \lambda) \, dt \right]^{1/2}$$

$$= \left[4w_1 \cos^2 \theta + w_2 \cos^2(\theta + \beta) + 4w_1^{1/2} w_2^{1/2} \right.$$

$$\left. \cdot \cos \theta \cos(\theta + \beta) \int_0^T a_1(t) \alpha_2(b_2^L, b_2^R, t - \tau_2) \, dt \right]^{1/2}. \qquad \text{(A.9)}$$

Since $|\int_0^T a_1(t) \alpha_2(b_2^L, b_2^R, t - \tau_2) \, dt| \le 1$, we can denote $\int_0^T a_1(t) \alpha_2(b_2^L, b_2^R, t - \tau_2) \, dt = \cos \alpha$, and using (A.9), the right-hand side of (A.8) can be lower bounded by

$$2w_1 + w_2 + w_1^{1/2} w_2^{1/2} \int_0^T 2a_1(t) \alpha_2(b_2^L, b_2^R, t - \tau_2) \cos \beta \, dt$$

$$- w_2^{1/2} \sum_{i=0}^{N} \left| \int_0^T (2w_1^{1/2} a_1(t) \cos \theta + w_2^{1/2} \right.$$

$$\left. \cdot \alpha_2(b_2^L, b_2^R, t - \tau_2) \cos(\theta + \beta)) \psi(t - iT_c + \lambda) \, dt \right|$$

$$\ge 2w_1 + w_2 + 2w_1^{1/2} \cos \alpha \cos \beta$$

$$- w_2^{1/2} (4w_1 \cos^2 \theta + w_2 \cos^2(\theta + \beta)$$

$$+ 4w_1^{1/2} w_2^{1/2} \cos \theta \cos(\theta + \beta) \cos \alpha)^{1/2}. \qquad \text{(A.10)}$$

Now, since $2w_1 + w_2 + 2w_1^{1/2} w_2^{1/2} \cos \alpha \cos \beta > 0$, the sign of the right-hand side of (A.10) is equal to the sign of

$$(2w_1 + w_2 + 2w_1^{1/2} w_2^{1/2} \cos \alpha \cos \beta)^2$$

$$- [4w_1 w_2 \cos^2 \theta + w_2^2 \cos^2(\theta + \beta)$$

$$+ 4w_2 w_1^{1/2} w_2^{1/2} \cos \theta \cos \alpha \cos(\theta + \beta)]$$

$$= (2w_1 + 2w_1^{1/2} w_2^{1/2} \cos \alpha \cos \beta)^2 + 4w_1 w_2 (1 - \cos^2 \theta)$$

$$+ w_2^2 (1 - \cos^2(\theta + \beta)) + 4w_2 w_1^{1/2} w_2^{1/2}$$

$$\cdot \cos \alpha \, [\cos \beta - \cos \theta \cos(\theta + \beta)]$$

$$= (2w_1 + 2w_1^{1/2} w_2^{1/2} \cos \alpha \cos \beta)^2 + 4w_1 w_2 \sin^2 \theta$$

$$+ w_2^2 \sin^2(\theta + \beta) + 4w_2 w_1^{1/2} w_2^{1/2} \sin \theta \sin(\theta + \beta) \cos \alpha$$

$$= (2w_1 + 2w_1^{1/2} w_2^{1/2} \cos \alpha \cos \beta)^2$$

$$+ (w_2 \sin(\theta + \beta) \cos \alpha + 2\sqrt{w_1 w_2} \sin \theta)^2$$

$$+ (w_2 \sin(\theta + \beta) \sin \alpha)^2. \qquad \text{(A.11)}$$

Therefore, we have shown that (A.10) and, consequently, the left-hand side of (3.22), are nonnegative. Moreover, the right-hand side of (A.11) is equal to zero only if

$$2w_1 + 2w_1^{1/2} w_2^{1/2} \cos \alpha \cos \beta = 0, \qquad \text{(A.12)}$$

but since $\beta = \theta_1 + \omega_c \tau_2 - \theta_2$ is uniformly distributed, (A.12) occurs with probability zero if $w_1 > 0$.

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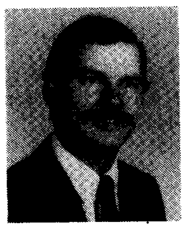

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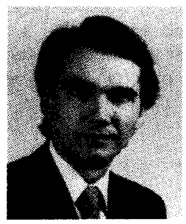

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