

# Scaling Laws in Random Heterogeneous Networks

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## I. INTRODUCTION

We analyze the effect that adding a wired infrastructure has on the transport capacity of a wireless network. The study of wireless ad-hoc networks was initiated by Gupta and Kumar in [1], where achievable scaling laws were demonstrated. A follow-up work by Kulkarni and Viswanath [3] gave a simple protocol that achieved the scaling laws of [1] while eliminating much of the complexity of the original analysis. Originally developed to address the problem of packet routing on a square grid of parallel processors [2],[5] the protocol in [3] is straightforward and deterministic. The inclusion of a cellular-like wired infrastructure into the ad-hoc wireless network was accomplished in [4]. Independent analysis yielding similar results can also be found in [6]. We build on the work in [3] and [4] by substituting a random wired point-to-point infrastructure on top of a wireless ad-hoc network in contrast to the cellular architecture used in [4].

## II. A SQUARE GRID WITH SHORTCUTS

Let us start with a square grid to which we randomly add shortcuts, i.e. direct connections between nodes. A shortcut of length  $k > 1$  is independently added to the grid with probability  $\frac{\phi}{k^p}$  where the length of a shortcut is the number of hops it takes to travel between its endpoints using the edges of the square grid. Here  $0 < \phi < 1$  and  $p \geq 0$  are parameters of our network. By varying  $p$ , we can discount longer shortcuts which otherwise dominate. This allows us to control the expected number of shortcuts per node of the square grid. Specifically, the expected number of shortcuts of length no more than  $R$  originating at any node is given by

$$\sum_{k=2}^R 4k \frac{\phi}{k^p} = \begin{cases} O(1) & \text{if } p > 2 \\ O(\log R) & \text{if } p = 2 \\ O(R^{2-p}) & \text{if } p < 2 \end{cases} \quad (1)$$

Let  $l(a, b)$  be the number of hops required to get from node  $a$  to node  $b$  utilizing at most one shortcut and let  $D$  be the grid distance between  $a$  and  $b$ . Additionally, let  $\{\alpha_n\}$  be any sequence such that  $\alpha_n \rightarrow \infty$ . Then we have the following

**Theorem 1** Fix  $p \geq 0$ . Then for any  $0 < \phi < 1$ ,

$$\mathbb{E}[l(a, b)] < D^{\min(\frac{p}{4}, 1)} \quad (2)$$

$$\mathbb{P} \left[ l(a, b) \geq \alpha_D D^{\min(\frac{p}{4}, 1)} \text{ i. o.} \right] = 0 \quad (3)$$

$$\forall \epsilon > 0 \quad \mathbb{P} \left[ \limsup_{D \rightarrow \infty} \frac{l(a, b)}{\sqrt[4]{\log(D) D^{\min(\frac{p}{4}, 1)}}} > \epsilon \right] \rightarrow 0 \quad (4)$$

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## III. A SCALING LAW WITH WIRED SHORTCUTS

Consider a network where the number of nodes,  $n$ , within the  $s_n$  squarelets are exchangeable random variables. Let  $c_n$  be the crowding factor as defined in [3]. Then

**Theorem 2**

i)

$$\mathbb{P} \left[ T_n < \min \left[ n s_n^{\min(p/4, 1)}, s_n^{-2} \right] / \alpha_n c_n \text{ i. o.} \right] = 0 \quad (5)$$

where  $T_n$  is the throughput in bit-meters per second (and in bits per second) in the resulting heterogeneous network

ii) For any wired link  $\lambda$ , let  $C_\lambda$  be the required capacity of that link

$$\mathbb{P} \left[ \max_{\lambda} C_{\lambda} > \alpha_n n s_n^2 c_n \text{ i. o.} \right] = 0 \quad (6)$$

We note, in contrast to [4], that in the majority of cases the transport capacity in our network is determined by the time it takes for packets to traverse the wireless network and not the access/exit queue wait times in the wired infrastructure. This is in fact the primary cause of our ability to improve upon the results in [4].

Let  $W_n$  be the number of point-to-point wired links. Then the network of [4] can deliver at best

$$\Theta(\sqrt{W_n} + \sqrt{n}) \quad (7)$$

growth in transport capacity. In fact, when the average number of links per node grows as  $O(1)$  with the number of nodes, [4] is only able to “break even” in terms of the usefulness of the wired infrastructure. For  $p < 4$ , our approach improves the transport capacity by replacing  $\sqrt{W_n}$  in (7) with  $W^{1-p/8}$ . Moreover, with  $p$  in the range between 2 and 4, the expected number of wired links per node is  $O(1)$ .

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