Sensitivity of the Rate-Distortion Function of Stationary Continuous-Time Gaussian Processes to non-Gaussian Contamination[1]

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Abstract — **Let $S(N,Z)$ be the sensitivity (cf. [1,2]) of the rate-distortion function (relative to the mean-square-error criterion) of a continuous-time stationary Gaussian process $N = N(t)$ to non-Gaussian contamination $Z = Z(t)$. We prove that for any entropy-regular process Z the equality $S(N,Z) = S(N,Z^*)$ holds, where $Z^* = Z^*(t)$ is a stationary Gaussian process with the same autocorrelation function as Z. An explicit expression for $S(N,Z)$ in the terms of the spectral densities of N and Z is also derived. We also prove that $S(N,Z) = 0$ for any entropy-singular process Z. Similar results for discrete-time processes have been obtained in [1,2].**

Let $R_D(X)$ be the rate-distortion function of a continuous-time stationary process $X = X(t)$ relative to the mean-square-error criterion. If X is Gaussian, then the rate-distortion function $R_D(X)$ admits the well-known water-filling solution

$$R_D(X) = \frac{1}{2}\int_{-\infty}^{\infty} \ln \max\left\{1, \frac{f_X(\lambda)}{\rho}\right\} d\lambda, \tag{1}$$

where $f_X(\lambda)$ is the spectral density of X, and ρ is defined by the equation

$$\int_{-\infty}^{\infty} \min\{\rho, f_X(\lambda)\}\, d\lambda = D, \tag{2}$$

if $0 < D < \sigma^2$, and $R_D(X) = 0$ if $D \geq \sigma^2 > 0$, where $\sigma^2 = \mathrm{var}X(t)$.

However, if $X = X(t)$ is non-Gaussian, the problem of explicit evaluation of $R_D(X)$ is rather difficult. Therefore, it is of interest to investigate the asymptotic behavior of $R_D(N + \theta Z)$ as $\theta \to 0$, where $N = N(t)$ is a Gaussian process and $Z = Z(t)$ is an arbitrary, independent of N, stationary continuous-time process, respectively.

To formulate our main result we need the following definitions. A stationary continuous-time process $X = X(t)$ is called entropy-regular if for any $-\infty < \Delta < \infty$, the discrete-time process $X_\Delta(k) = X(k\Delta), k = 0, \pm 1, \dots$ is entropy-regular (or a process with completely positive entropy) (cf. [2]). Similarly, a stationary continuous-time process $X = X(t)$ is called entropy-singular if for any $-\infty < \Delta < \infty$, the discrete-time process $X_\Delta(k) = X(k\Delta), k = 0, \pm 1, \dots$ is entropy-singular [3]. Some properties of entropy-regular and entropy-singular processes were pointed out in [4].

Theorem: *Let $N = N(t)$ and $Z = Z(t)$ be arbitrary independent second-order continuous-time stationary processes, and let N be an entropy-regular Gaussian process. If Z is an entropy-regular process then the equality*

$$S(N,Z) = S(N,Z^*) \tag{3}$$

holds. Moreover, in this case the sensitivity admits the expression (for any D, $0 < D < \sigma^2$)

$$S(N,Z) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{f_Z(\lambda)}{\max\{\rho, f_N(\lambda)\}}\, d\lambda, \tag{4}$$

where ρ is defined by (2). If Z is an entropy-singular process, then for any D and θ the equality

$$R_D(N + \theta Z) = R_D(N) \tag{5}$$

holds, and, in particular, $S(N,Z) = 0$.

Similar results for discrete-time processes have been obtained in [1, 2]. It should be noted that we can not directly apply the discrete-time results, mentioned above, to prove the continuous-time counterparts because of the following reason. In our proof we essentially use the formula

$$\overline{I}(Z^*; N + Z^*) = \frac{1}{2}\int_{-\infty}^{\infty} \ln\left[1 + \frac{f_Z(\lambda)}{f_N(\lambda)}\right] d\lambda \tag{6}$$

(where $\overline{I}(\cdot,\cdot)$ is the information per unit time) which has been proved only under some additional conditions on spectral densities of $N(t)$ and $Z(t)$ in contrast with discrete-time case where the analog of (6) is always valid if N is nonsingular process. Some sufficient conditions for (6) are known. For example, it holds if $f_Z(\lambda) = R(\lambda)\Psi(\lambda)$, where $R(\lambda)$ is a rational function and $\Psi(\lambda)$ is a mesurable function, such that $0 \leq \Psi(\lambda) \leq 1$ for all $\lambda \in (-\infty, \infty)$ and $\int_{-\infty}^{\infty} |\ln \Psi(\lambda)|\, d\lambda < \infty$. To prove the Theorem in this paper we have a rather nontrivial proof that avoids some very strong additional restrictions on N and Z.

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