Sensitivity of Gaussian Channel Capacity and Rate-Distortion Function to nonGaussian Contamination

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Abstract — **In some applications, channel noise is the sum of a Gaussian noise and a relatively weak non-Gaussian contaminating noise. Although the capacity of such channels cannot be evaluated in general, we analyze the decrease in capacity, or sensitivity of the channel capacity to the weak contaminating noise. We show that for a very large class of contaminating noise processes, explicit expressions for the sensitivity of a discrete-time channel capacity do exist. Sensitivity is shown to depend on the contaminating process distribution only through its autocorrelation function and so it coincides with the sensitivity with respect to a Gaussian contaminating noise with the same autocorrelation function. A key result is a formula for the derivative of the water-filling capacity with respect to the contaminating noise power.**

Parallel results for the sensitivity of rate-distortion function relative to a mean-square-error criterion of almost Gaussian processes are obtained.

I. SENSITIVITY OF CHANNEL CAPACITY

Consider a discrete-time stationary channel:

$$Y_j = X_j + N_j + \theta Z_j \tag{1}$$

We assume that the random sequences $X = \{X_j\}$, $N = \{N_j\}$ and $Z = \{Z_j\}$ are second-order and mutually independent. The nominal noise N is Gaussian, $\mathbf{E}N_j = \mathbf{E}Z_j = 0$, $\mathbf{E}N_j^2 = \sigma^2$, $\mathbf{E}Z_j^2 = 1$. Denote by $C_P(\theta)$ the capacity of channel (1) under the assumption that the input power is constrained to some fixed constant P. The sensitivity of channel capacity with respect to the contaminating noise power is defined as

$$\mathbf{S}_P = \lim_{\theta \to 0} \frac{C_P(0) - C_P(\theta)}{\theta^2} \tag{2}$$

II. GAUSSIAN CONTAMINATION

If the contaminating process $\{Z_i\}$ is Gaussian, then the capacity of (1) admits the well-known water-filling solution

$$C(\theta) = \frac{1}{2} \int_{-1/2}^{1/2} \ln \left(1 + \frac{[K_\theta - N_0(f) - \theta^2 Z(f)]^+}{N_0(f) + \theta^2 Z(f)} \right) df , \tag{3}$$

where $N_0(f)$ and $Z(f)$ are the power spectral densities of the nominal and contaminating noises, respectively, and the water level K_θ is adjusted so that the integral of the optimum input power spectral density $S_\theta(f)$ is equal to P, where $S_\theta(f)$ is the numerator in (3).

We show in this paper that the sensitivity of the water-filling channel capacity formula admits the following simple expression:

$$\mathbf{S}_P = \frac{1}{2K_0} \int_{-1/2}^{1/2} Z(f) \frac{S_0(f)}{N_0(f)} df , \tag{4}$$

where K_0 is the nominal water level. It follows that the sensitivity is maximized by a contaminating random process which concentrates its power at those frequencies where the nominal noise spectral density is minimum. Note that the worst-case sensitivity is minimized over the nominal noise spectral density by white noise, in which case the sensitivity is equal to

$$\mathbf{S} = \frac{P}{2\sigma^2} \frac{1}{P + \sigma^2} , \tag{5}$$

regardless of the power spectral density of the contaminating process.

III. NONGAUSSIAN CONTAMINATION

Since Gaussian noise minimizes capacity for a given power spectral density, the expression in (4) is an upper bound to sensitivity for nonGaussian contamination. Despite the lack of an expression for $C(\theta)$ in the nonGaussian case, this paper shows that

- The sensitivity is equal to (5) if the nominal Gaussian noise is white and the contaminating noise is regular (cf. [2]).
- The sensitivity is equal to (4) if both the nominal and contaminating noises are regular and if the ratio of spectral densities of contaminating to nominal noises: $Z(f)/N_0(f)$ is bounded on $\left[0, \frac{1}{2}\right]$.
- The sensitivity is equal to 0 if the nominal noise is regular and the contaminating noise is entropy-singular.

IV. RATE-DISTORTION FUNCTION

Consider the random process $N + \theta Z$ and denote by $R_D(\theta)$ its rate-distortion function relative to the mean-square-error criterion. We have shown (under the same conditions as above) that if $D \leq \sigma^2$, then the sensitivity of the rate-distortion function is

$$\lim_{\theta \to 0} \frac{R_D(\theta) - R_D(0)}{\theta^2} = \int_0^{1/2} \frac{Z(f)}{\max\{\lambda_0, N_0(f)\}} df \tag{6}$$

where λ_0 is defined by the equation

$$2 \int_0^{1/2} \min\{\lambda_0, N_0(f)\} \, df = D. \tag{7}$$

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