Power Efficiency of Joint Frequency-Phase Modulation in the Low-SNR Regime over Noncoherent Rician Channels

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Abstract—[1] **Transmission of information over noncoherent Rician fading channels using M-ary orthogonal on/off FSK (OOFSK) signaling with phase modulation is considered. The capacity of this signaling scheme is obtained for both finite M and as M tends to infinity. Power efficiency is investigated when the transmitter is subject to a peak-to-average power ratio (PAR) limitation or a peak power limitation. It is shown that under PAR limitation, in contrast to average power limited systems, the minimum bit energy is not always achieved at zero spectral efficiency. It is concluded that, in these cases, operating at very low spectral efficiency should be avoided. On the other hand, it is demonstrated that if there is only a peak power limitation, power efficiency improves as one operates with smaller SNR and vanishing duty factor.**

I. Introduction

A wide range of digital communication systems in wireless, deep-space, and sensor networks operate in the low-power regime where power consumption rather than bandwidth is the limiting factor. In these systems, power-efficient transmission schemes are needed in order to make effective use of scarce energy resources. Ultrawideband systems also employ low-power pulses of very short duration subject to strict peak power requirements.

The power efficiency of a communication system can be measured by the energy required for reliable communication of one bit. When communicating at rate R bits/s with power P, the transmitted energy per bit is $E_b = \frac{P}{R}$. Since the maximum rate is given by the channel capacity, $C(P)$, the least amount of bit energy required for reliable communication is $E_b = \frac{P}{C(P)}$. In his seminal work [1], Shannon showed that the capacity of an ideal bandlimited additive white Gaussian noise channel is $C = B \log_2 \left(1 + \frac{P}{BN_0}\right)$ bits/s where P is the received power, B is the channel bandwidth and N_0 is the one-sided noise spectral level. Notice that as the bandwidth grows to infinity, the capacity monotonically increases to $\frac{P}{N_0} \log_2 e$ bits/s, therefore decreasing the required received bit-energy normalized to the noise power to

$$\frac{E_b^r}{N_0} = \frac{P/N_0}{C} \underset{B\to\infty}{\longrightarrow} \log_e 2 = -1.59 \text{ dB}. \tag{1}$$

This minimum bit energy (1) can be approached by pulse position modulation with vanishing duty cycle [2] or by M-ary orthogonal signaling as M becomes large [3]. In the presence of unknown fading, Jacobs [4] and Pierce [5] have noted that M-ary orthogonal signaling obtained by frequency shift keying (FSK) modulation can still approach (1) for large values of M. Gallager [11, Sec. 8.6] has also demonstrated that over fading channels M-ary orthogonal FSK signaling with vanishing duty cycle approaches the infinite bandwidth capacity of unfaded Gaussian channels as $M \to \infty$ thereby achieving (1). Luo and Médard [10] have shown that FSK with small duty cycle can achieve rates of the order of capacity in ultrawideband systems with limits on bandwidth and peak power. More recently, Verdú [6] has proven in considerably wider generality than was previously known that the minimum received bit energy normalized to the noise level in a Gaussian channel is -1.59 dB regardless of the knowledge of the fading at the receiver and/or transmitter. In particular, it is shown in [6] that if the receiver does not have perfect knowledge of the fading, flash signaling is required to achieve the minimum bit energy.

Besides approaching the minimum energy per bit, FSK modulation is particularly suitable for noncoherent communications. Butman *et al.* [7] studied the performance of M-ary FSK, which has unit peak-to-average power ratio, over noncoherent Gaussian channels by computing the capacity and computational cut-off rate. Stark [8] analyzed the capacity and cut-off rate of M-ary FSK signaling with both hard and soft decisions in the presence of Rician fading and noted that there exists an optimal code rate for which the required bit energy is minimized.

In this paper, we analyze the capacity and power efficiency of M-ary on/off FSK (OOFSK) signaling with phase modulation, in which M-ary FSK signaling is overlaid on on/off keying, enabling us to introduce peakiness in both time and frequency. As joint frequency-phase modulation is

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considered, information is also carried over the phase of the FSK signals. As discussed above, approaching the minimum bit energy demands not only infinite bandwidth but also, in the case of unknown fading, input signals that are peaky in time or frequency. Signals should either have increasingly higher peak power or concentrate all the power in one frequency slot. In this work, motivated by practical considerations, we limit the peakedness of input signals by imposing peak power constraints.

II. System Model

We assume that M-ary orthogonal OOFSK signaling with phase modulation, in which FSK signaling is overlaid on on-off keying with a fixed duty factor, $\nu \leq 1$, is employed at the transmitter for communication over Rician fading channels. In this signaling scheme, over the time interval of $[0, T]$ the transmitter either sends no signal with probability $1-\nu$, or sends one of M orthogonal sinusoidal signals,

$$s_{i,\theta_i}(t) = \sqrt{\frac{P}{\nu}}\, e^{j(\omega_i t + \theta_i)} \quad 0 \leq t \leq T, \quad 1 \leq i \leq M, \quad (2)$$

with probability ν. To ensure orthogonality, adjacent frequency slots satisfy $|\omega_{i+1} - \omega_i| = \frac{2\pi}{T}$. Since the transmitter uses phase modulation as well to increase the capacity and improve the efficiency especially in the low-SNR regime, the phase θ_i of the sinusoidal signals are also random variables carrying information. Choosing ν=1, we obtain ordinary FSK signaling with phase modulation. The channel input can be represented by the pair (X, θ). If $X = i$ for $1 \leq i \leq M$, and $\theta = \theta_i$, the transmitter sends the sine wave $s_{i,\theta_i}(t)$, while no transmission is denoted by $X = 0$, and hence $s_0(t) = 0$. Note that OOFSK signaling has average power P, and peak power P/ν.

We assume that the transmitted signal undergoes fast frequency nonselective Rician fading. In particular, we assume that while the fading stays constant in each symbol interval, it changes independently from symbol to symbol. The received signal can be modeled as follows:

$$r(t) = h_k\, s_{X_k,\theta_k}(t-(k-1)T) + n(t), \; (k-1)T \leq t \leq kT, \quad (3)$$

where $\{X_k, \theta_k\}_{k=1}^{\infty}$ is the input sequence with $X_k \in \{0, 1, 2, \ldots, M\}$ and $\theta_k \in [-\pi, \pi)$, $\{h_k\}_{k=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) proper complex Gaussian random variables with $E\{h_k\} = d$ and $\text{var}(h_k) = \gamma^2$, and $n(t)$ is a zero-mean circularly symmetric complex white Gaussian noise process with single-sided spectral density N_0. Due to the fast fading conditions, we assume the *noncoherent scenario* where neither the receiver nor the transmitter knows the fading coefficients $\{h_k\}$.

At the receiver, a bank of correlators is employed to obtain in each symbol interval the M-dimensional vector $\mathbf{Y}_k = (Y_{k,1}, \ldots, Y_{k,M})$ where

$$Y_{k,i} = \frac{1}{\sqrt{N_0 T}} \int_{(k-1)T}^{kT} r(t) e^{-j\omega_i t}\, \mathrm{d}t, \quad i = 1, 2, \ldots, M. \quad (4)$$

It is easily seen that, given the symbol $X_k = i$, phase θ_i and fading coefficient h_k, $Y_{k,j}$ is a proper complex Gaussian random variable with $E\{Y_{k,j} | X_k = i, \theta_i, h_k\} = \alpha\, h_k\, e^{j\theta_i} \delta_{ij}$ and $\text{var}(Y_{k,j} | X_k = i, \theta_i, h_k) = 1$, where $\delta_{ij} = 1$ if $i = j$ and is zero otherwise, and $\alpha^2 = \frac{PT}{\nu N_0} = \frac{\text{SNR}}{\nu}$ with SNR denoting the signal-to-noise ratio per symbol.

III. Channel Capacity

We first analyze the capacity of the OOFSK signaling over Rician fading channels. While providing the maximum rates at which reliable communication is possible, capacity analysis also enables us to investigate the power efficiency by facilitating the computation of the minimum energy required to send one information bit.

Proposition 1: For the noncoherent Rician fading channel model (3), the capacity of M-ary orthogonal OOFSK signaling with phase modulation and duty factor $\nu \leq 1$ is given by

$$\begin{aligned} C_M(\text{SNR}) &= -M - \nu \log(\gamma^2 \frac{\text{SNR}}{\nu} + 1) \\ &- (1-\nu) \int p_{\mathbf{R}|X=0} \log p_{\mathbf{R}}\, \mathrm{d}\mathbf{R} - \nu \int p_{\mathbf{R}|X=1} \log p_{\mathbf{R}}\, \mathrm{d}\mathbf{R} \quad (5) \end{aligned}$$

where

$$p_{\mathbf{R}} = (1-\nu) p_{\mathbf{R}|X=0} + \frac{\nu}{M} \sum_{i=1}^{M} p_{\mathbf{R}|X=i},$$

$$p_{\mathbf{R}|X=0} = e^{-\sum_{j=1}^{M} R_j},$$

$$p_{\mathbf{R}|X=i} = e^{-\sum_{j=1}^{M} R_j} f(R_i, \text{SNR}) \quad 1 \leq i \leq M,$$

and

$$f(R_i, \text{SNR}) = \frac{e^{\frac{\frac{\text{SNR}}{\nu}(\gamma^2 R_i - |d|^2)}{\gamma^2 \frac{\text{SNR}}{\nu} + 1}}}{\gamma^2 \frac{\text{SNR}}{\nu} + 1} I_0 \left(\frac{2\sqrt{\frac{\text{SNR}}{\nu} |d|^2 R_i}}{\gamma^2 \frac{\text{SNR}}{\nu} + 1} \right).$$

Proof: Note that since memoryless channel is assumed, channel capacity is formulated as follows:

$$\begin{aligned} C_M &= \max_{X,\theta} I(X, \theta; \mathbf{Y}) \\ &= \max_{X,\theta}, (1-\nu) \int p_{\mathbf{Y}|X=0,\theta} \log \frac{p_{\mathbf{Y}|X=0,\theta}}{p_{\mathbf{Y}}}\, \mathrm{d}\mathbf{Y}\, p(\theta)\, \mathrm{d}\theta \\ &+ \sum_{i=1}^{M} P(X=i) \int p_{\mathbf{Y}|X=i,\theta} \log \frac{p_{\mathbf{Y}|X=i,\theta}}{p_{\mathbf{Y}}}\, \mathrm{d}\mathbf{Y}\, p(\theta)\, \mathrm{d}\theta. \end{aligned}$$

Due to the symmetry of the channel, it can be easily verified that the capacity-achieving X is uniformly distributed over the nonzero values, i.e., $P(X_k = i) = \frac{\nu}{M}$ for $1 \leq i \leq M$ where $P(X_k = 0) = 1 - \nu$; and the optimal θ is uniformly distributed on $[-\pi, \pi)$. Note that in this case,

$$\begin{aligned} C_M(\text{SNR}) =& (1-\nu) \int p_{\mathbf{Y}|X=0,\theta} \log \frac{p_{\mathbf{Y}|X=0,\theta}}{p_{\mathbf{Y}}}\, \mathrm{d}\mathbf{Y} \frac{1}{2\pi}\, \mathrm{d}\theta \\ &+ \nu \int p_{\mathbf{Y}|X=1,\theta} \log \frac{p_{\mathbf{Y}|X=1,\theta}}{p_{\mathbf{Y}}}\, \mathrm{d}\mathbf{Y} \frac{1}{2\pi}\, \mathrm{d}\theta \end{aligned}$$

where

$$p_{\mathbf{Y}|X=i,\theta_i} =$$

$$= \begin{cases} \frac{1}{\pi^{M-1}} e^{-\sum_{j\neq i} |Y_j|^2} \frac{1}{\pi(\gamma^2\alpha^2+1)} e^{-\frac{|Y_i - \alpha d e^{j\theta_i}|^2}{\gamma^2\alpha^2+1}} & 1 \leq i \leq M \\ \frac{1}{\pi^M} e^{-\sum_{j=1}^{M} |Y_j|^2} & i = 0 \end{cases}.$$

The capacity expression in (5) is obtained by first integrating with respect to θ and then making a change of variables, $R_j = |Y_j|^2$. □

Not having a closed-form expression, channel capacity (5) has to be numerically evaluated for a given number of orthogonal frequencies M, and SNR. Note that the number of integral expressions increases linearly in M, leading to a higher computational complexity for large M. On the other hand, we reach to a simple closed-form expression of the capacity as M tends to infinity.

Corollary 1: The capacity expression (5) of M-ary OOFSK signaling in the limit as $M \uparrow \infty$ becomes

$$C_\infty(\text{SNR}) = (\gamma^2 + |d|^2)\,\text{SNR} - \nu \log\left(\gamma^2 \frac{\text{SNR}}{\nu} + 1\right). \quad (6)$$

Proof: The method of proof follows primarily from [9] where martingale theory is used to establish a similar result for M-ary FSK signaling over the noncoherent Gaussian channel. The capacity expression in (5) can be rewritten as

$$\begin{aligned} C_M(\text{SNR}) &= (\gamma^2 + |d|^2)\,\text{SNR} - \nu \log\left(\gamma^2 \frac{\text{SNR}}{\nu} + 1\right) \\ &\quad - \int e^{-\sum_{i=1}^{M} R_i}\, \frac{S_M(\mathbf{R})}{M} \log \frac{S_M(\mathbf{R})}{M}\, d\mathbf{R} \end{aligned} \quad (7)$$

where $S_M(\mathbf{R}) = \sum_{i=1}^{M}\left(\nu f(R_i, \text{SNR}) + (1-\nu)\right)$ is a sum of i.i.d. random variables. From [9, Corollary 1], we conclude that $\left\{\chi_M = g\left(\frac{S_M(\mathbf{R})}{M}\right) = \frac{S_M(\mathbf{R})}{M} \log \frac{S_M(\mathbf{R})}{M}\right\}_{M=1}^{\infty}$ is a submartingale, and hence from the martingale convergence theorem [12], χ_M converges to a limit χ_∞ almost surely and in mean. Therefore $\lim_{M\to\infty} E\{\chi_M\} = E\{\lim_{M\to\infty} \chi_M\} = E\{\chi_\infty\}$. Note also that from the strong law of large numbers and continuity of the function $g(x) = x \log x$,

$$\begin{aligned} \lim_{M\to\infty} \chi_M &= \lim_{M\to\infty} g\left(\frac{S_M(\mathbf{R})}{M}\right) \\ &= g\left(\lim_{M\to\infty} \frac{S_M(\mathbf{R})}{M}\right) \\ &= g\left(E_R\{\nu f(R, \text{SNR}) + (1-\nu)\}\right) \\ &= g\left(\int e^{-R}\left(\nu f(R, \text{SNR}) + (1-\nu)\right) dR\right) \\ &= g(1) = 1 \log 1 = 0. \end{aligned}$$

Therefore we have $\lim_{M\to\infty} E_{\mathbf{R}}\left\{\frac{S_M(\mathbf{R})}{M} \log \frac{S_M(\mathbf{R})}{M}\right\} = 0$, which immediately leads to

$$\lim_{M\to\infty} C_M(\text{SNR}) = (\gamma^2 + |d|^2)\,\text{SNR} - \nu \log\left(\gamma^2 \frac{\text{SNR}}{\nu} + 1\right).$$

□

Note that in the absence of any limitations on the peak power, as $\nu \downarrow 0$, $\frac{C_\infty(\text{SNR})}{T} \to (\gamma^2 + |d|^2)\frac{P}{N_0}$ nats/s which is equal to the infinite bandwidth capacity of the AWGN channel with the same received signal power.

IV. Power Efficiency

In this section, we analyze the power efficiency of OOFSK signaling with phase modulation by studying the energy per information bit requirement in the low-SNR regime. In the low-power regime, the spectral-efficiency/bit-energy tradeoff reflects the fundamental tradeoff between bandwidth and power. Assuming that the bandwidth of M-ary OOFSK modulation is $\frac{M}{T}$ where T is the symbol duration, the maximum achievable spectral efficiency is

$$\mathsf{C}\left(\frac{E_b}{N_0}\right) = \frac{1}{M}\, C(\text{SNR}) \quad \text{bits/s/Hz} \quad (8)$$

where $C(\text{SNR})$ is the capacity in bits/symbol, and

$$\frac{E_b}{N_0} = \frac{\text{SNR}}{C(\text{SNR})} \quad (9)$$

is the bit energy normalized to the noise power. For average power limited channels, the bit energy required for reliable communications decreases monotonically with decreasing spectral efficiency, and the minimum bit energy is achieved at zero spectral efficiency, i.e.,

$$\frac{E_b}{N_0}_{\min} = \lim_{\text{SNR}\to 0} \frac{\text{SNR}}{C(\text{SNR})} \log_e 2 = \frac{\log_e 2}{\dot{C}(0)}.$$

Hence for fixed rate transmission, reduction in the required power comes only at the expense of increased bandwidth, and the minimum bit energy is achieved only in the asymptotic regime of infinite bandwidth. If one is willing to spend more power, then reliable communication over a finite bandwidth is possible. Hence achieving the minimum bit energy is not a sufficient criterion for finite bandwidth analysis. Verdú [6] has recently given the following formula for the wideband slope, defined as the slope of the spectral efficiency curve $\mathsf{C}\left(\frac{E_b}{N_0}\right)$ in bits/s/Hz/3dB at zero spectral efficiency:

$$\begin{aligned} S_0 &\overset{\text{def}}{=} \lim_{\frac{E_b}{N_0} \downarrow \frac{E_b}{N_0}\big|_{\mathsf{C}=0}} \frac{\mathsf{C}\left(\frac{E_b}{N_0}\right)}{10\log_{10}\frac{E_b}{N_0} - 10\log_{10}\frac{E_b}{N_0}\big|_{\mathsf{C}=0}}\, 10\log_{10} 2 \\ &= \frac{1}{M} \frac{2\left(\dot{C}(0)\right)^2}{-\ddot{C}(0)}, \end{aligned} \quad (10)$$

where $\dot{C}(0)$ and $\ddot{C}(0)$ denote the first and second derivatives of the capacity in nats. The wideband slope closely approximates the growth of the spectral efficiency curve in the low-power regime and hence is a useful tool providing insightful results when bandwidth is a resource to be conserved.

We first consider the case in which the input peak-to-average power ratio (PAR) is limited, and investigate the spectral-efficiency/bit-energy tradeoff in the low power regime. Note that since OOFSK signaling has average power P, and peak power P/ν, PAR constraint controls the duty cycle parameter ν.

Proposition 2: Assume that the transmitter is constrained to have limited peak to average power ratio, and the PAR of M-ary OOFSK signaling, $1/\nu$, is kept fixed at its maximum level. Then the received bit energy required at zero spectral efficiency and the wideband slope are

$$\left.\frac{E_b^r}{N_0}\right|_{\mathsf{C}=0} = \left(1+\frac{1}{\mathsf{K}}\right)\log_e 2 \text{ and } S_0 = \frac{2\mathsf{K}^2}{(1+\mathsf{K})^2 - \frac{M}{\nu}},$$

respectively, where $\mathsf{K} = \frac{|d|^2}{\gamma^2}$ is the Rician factor.

Proof: Note that in the capacity expression (5), the only term that depends on SNR is $f(R_i, \text{SNR})$. Using the facts $\lim_{x\to 0}\frac{I_1(a\sqrt{x})}{\sqrt{x}} = \frac{a}{2}$ and $\lim_{x\to 0}\frac{I_0(a\sqrt{x})}{x} - \frac{2I_1(a\sqrt{x})}{ax^{3/2}} = \frac{a^2}{8}$, one can easily show that the first and second derivatives with respect to SNR of $f(R_i, \text{SNR})$ at zero SNR are

$$\dot{f}(R_i, 0) = \frac{1}{\nu}(\gamma^2 + |d|^2)(-1+R_i)$$

and

$$\ddot{f}(R_i, 0) = \frac{1}{\nu^2}(|d|^4 + 2\gamma^4 + 4\gamma^2|d|^2)\left(1 - 2R_i + \frac{R_i^2}{2}\right),$$

respectively. Then, differentiating the capacity (5) with respect to SNR we have

$$\dot{C}(0) = |d|^2 \quad \text{and} \quad \ddot{C}(0) = -\frac{(\gamma^2+|d|^2)^2}{M} + \frac{\gamma^4}{\nu}. \qquad (11)$$

The received bit energy required at zero spectral efficiency is obtained from the formula $\left.\frac{E_b^r}{N_0}\right|_{\mathsf{C}=0} = \frac{(\gamma^2+|d|^2)\log_e 2}{\dot{C}(0)}$ and the wideband slope is found by inserting the derivative expressions in (11) into (10). □

In the limited PAR case, in contrast to average power limited systems, the minimum bit energy is not always achieved at zero spectral efficiency. Note that in the Rayleigh fading channel where $\mathsf{K} = 0$, $\left.\frac{E_b^r}{N_0}\right|_{\mathsf{C}=0} = \infty$, and hence operating at very low spectral efficiency (or equivalently at very low SNR) should be avoided. Note also that in the Rician channel, if $(1+\mathsf{K})^2 < \frac{M}{\nu}$, then the wideband slope is negative, and the minimum bit energy is achieved at a nonzero spectral efficiency. Figure 1 plots the bit energy curves as a function of spectral efficiency in bits/s/Hz for 2-FSK signaling with uniform phase ($\nu = 1$). Note that for $\mathsf{K} = 0.25$, the wideband slope is negative, and hence the minimum bit energy is achieved at a nonzero spectral efficiency. On the other hand for $K = 0.5, 1, 2$, the wideband slope is positive, and hence higher power efficiency is achieved as one operates at lower spectral efficiency. Fig. 2 plots the bit energy curves for 2-OOFSK signaling with uniform phase and different duty cycle parameters over the Rician channel with $\mathsf{K} = 1$. We observe that the required minimum bit energy is decreasing with decreasing duty cycle. For instance, when $\nu = 0.01$, the minimum bit energy of ~ 0.46dB is achieved at the cost of a peak-to-average ratio of 100. Note also that since the received bit energy at zero spectral efficiency (2) depends only on the Rician factor K, all the curves in Fig. 2 meet at the same point on the y-axis.

We also consider the case in which the transmitter is subject to a fixed peak power limit and there is no constraint on the peak-to-average ratio. The peak power is kept fixed at $P/\nu =$

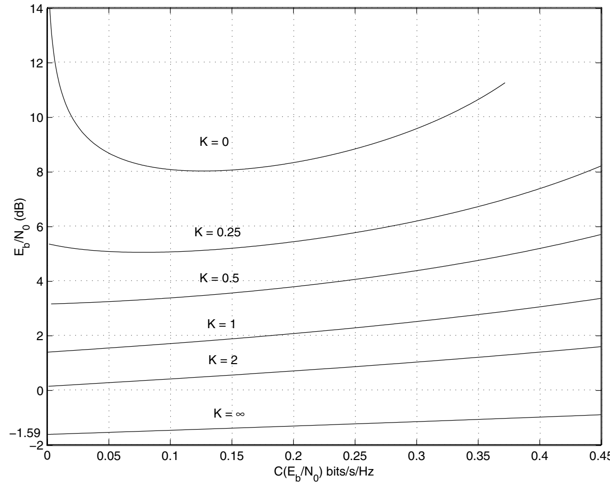

Fig. 1. $\frac{E_b}{N_0}$ (dB) vs. Spectral Efficiency $C(\frac{E_b}{N_0})$ bits/s/Hz for the Rayleigh channel ($\mathsf{K} = 0$), Rician channels ($\mathsf{K} = 0.25, 0.5, 1, 2$), and the unfaded Gaussian channel ($\mathsf{K} = \infty$) when $M = 2$ and $\nu = 1$.

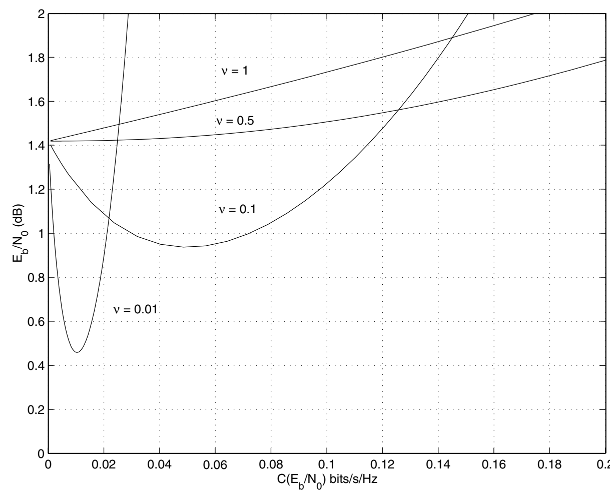

Fig. 2. $\frac{E_b}{N_0}$ (dB) vs. Spectral Efficiency $C(\frac{E_b}{N_0})$ bits/s/Hz for the Rician channels with $\mathsf{K} = 1$ for $\nu = 1, 0.5, 0.1, 0.01$ when $M = 2$.

A, and hence as P decreases to zero, the duty factor has to vanish as well increasing the PAR.

Proposition 3: Assume that the transmitter is limited in peak power, $\frac{P}{\nu} \leq A$, and the symbol duration T is fixed. Then the minimum received bit energy, which is achieved at zero spectral efficiency, and the wideband slope are

$$\frac{E_b^r}{N_{0\,\min}} = \frac{\log_e 2}{1 - \frac{\log(\gamma^2\eta+1)}{(\gamma^2+|d|^2)\eta}} \qquad (12)$$

and

$$S_0 = \begin{cases} \frac{2\left(\eta(\gamma^2+|d|^2) - \log(\eta\gamma^2+1)\right)^2}{\frac{1}{1-\eta^2\gamma^4}\exp\left(\frac{2\eta^2\gamma^2|d|^2}{1-\eta^2\gamma^4}\right) I_0\left(\frac{2\eta|d|^2}{1-\eta^2\gamma^4}\right) - 1} & \eta\gamma^2 < 1 \\ 0 & \eta\gamma^2 \geq 1 \end{cases} \qquad (13)$$

respectively, where $\eta = A\frac{T}{N_0}$ is the normalized peak power.
Proof: When we fix the peak power $A = \frac{P}{v}$, we have $v = \frac{\text{SNR}}{\eta}$ and the capacity becomes

$$C_M(\text{SNR}) = -M - \frac{\text{SNR}}{\eta}\log(\gamma^2\eta + 1)$$
$$-\left(1 - \frac{\text{SNR}}{\eta}\right)\int p_{\mathbf{R}|X=0}\log p_{\mathbf{R}}\,\mathrm{d}\mathbf{R} - \frac{\text{SNR}}{\eta}\int p_{\mathbf{R}|X=1}\log p_{\mathbf{R}}\,\mathrm{d}\mathbf{R}.$$

In the above capacity expression

$$p_{\mathbf{R}} = \left(1 - \frac{\text{SNR}}{\eta}\right)p_{\mathbf{R}|X=0} + \frac{\text{SNR}}{\eta}\sum_{i=1}^{M} p_{\mathbf{R}|X=i}$$

where $p_{\mathbf{R}|X=0}$ and $p_{\mathbf{R}|X=i}$ for $1 \le i \le M$ do not depend on SNR because the ratio $\frac{\text{SNR}}{\nu} = \eta$ is a constant. Due to concavity of the capacity curve, which follows from the concavity of $-x\log x$ and the fact that $p_{\mathbf{R}}$ is a linear function of SNR, the minimum bit energy is achieved at zero spectral efficiency. Differentiating the capacity with respect to SNR, we get

$$\dot{C}(0) = \gamma^2 + |d|^2 - \frac{\log\gamma^2\eta + 1}{\eta},$$

and

$$\ddot{C}(0) = \begin{cases} \dfrac{1 - \frac{1}{1-\eta^2\gamma^4}\exp\left(\frac{2\eta^2\gamma^2|d|^2}{1-\eta^2\gamma^4}\right) I_0\left(\frac{2\eta|d|^2}{1-\eta^2\gamma^4}\right)}{\eta^2 M} & \eta\gamma^2 < 1 \\ -\infty & \eta\gamma^2 \ge 1. \end{cases}$$

Then, (12) and (13) are easily obtained using aforementioned formulas for the minimum bit energy and the wideband slope. □

In this case, power efficiency improves as one operates at lower SNR values. However, note that if $\eta\gamma^2 \ge 1$, approaching the minimum bit energy is very slow. Note also that as the peak limit is relaxed, i.e., $\eta \uparrow \infty$, $\frac{E_b^r}{N_0}_{\min} \to -1.59$ dB.

Fig. 3 plots the bit energy curves as a function of spectral efficiency for the Rayleigh channel ($\mathsf{K} = 0$), Rician channels ($\mathsf{K} = 0.25, 0.5, 1, 2$), and the unfaded Gaussian channel ($\mathsf{K} = \infty$) when normalized peak power limit is $\eta = 1$. We observe that for all cases the required bit energy decreases with decreasing spectral efficiency, and therefore the minimum bit energy is achieved at zero spectral efficiency.

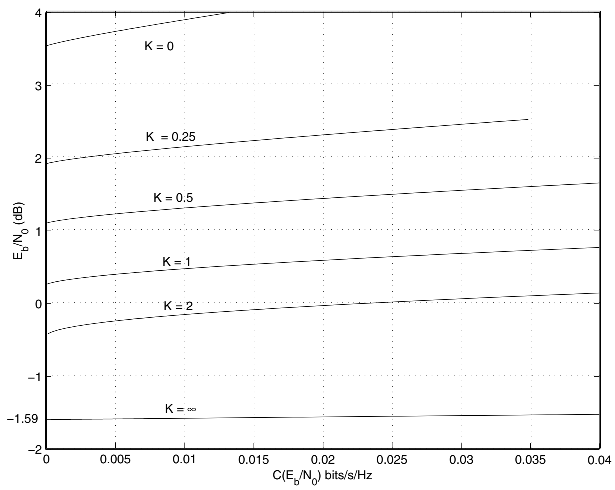

Fig. 3. $\frac{E_b}{N_0}$ (dB) vs. Spectral Efficiency $C(\frac{E_b}{N_0})$ bits/s/Hz for the Rayleigh channel ($\mathsf{K} = 0$), Rician channels ($\mathsf{K} = 0.25, 0.5, 1, 2$), and the unfaded Gaussian channel ($\mathsf{K} = \infty$) when $M = 2$ and fixed peak limit $\eta = 1$.

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