The Impact of Frequency-Flat Fading on the Spectral Efficiency of CDMA

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***Abstract*—The capacity of the randomly spread synchronous code-division multiple-access (CDMA) channel subject to frequency-flat fading is studied in the wide-band limit of large number of users. We find the spectral efficiency as a function of the number of users per chip, the distribution of the flat fading, and the signal-to-noise ratio (SNR), for the optimum receiver as well as linear receivers (single-user matched filter, decorrelator, and minimum mean-square errror (MMSE)). The potential improvements due to both decentralized transmitter power control and multiantenna receivers are also analyzed.**

***Index Terms*—Antenna diversity, channel capacity, code-division multiple access (CDMA), fading channels, Gaussian channels, multiuser detection, multiuser information theory, power control, spread spectrum**

I. INTRODUCTION

RECENT results on the asymptotic distribution of the singular values of certain random matrices have enabled the study of the asymptotic limits of randomly-spread code-division multiple-access (CDMA) channels [1]–[3]. In those multiple-access channels, signature waveforms are assigned randomly and independently to the users. The analysis applies to both pseudonoise sequences spanning many symbol periods as well as randomly selected signature waveforms that remain fixed from symbol to symbol as in the basic CDMA model [4]. Since it depends on the correlations between the signature waveforms, the capacity of the randomly spread basic CDMA channel model is random. However, as the number of users grows, the capacity converges to a deterministic limit.

In [1] we analyzed the total capacity (rate-sum) per chip, or spectral efficiency in bits per second per hertz (b/s/Hz), of the randomly spread synchronous CDMA channel when all the users are received with identical powers in the presence of additive white Gaussian noise. In that simple model, the spectral efficiency is a function of the energy-per-bit relative to one-sided noise spectral level $\frac{E_b}{N_0}$ and the number of users per chip denoted by

$$\beta = \frac{K}{N} \tag{1}$$

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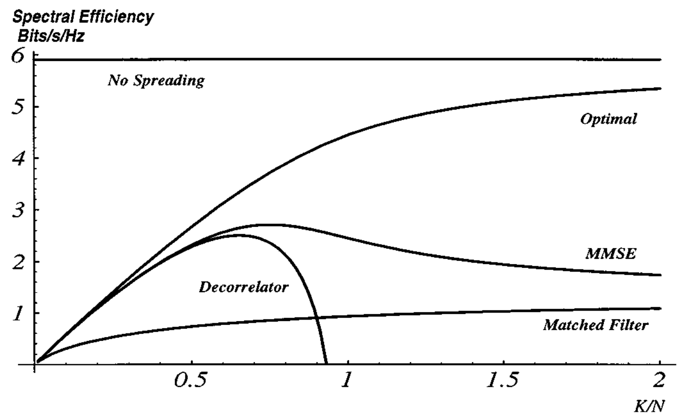

Fig. 1. Spectral efficiencies for $\frac{E_b}{N_0} = 10$ dB without fading.

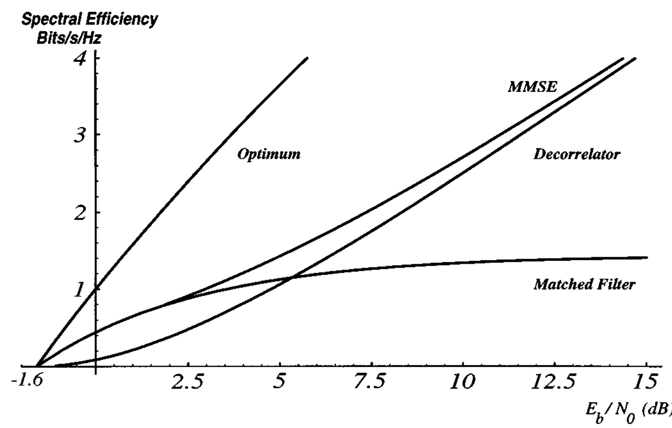

Fig. 2. Large-K spectral efficiencies with optimum K/N without fading.

where K and N are the number of users and the spreading factor (number of chips per symbol), respectively. The major findings of [1] are illustrated in Figs. 1 and 2.[1]

- For low β, linear multiuser detectors (such as the decorrelator and the minimum mean-square error (MMSE) [4]) achieve near-optimum spectral efficiency. Otherwise, important gains in efficiency can be realized by nonlinear multiuser receivers.
- Randomly spread CDMA incurs negligible spectral efficiency loss relative to no-spreading if an optimum receiver is used and β is sufficiently high.

[1]For comparison with the other figures in this paper, we reproduce [1, Figs 1 and 2] with spectral efficiency measured in b/s/Hz which corresponds to twice the bits per chip (b/chip) of the real-valued model analyzed in [1] (cf. [1, Footnote 1]).

- Regardless of β and $\frac{E_b}{N_0}$, the single-user matched filter is far from optimal. In particular, even at $\frac{E_b}{N_0} \downarrow -1.59175$ dB it achieves only half of the available capacity.
- Unlike the single-user matched filter, the spectral efficiency of linear and nonlinear multiuser detectors grows without bound with $\frac{E_b}{N_0}$.

How do those conclusions change in the presence of frequency-flat fading? The complex-valued channel model we analyze is

$$\mathsf{y}(t) = \sum_{k=1}^{K} \sum_{i=1}^{n} \mathsf{A}_k[i] b_k[i] \mathsf{s}_k(t - iT) + \mathsf{n}(t) \tag{2}$$

where the real and imaginary components of the complex random process n are independent white Gaussian processes each with double-sided power spectral density equal to $1/2$. The signature waveform of the kth user, s_k, is normalized to have unit energy. The codewords sent by the kth user satisfy the power constraint

$$\frac{1}{n} \sum_{i=1}^{n} b_k^2[i] \leq \mathsf{SNR}. \tag{3}$$

Thus, SNR is the per-symbol signal-to-noise ratio (SNR) constraint common to all K users. The complex-valued fading parameters $\mathsf{A}_k[i]$ are independent from user to user and follow a common distribution (for all i and k) such that[2]

$$E[|\mathsf{A}_k[i]|^2] = 1. \tag{4}$$

We will assume throughout that the receiver knows

$$\{\mathsf{A}_k[i],\, i = 1, \ldots, n;\, k = 1, \ldots, K\}.$$

Denote the unit-norm signature of the kth user by

$$\boldsymbol{s}_k = [s_{1k}, \ldots, s_{Nk}]^T \tag{5}$$

and assume that[3] $s_{jk} \in \{-1/\sqrt{N}, +1/\sqrt{N}\}$ are chosen equally likely and independently for all (k, j). As usual [4], the $N \times K$ matrix $\boldsymbol{S}$ denotes the matrix of signature vectors, whose kth column is $\boldsymbol{s}_k$. The signature waveform is then given by

$$\mathsf{s}_k(t) = \sum_{i=1}^{N} s_{ik} \psi_i(t) \tag{6}$$

where $\{\psi_i(t),\, i = 1, \ldots, N\}$ are orthonormal waveforms. Projecting (2) along those orthonormal waveforms we get the N-dimensional multiaccess channel model [4, eq. (2.97)]

$$\boldsymbol{r} = \boldsymbol{S}\boldsymbol{A}\boldsymbol{b} + \boldsymbol{m} \tag{7}$$

where the K channel inputs form the vector

$$\boldsymbol{b} = [b_1, \ldots, b_K]^T \tag{8}$$

[2]The identical-distribution assumption on the fading coefficients can be eliminated as we will see below.

[3]The same spectral efficiency results are obtained for nonbinary (e.g., Gaussian) sequences.

the diagonal matrix of received fading coefficients is denoted by

$$\boldsymbol{A} = \mathrm{diag}\{\mathsf{A}_1, \ldots, \mathsf{A}_K\} \tag{9}$$

and $\boldsymbol{m}$ is a complex Gaussian vector all whose real and imaginary components have variance $1/2$ and are independent.

A system that achieves total capacity $\mathcal{C}(\beta, \mathsf{SNR})$ given in bits per degree of freedom (b/s/Hz), has an energy per bit per noise level equal to

$$\frac{E_b}{N_0} = \frac{\beta \mathsf{SNR}}{\mathcal{C}(\beta, \mathsf{SNR})} \tag{10}$$

because $\mathcal{C}(\beta, \mathsf{SNR})$ is the overall number of bits transmitted per chip, and the energy each user spends per symbol relative to a noise spectral level $N_0 = 1$ is equal to SNR. Note that (10) differs from the corresponding relationship in [1] by a factor of 2 because the real-valued channel version was considered in [1].

Previous work on the capacity of multiple-access channels subject to frequency-flat fading is surveyed in [5]. Several studies addressed randomly spread direct-sequence CDMA (DS-CDMA) fading channels similar to that in the present paper. In [1], the homogeneous fading process, having dynamics in scales of chip durations (rather than coded symbols as is the case here) is considered and it is concluded that homogeneous fading has no effect on spectral efficiency.

Recently, the analysis of DS-CDMA systems with random signature assignments has garnered much attention. Furthermore, the model is isomorphic to single-user multiantenna systems fed by independent data streams (where the number of transmitting and receiving elements take the role of K and N, respectively) [6]–[8]. A fundamental result is the Tse–Hanly equation [2] whose solution is the output SNR of the MMSE multiuser detector for an asymptotically large system. In the present paper, we investigate the analytical properties of the solution to the Tse–Hanly equation, and we show that it plays a key role in the evaluation of the capacity of optimum (nonlinear) receivers. Extending the results of [2], the effect of fading, and, in particular, channel estimation error, on the MMSE receiver SNR is examined in [9]. Other relevant extensions of [2] are pursued in [10]–[12]. Results related to admissibility associated with a "quality of service" per user in the presence of fading are obtained in [13]. Optimal centralized power control which maximizes the total capacity as a function of all fading coefficients is studied in [14] where it is shown that the optimal law does not depend on the realization of the signature sequences and is decentralized in nature. A water-filling strategy, controlled by the instantaneous fade level experienced by each user, is asymptotically ($K \to \infty$) optimal.

In Section II, we define the key quantities of interest in the context of the single-user channel, for which we obtain several new observations, such as unveiling the role of the kurtosis of the fading distribution and the infinite capacity per joule of the optimum power controlled channel (even if the fadel levels are only partially known). In Section III, we examine the spectral efficiency (and outage probability) achievable when the front-end of the receiver consists of linear processors (the single-user matched filter, the decorrelator, and the MMSE receiver) followed by a single-user decoder.

TABLE I
FADING DISTRIBUTIONS AND THEIR KURTOSIS

Fading	$f_R(r), r \geq 0$	$E[R^2]$	$\kappa(R) = \kappa(\lvert\mathsf{A}\rvert)$
Deterministic	$\delta(r-1)$	1	1
Rayleigh	$re^{-r^2/2}$	2	2
Rice	$re^{-\frac{r^2+d^2}{2}} I_0(rd)$	$2+d^2$	$2 - \frac{d^4}{(2+d^2)^2}$
Nakagami	$c(d) r^{2d-1} e^{-dr^2/2}$	$2 + \frac{2}{d}$	$1 + \frac{1}{d}$
Log-normal	$\frac{20 \log_{10} e}{r \sigma_P \sqrt{2\pi}} \exp\left(-\frac{(20 \log_{10} r)^2}{2\sigma_P^2}\right)$	$10^{\sigma_P^2 (\log_e 10)/200}$	$10^{\sigma_P^2 (\log_e 10)/100}$

The main result of the paper is given in Section IV, where we show that the increase in spectral efficiency afforded by nonlinear processing can be characterized compactly in terms of the optimum linear performance. Tightly framing achievable performance, particular emphasis throughout our spectral efficiency analysis is placed on the regions of low and high $\frac{E_b}{N_0}$. Section V generalizes the main results in Sections III and IV to the multiantenna case. Section VI summarizes the main findings of this paper. Appendix F briefly addresses a different performance measure, i.e., outage probability which is particularly relevant to slow-fading channels.

II. SINGLE-USER CHANNEL

In this section, we address the spectral efficiency of the single-user channel ($K = 1$) without spreading ($N = 1$). Assuming that the fading coefficients are ergodic and unknown to the transmitter, the single-user capacity is given by (e.g., [5])

$$\mathcal{C}(\mathsf{SNR}) = E[\log_2(1 + \mathsf{SNR}|\mathsf{A}|^2)] \tag{11}$$

where the expectation is with respect to the random variable $|\mathsf{A}|$, which is assumed throughout the paper to be normalized

$$E[|\mathsf{A}|^2] = 1. \tag{12}$$

Because of Jensen's inequality, (11) lies below the unfaded capacity $\log_2(1 + \mathsf{SNR})$.

To obtain the spectral efficiency as a function of $\frac{E_b}{N_0}$, (11) is used in conjunction with (10) with $\beta = 1$, since in this section we are assuming $N = 1$ and $K = 1$.

In the very noisy region, it is easy to see that the $\frac{E_b}{N_0}_{\min}$ required for reliable communication does not change with fading, because (cf. [15])

$$\begin{aligned} \frac{E_b}{N_0}_{\min} &= \lim_{\mathsf{SNR} \downarrow 0} \frac{\mathsf{SNR}}{\mathcal{C}(\mathsf{SNR})} && (13)\\ &= \log_e 2 / E[|\mathsf{A}|^2] && (14)\\ &= \log_e 2 && (15)\\ &= -1.59175 \text{ dB}. && (16) \end{aligned}$$

The reason for the insensitivity of $\frac{E_b}{N_0}_{\min}$ to the fading distribution lies in the fact that to achieve minimum energy per transmitted bit, ON–OFF signaling is optimum (e.g., [15]–[17]), and the optimum receiver averages the received signal in the on-periods of each of the possible codewords and weighs each matched filter output by its corresponding fading coefficient. Thus, performance is a function of the second moment of the fading coefficients.

At $\frac{E_b}{N_0}_{\min}$, the slope of the spectral efficiency versus $\frac{E_b}{N_0}$ in b/s/Hz per 3 dB is given by [18]

$$\begin{aligned} \mathcal{S}_0 &\stackrel{\text{def}}{=} \lim_{\frac{E_b}{N_0} \downarrow \frac{E_b}{N_0}_{\min}} \frac{\mathcal{C}\left(\frac{E_b}{N_0}\right)}{10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_0}_{\min}} 10 \log_{10} 2 && (17)\\ &= \frac{2 \dot{\mathcal{C}}(0)^2}{-\ddot{\mathcal{C}}(0)} && (18) \end{aligned}$$

with $\dot{\mathcal{C}}$ and $\ddot{\mathcal{C}}$, the first and second derivative, respectively, of the function $\mathcal{C}(\mathsf{SNR})$ computed in nats.[4] With this notation, we can write the dependence of the $\frac{E_b}{N_0}$ required to obtain spectral efficiency $\mathcal{C}$ as

$$10 \log_{10} \frac{E_b}{N_0}(\mathcal{C}) = 10 \log_{10} \frac{E_b}{N_0}_{\min} + \frac{\mathcal{C}}{\mathcal{S}_0} 10 \log_{10} 2 + o(\mathcal{C}), \quad \mathcal{C} \to 0. \tag{19}$$

Evaluating (18) with (11) we obtain that the spectral efficiency slope at $\frac{E_b}{N_0}_{\min}$ is equal to

$$\mathcal{S}_0 = \frac{2}{\kappa(|\mathsf{A}|)} \tag{20}$$

with the kurtosis of $|\mathsf{A}|$ defined as

$$\kappa(|\mathsf{A}|) = \frac{E[|\mathsf{A}|^4]}{E^2[|\mathsf{A}|^2]}. \tag{21}$$

Table I gives the kurtosis of the major fading distributions. In that table, $|\mathsf{A}| = R/\sqrt{E[R^2]}$. Kurtosis[5] (introduced by Karl Pearson in 1906) is a measure of the "peakedness" of a distribution, namely, the higher the kurtosis the lower the concentration of the density function around its mode. Because the variance of $|\mathsf{A}|^2$ is nonnegative, kurtosis achieves its minimum value (equal to 1) for deterministic $|\mathsf{A}|^2$.

Recall that we should expect that the higher the "randomness" of the normalized fading distribution the higher the Jensen inequality penalty suffered by capacity. Kurtosis emerges as the right measure of "randomness" in the low $\frac{E_b}{N_0}$ regime. Some intuition can be gained by noticing that large channel gains contribute much more to the received power (parabola) than to channel capacity (logarithm). However, since $\frac{E_b}{N_0}_{\min}$ depends exclusively on the second moment (but not on the shape) of the fading distribution, the initial slope of the spectral efficiency achieved by a high kurtosis fading distribution (which puts some mass on large values) is very low. For example, consider a two-mass distribution with very

[4]Although we use the same notation for the function $\mathcal{C}$ as a function of SNR and as a function of $\frac{E_b}{N_0}$, we explicitly denote the argument when relevant.

[5]A related unnormalized measure also called kurtosis in some signal processing works (e.g., [19]) is given by $E[|\mathsf{A}|^4] - 2E^2[|\mathsf{A}|^2] - |E[\mathsf{A}^2]|^2$.

large kurtosis: $|\mathsf{A}|^2 = 0.99$ with probability $1 - 10^{-8}$ and $|\mathsf{A}|^2 = 1,000,000.99$ with probability 10^{-8}. The capacity is very similar if we drastically reduce the kurtosis by eliminating the small mass and we concentrate all the mass at 0.99. The only difference is that now $\frac{E_b}{N_0}_{\min}$ is slightly higher, but the slope at $\frac{E_b}{N_0}_{\min}$ changes from 2×10^{-4} to 1 b/s/Hz/3 dB.

In the high spectral efficiency region, the required $\frac{E_b}{N_0}$ behaves as

$$10 \log_{10} \frac{E_b}{N_0}(\mathcal{C}) = \frac{\mathcal{C}}{\mathcal{S}_\infty} 10 \log_{10} 2 - 10 \log_{10}(\mathcal{C}) + \mathcal{L}_\infty 10 \log_{10} 2 + o(1), \qquad \mathcal{C} \to \infty \tag{22}$$

where $\mathcal{S}_\infty$ is the asymptotic slope of the spectral efficiency in b/s/Hz per 3 dB

$$\mathcal{S}_\infty \stackrel{\text{def}}{=} \lim_{\frac{E_b}{N_0} \to \infty} \frac{\mathcal{C}\left(\frac{E_b}{N_0}\right)}{10 \log_{10} \frac{E_b}{N_0}} 10 \log_{10} 2 \tag{23}$$

$$= \lim_{x \to \infty} x \dot{\mathcal{C}}(x) \tag{24}$$

with $\dot{\mathcal{C}}(x)$ denoting, as above, the derivative of the per-symbol capacity in nats per degree of freedom. The horizontal $\frac{E_b}{N_0}$-penalty is equal to 3 dB times

$$\mathcal{L}_\infty = \lim_{x \to \infty} \left(\log_2 x - \frac{\mathcal{C}(x)}{\mathcal{S}_\infty} \right). \tag{25}$$

For reference purposes, it is useful to note that for the additive white Gaussian noise channel, $\mathcal{C}(x) = \log_2(1 + x)$

$$\frac{E_b}{N_0}(\mathcal{C}) = \frac{2^{\mathcal{C}} - 1}{\mathcal{C}} \tag{26}$$

$$\mathcal{S}_0 = 2 \tag{27}$$

$$\mathcal{S}_\infty = 1 \tag{28}$$

$$\mathcal{L}_\infty = 0. \tag{29}$$

We also note for future use that in the multiuser case, applying (10), we get that (19), (20), (22), and (24) hold verbatim, and (25) is replaced by

$$\mathcal{L}_\infty = \log_2 \beta + \lim_{x \to \infty} \left(\log_2 x - \frac{\mathcal{C}(x)}{\mathcal{S}_\infty} \right). \tag{30}$$

Applying the general formula (24) to (11), we see that the asymptotic growth is given by

$$\mathcal{S}_\infty = P[|\mathsf{A}| > 0]. \tag{31}$$

So, unless the fading distribution has a point mass at 0,[6] the asymptotic slope of the spectral efficiency versus $\frac{E_b}{N_0}$ is not degraded by fading. However, there is indeed a penalty in $\frac{E_b}{N_0}$. According to (25), the $\frac{E_b}{N_0}$-penalty due to fading is given by

$$\mathcal{L}_\infty = -\lim_{x \to \infty} E\left[\log_2 \left(\frac{1}{x} + Z \right) \right] \tag{32}$$

$$= -\frac{E[1\{|\mathsf{A}| > 0\} \log_2 |\mathsf{A}|^2]}{P[|\mathsf{A}| > 0]} \tag{33}$$

where Z denotes the random variable $|\mathsf{A}|^2$ restricted to $(0, \infty)$.

In the particular case of Rayleigh fading, $\sqrt{2}|\mathsf{A}|$ follows the distribution

$$f_R(r) = r e^{-r^2/2}, \qquad r \geq 0 \tag{34}$$

[6]Relevant to ON–OFF transmitting strategies.

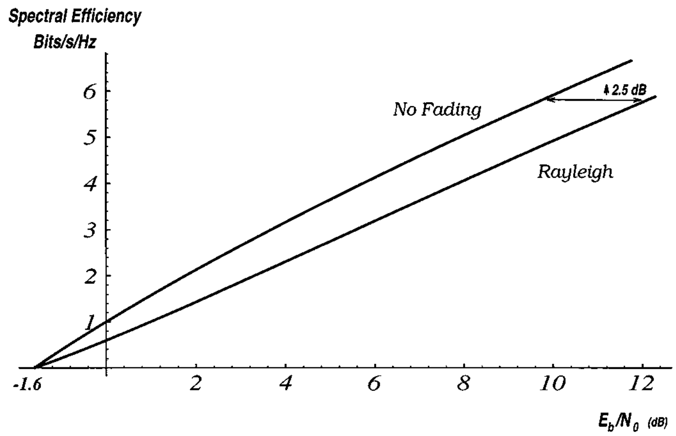

Fig. 3. Single-user spectral efficiency with Rayleigh fading.

or, equivalently, $|\mathsf{A}|^2$ has the exponential density e^{-x}, $x \geq 0$. Expressing (11) in bits we have

$$\mathcal{C}^1(\mathsf{SNR}) = \Lambda(\mathsf{SNR}) \log_2 e \tag{35}$$

where

$$\Lambda(\mathsf{SNR}) \stackrel{\text{def}}{=} \int_0^\infty \log_e(1 + \mathsf{SNR} f) e^{-f}\, df$$

$$= \int_{1/\mathsf{SNR}}^\infty \frac{e^{1/\mathsf{SNR} - v}}{v}\, dv. \tag{36}$$

The decrease in spectral efficiency due to Rayleigh fading is shown in Fig. 3. Since the kurtosis of the Rayleigh distribution is 2, near $\frac{E_b}{N_0}_{\min}$, the spectral efficiency slope is 1/3 b/s/Hz/dB, or half of its value in the unfaded channel. The asymptotic penalty in the large SNR region is obtained by particularizing (33)

$$\mathcal{L}_\infty 10 \log_{10} 2 = -E[\log_2 |\mathsf{A}|^2] 10 \log_{10} 2 \tag{37}$$

$$= -10 \gamma \log_{10} e \tag{38}$$

$$= 2.50682 \text{ dB} \tag{39}$$

where γ is the Euler–Mascheroni constant

$$\gamma = -\int_0^\infty e^{-f} \log_e f\, df \tag{40}$$

$$= \lim_{m \to \infty} \left(\sum_{i=1}^m \frac{1}{i} - \log_e m \right). \tag{41}$$

If the transmitter knows the instantaneous fading coefficients, then it may apply various forms of power control while still satisfying (3). Instead of (2), the receiver observes

$$\mathsf{y}(t) = \sum_{k=1}^K \sum_{i=1}^n \sqrt{p[|\mathsf{A}[i]|^2]}\, \mathsf{A}_k[i] b_k[i] \mathsf{s}_k(t - iT) + \mathsf{n}(t) \tag{42}$$

where the power control law is constrained to satisfy

$$E[p[|\mathsf{A}|^2]] = \mathsf{SNR}. \tag{43}$$

For example, under the *power equalization* strategy (also known as "perfect power control") the instantaneous transmitted power is

$$p[|\mathsf{A}[i]|^2] = \frac{\varrho \mathsf{SNR}}{|\mathsf{A}[i]|^2} \tag{44}$$

where the harmonic mean of $|\mathsf{A}|^2$ is denoted by

$$\varrho = \left(E\left[\frac{1}{|\mathsf{A}|^2} \right] \right)^{-1} \tag{45}$$

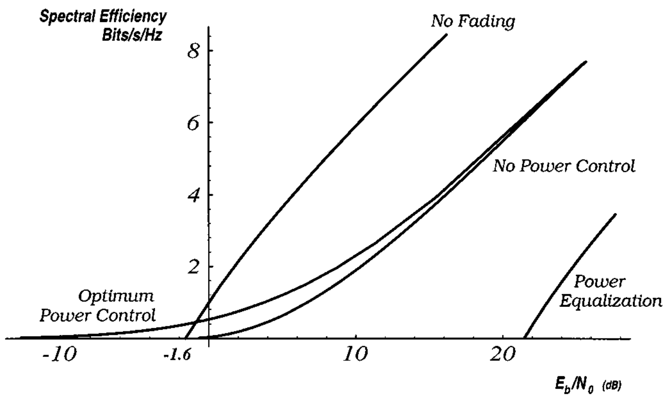

Fig. 4. Spectral efficiencies for single-user channel with log-normal fading $\sigma_P = 10$ dB.

and the ensuing capacity is given by

$$\mathcal{C}(\mathsf{SNR}) = \log_2(1 + \mathsf{SNR}\varrho) \tag{46}$$

which, in addition, has the interpretation of "delay-limited" capacity introduced in [20]. The capacity-maximizing strategy is the *water-filling* law [21]

$$p[|\mathsf{A}[i]|^2] = \left[\frac{1}{\lambda} - \frac{1}{|\mathsf{A}[i]|^2}\right]^+ \tag{47}$$

with $[a]^+ = \max\{0, a\}$ and λ chosen so as to satisfy

$$E\left[\left(\frac{1}{\lambda} - \frac{1}{|\mathsf{A}|^2}\right)^+\right] = \mathsf{SNR}. \tag{48}$$

The resulting capacity with optimum power control is equal to

$$\mathcal{C}(\mathsf{SNR}) = E[\log_2(1 + p(|\mathsf{A}|^2)|\mathsf{A}|^2)] \tag{49}$$

$$= E\left[\left[\log_2\left(\frac{|\mathsf{A}|^2}{\lambda}\right)\right]^+\right]. \tag{50}$$

In the special case of Rayleigh fading, (48) and (50) lead to the following expression for the $\frac{E_b}{N_0}$ required to achieve capacity $\mathcal{C}$ nats/s/Hz:

$$\frac{1}{\log_e 2}\frac{E_b}{N_0} = \frac{\exp(-\varphi(\mathcal{C}))}{\mathcal{C}\varphi(\mathcal{C})} \tag{51}$$

where φ denotes the inverse of the exponential integral function [22]

$$h(\alpha) = \alpha\int_1^\infty \log_e x e^{-\alpha x}\, dx \tag{52}$$

$$= \int_1^\infty \frac{e^{-\alpha x}}{x}\, dx. \tag{53}$$

Fig. 4 shows the spectral efficiencies as a function of $\frac{E_b}{N_0}$ attained with no power control, optimum power control, and power equalization when the fading amplitude follows a *log-normal* distribution

$$f_R(r) = \frac{20\log_{10} e}{r\sigma_P\sqrt{2\pi}} \exp\left(-\frac{(20\log_{10} r)^2}{2\sigma_P^2}\right) \tag{54}$$

suitably scaled (cf. Table I) to have unit second-moment.

We see in Fig. 4 that power equalization is a particularly poor strategy, certainly much worse than no power control at all. Constant average transmitted power (i.e., no power control) is near-optimal for sufficiently high $\frac{E_b}{N_0}$, because in that case water-filling has a minor effect on the instantaneous transmitted levels.

For sufficiently low $\frac{E_b}{N_0}$, we see that fading in conjunction with optimum power control is actually beneficial relative to no fading. In fact, if water-filling is used in the presence of log-normal fading or any other unbounded fading distribution, then reliable communication is achievable at *any* $\frac{E_b}{N_0}$. This is in stark contrast to the behavior in the absence of power control: regardless of whether fading is present, reliable communication is impossible for $\frac{E_b}{N_0} < -1.59175$ dB. Intuitively, one can see that it is possible to transmit any amount of information reliably with as little energy as desired by simply concentrating all the available transmitter energy at those rare moments in which the fading level is exceedingly large. In general (cf. Appendix I)

$$10\log_{10}\frac{E_b}{N_0}_{\min} = -1.59175 - 20\log_{10}|\mathsf{A}|_{\max} \tag{55}$$

where $|\mathsf{A}|_{\max}$ is the essential supremum of the random variable $|\mathsf{A}|$.

Thus, if the transmitter uses power control, fading is actually beneficial at low SNRs. Note that this does not violate the proverbial Jensen's inequality fading penalty because, with power control, the received power is higher than the transmitted power times the channel attenuation.

A modified form of power equalization, whereby no power is transmitted below a certain channel attenuation threshold is known to be optimal in terms of outage performance [5].

Often, the fading coefficients are the product of slowly varying fading (due to shadowing and other forms of path loss) and Rayleigh fast fading. It is usually reasonable to assume that the transmitter can track the slow, but not the fast, component of the fading distribution, and that both components are independent. The optimum power control law for this scenario of partially known fading at the transmitter is derived in Appendix A. Surprisingly, it turns out that as long as the distribution of the trackable (slowly varying) fading component is unbounded, reliable communication is achievable for any $\frac{E_b}{N_0}$.

III. Linear Multiuser Detection

In this section, we investigate the spectral efficiency achievable with three receivers that have identical structure: for each user, a linear transformation produces observables at the symbol rate which are then fed to a single-user error control decoder.

A. Single-User Matched Filter

In this subsection, we consider a suboptimum receiver whose front end is a bank of single-user matched filters followed by respective single-user decoders. If the channel is not subject to fading, the spectral efficiency is [1]

$$\mathsf{C}^{\mathsf{sumf}}(\beta, \mathsf{SNR}) = \beta\log_2\left(1 + \frac{\mathsf{SNR}}{1 + \mathsf{SNR}\beta}\right) \tag{56}$$

regardless of whether the error control decoder knows the spreading codes of the interferers. Equation (56) was shown in [1] by means of a central limit theorem which holds in the model employed in this paper in which the number of users and the spreading gain grow without bound with a fixed ratio (see also [4], [23]). Similarly, for the fading channel, we can show that the effect of the multiaccess interference at the output of the matched filter is a Gaussian random variable.[7] The asymptotic spectral efficiency of the single-user matched filter is then given by (11) with SNR reduced by a suitable factor

$$\mathcal{C}^{\mathsf{sumf}}(\beta,\,\mathsf{SNR}) = \beta E\left[\log_2\left(1+\frac{\mathsf{SNR}|\mathsf{A}|^2}{1+\mathsf{SNR}\beta}\right)\right]. \tag{57}$$

Comparing (56) and (57) we see that, as $K\to\infty$, the only effect fading has is in the received power of the user to whose signature waveform the filter is matched. Recall that (57) represents the total system capacity (not the individual capacity of any given user), and, thus, it is not necessary to assume ergodicity of the fading processes affecting individual users for (57) to hold. However, for nonergodic fading, we can only guarantee the total capacity in (57) if the transmitters are informed by the receiver of the individual rates at which they can transmit. Furthermore, for (57) to hold it is not necessary to assume that the fading coefficients of different users are identically distributed. It is sufficient to assume that for all i, the mixture distribution of $\{|\mathsf{A}_1[i]|^2,\,\ldots,\,|\mathsf{A}_K[i]|^2\}$ is identical. Then, the average in (57) is taken with respect to that mixture distribution.

Using

$$\beta\mathsf{SNR} = \frac{E_b}{N_0}\mathcal{C}^{\mathsf{sumf}}(\beta,\,\mathsf{SNR}) \tag{58}$$

the $\frac{E_b}{N_0}{}^{\mathsf{sumf}}_{\min}$ required for reliable communication remains the same as in the single-user case (16)

$$\frac{E_b}{N_0}{}^{\mathsf{sumf}}_{\min} = \lim_{\mathsf{SNR}\downarrow 0}\frac{\beta\mathsf{SNR}}{\mathcal{C}^{\mathsf{sumf}}(\beta,\mathsf{SNR})} \tag{59}$$

$$= \log_e 2. \tag{60}$$

To obtain the slope of the spectral efficiency at $\frac{E_b}{N_0}{}^{\mathsf{sumf}}_{\min}$, we note first that formula (18) holds with (58)

$$\mathcal{S}_0^{\mathsf{sumf}} = \frac{2\dot{\mathcal{C}}^{\mathsf{sumf}}(\beta,\,0)^2}{-\ddot{\mathcal{C}}^{\mathsf{sumf}}(\beta,0)} \tag{61}$$

$$= \frac{2\beta}{2\beta+\kappa(|\mathsf{A}|)}. \tag{62}$$

The spectral efficiency in (57) is monotonic increasing with β. The limiting value is

$$\lim_{\beta\to\infty}\mathcal{C}^{\mathsf{sumf}}\left(\beta,\,\frac{E_b}{N_0}\right) = \log_2 e - \frac{N_0}{E_b}\ \text{b/s/Hz} \tag{63}$$

which is independent of the fading distribution. Since (63) encompasses the unfaded case, fading does not degrade the spectral efficiency of the single-user matched filter as $\beta\to\infty$. The reason is that when many users are packed in one dimension, the per-symbol SNR goes to 0 (cf. (58)). Thus, for any fixed $\frac{E_b}{N_0} > \frac{E_b}{N_0}{}^{\mathsf{sumf}}_{\min}$, $\beta\mathsf{SNR}$ remains bounded away from both 0 and ∞, and the argument of the $\log$ in (57) has very small variance, and, thus, the Jensen inequality penalty is very small.

Regardless of the value of β and the fading distribution, the slope of the spectral efficiency goes to 0 as $\frac{E_b}{N_0}$ grows. The reason is that the single-user matched filter is interference-limited: even if there is no additive white Gaussian noise it achieves bounded capacity.

[7]Note that this does *not* imply that the single-user matched filter is asymptotically optimal [4].

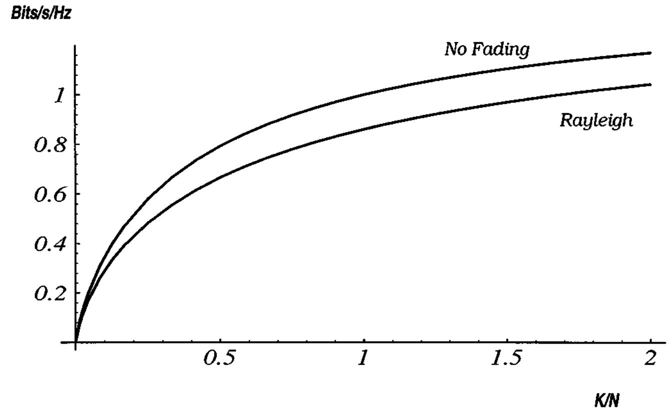

Fig. 5. Matched-filter spectral efficiency for $\frac{E_b}{N_0}\to\infty$.

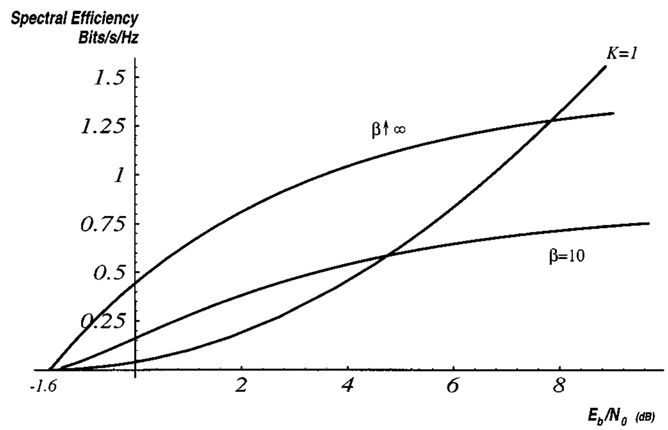

Fig. 6. Single-user versus multiuser spectral efficiencies with matched filter and log-normal fading $\sigma_P = 10$ dB.

Fixing β and letting $\frac{E_b}{N_0}\to\infty$, (57) approaches

$$\lim_{\mathsf{SNR}\to\infty}\mathcal{C}^{\mathsf{sumf}}(\beta,\,\mathsf{SNR}) = \beta E\left[\log_2\left(1+\frac{|\mathsf{A}|^2}{\beta}\right)\right]. \tag{64}$$

Fig. 5 shows (64) for both deterministic and Rayleigh $|\mathsf{A}|$. From (63) we know that the limit of both curves as $\beta\to\infty$ is equal to 1.44 b/s/Hz. However, the approach to the limit is very slow, particularly for Rayleigh fading.

Let us now compare the spectral efficiency achievable by the bank of matched filters to the single-user spectral efficiency achievable by an optimum receiver. Fig. 6 shows the matched filter behavior for $\beta = 10$ and $\beta\to\infty$ (63) in the special case of log-normal fading. Remarkably, at low $\frac{E_b}{N_0}$, the highly suboptimum matched-filter-bank receiver achieves *higher* capacity in the multiuser case $(K\to\infty)$ than an optimum receiver in the single-user case. Comparing (62) with the single-user expression (20), we see that if the kurtosis of the fading distribution is greater than 2, then, for sufficiently low $\frac{E_b}{N_0}$, the spectral efficiency is higher in the multiuser case than in the single-user case as long as

$$\beta > \frac{1}{1-2/\kappa(|\mathsf{A}|)}. \tag{65}$$

Recall that in the absence of fading, and for any receiver, the single-user channel has always larger or equal spectral efficiency than the multiuser channel that uses the same energy-per-bit. The beneficial effect of *multiuser diversity* in the presence of fading is a phenomenon that was first pointed out in [24] in the context of an optimum receiver without spreading. It was also observed in [25] and in [26], in the context of power-controlled narrow-band communication, and a cellular system, respectively. However, our novel observation is that multiuser diversity may be so important that it even overcomes the suboptimality of the receiver which totally neglects the presence of multiaccess interference. A few observations are helpful to gain insight into this behavior. First, the shape of the fading distribution is irrelevant as far as the effect of the interferers is concerned because of the central limit theorem. Second, the individual rate achieved by each user is much lower than the one achieved in the single-user case—it is the aggregate rate sum that is higher than in the single-user case. Third, the single-user transmitter is free to mimic the sum of the signals transmitted by the K users (even adding artificial fluctuating gains), however, the resulting signal is then subject to a single fading process rather than K independent ones, thereby not benefiting from diversity. Fourth, for given spectral efficiency and $\frac{E_b}{N_0}$, the per-symbol SNR is lower in the multiuser case by a factor of β. For given $\frac{E_b}{N_0}$, the equations that determine the spectral efficiency of the single-user channel and multiuser channels are, respectively,

$$\mathcal{C} = E\left[\log\left(1 + |\mathsf{A}|^2 \mathcal{C}\frac{E_b}{N_0}\right)\right] \tag{66}$$

$$\mathcal{C} = \beta E\left[\log\left(1 + \frac{|\mathsf{A}|^2 \mathcal{C}\frac{E_b}{N_0}}{\beta\left(1 + \mathcal{C}\frac{E_b}{N_0}\right)}\right)\right]. \tag{67}$$

Thus, by making β very large we can turn the right side of (67) into essentially

$$E\left[\frac{|\mathsf{A}|^2 \mathcal{C}\frac{E_b}{N_0}}{\left(1 + \mathcal{C}\frac{E_b}{N_0}\right)}\right] \log e$$

where we see that the contribution to capacity of large values of $|\mathsf{A}|^2$ is essentially equivalent to their contribution to power, and thus the Jensen penalty and the associated sensitivity to the fading distribution vanishes for large β. Recall that this is in complete contrast to the single-user behavior, where the impact of large channel gains on capacity and on received power are very different. By operating at a much lower per-symbol SNR, the multiuser system can take advantage of sporadic large channel gains, which are largely wasted in the single-user case, even if the times at which those favorable channel gains occur are unknown to the transmitters. Another observation is that for both the single-user capacity and the multiuser capacity with the matched filter bank, fading can only lower capacity (this is in contrast to the MMSE receiver as we will see in the sequel). While the multiuser diversity mechanism serves to alleviate this detrimental effect, fading hurts the single-user capacity much more, to the point that it even manages to make the single-user capacity lower than the multiuser capacity in the large-kurtosis low-$\frac{E_b}{N_0}$ large-β scenario.

Power Control: We now derive the optimum power control law when the transmitters know the fading coefficients and the receiver is the single-user matched filter. Note that the power control affects the payoff function in two different ways: a) shaping the distribution of received powers and b) changing the received multiaccess interference power. The optimum power-control problem can be posed as the following two-step maximization:

$$\overline{\mathcal{C}}^{\mathsf{sumf}}(\beta, \mathsf{SNR}) = \max_{\zeta} \max_{\substack{p:\\ E[p(|\mathsf{A}|^2)|\mathsf{A}|^2]=\zeta \\ E[p(|\mathsf{A}|^2)]=\mathsf{SNR}}} E\left[\beta \log_2\left(1 + \frac{p(|\mathsf{A}|^2)|\mathsf{A}|^2}{1+\zeta\beta}\right)\right]. \tag{68}$$

The solution to the inner maximization problem is given by the optimum power control law

$$p(\theta) = \left[\frac{1}{\lambda_1 + \lambda_2\theta} - \frac{1+\zeta\beta}{\theta}\right]^+ \tag{69}$$

where λ_1 and λ_2 are chosen to satisfy

$$E\left[\left[\frac{1}{\lambda_1 + \lambda_2|\mathsf{A}|^2} - \frac{1+\zeta\beta}{|\mathsf{A}|^2}\right]^+\right] = \mathsf{SNR} \tag{70}$$

$$E\left[\left[\frac{|\mathsf{A}|^2}{\lambda_1 + \lambda_2|\mathsf{A}|^2} - 1 - \zeta\beta\right]^+\right] = \zeta. \tag{71}$$

In this form, we get a parametric family of curves in terms of the auxiliary variable ζ. Numerical optimization yields the optimum value of ζ as a function of SNR and β.

Allowing different power control policies for different transmitters does not buy anything if the fading coefficients are identically distributed because the power control policies of the interferers enter (68) only through ζ (average received power), which depends only on a policy which is the mixture of all the policies. On the other hand, if different transmitters face different fading distributions, then using the same power control policy would be suboptimal. It can be shown, however, that the structure of (69) is retained, with the Lagrange multipliers being different for users who face different distributions.

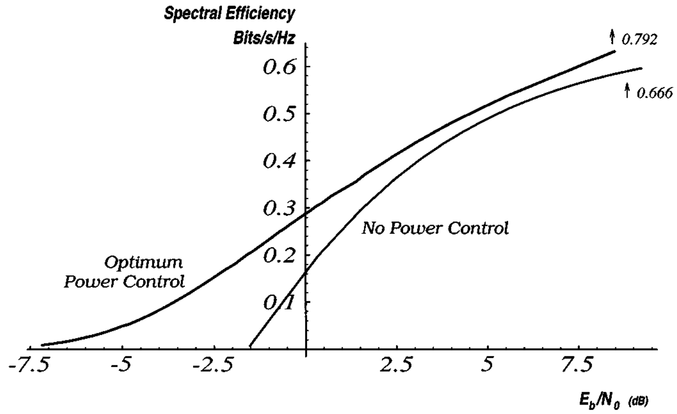

Fig. 7. Matched filter in Rayleigh fading and $\beta = 0.5$.

Fig. 7 compares the spectral efficiency achieved with and without power control. At low $\frac{E_b}{N_0}$, we see that, as in the single-user case, power control allows reliable communication below -1.59175 dB. Surprisingly, at high $\frac{E_b}{N_0}$, power control also helps: in Fig. 7 (Rayleigh fading) the spectral efficiency with [resp., without] power control tends to 0.792 [resp., 0.666] b/s/Hz. To understand this behavior, note that Jensen's inequality applied to (68) yields

$$\overline{\mathcal{C}}^{\mathsf{sumf}}(\beta,\,\mathsf{SNR}) \le \max_{\zeta} \beta \log_2\left(1+\frac{\zeta}{1+\zeta\beta}\right) \tag{72}$$

$$= \beta \log_2\left(1+\frac{1}{\beta}\right). \tag{73}$$

As $\frac{E_b}{N_0} \to \infty$, $\zeta \to \infty$ and the inequality in (72) becomes tight. We can see this using a truncated power equalization law (cf. [21])

$$p(|\mathsf{A}|^2) = \begin{cases} \frac{\nu \mathsf{SNR}}{|\mathsf{A}|^2}, & \text{if } |\mathsf{A}|^2 > \overline{\theta} \\ 0, & \text{if } |\mathsf{A}|^2 \le \overline{\theta} \end{cases} \tag{74}$$

where ν is chosen so that the expectation of (74) is equal to SNR. The variance of the unit-mean random variable

$$p(|\mathsf{A}|^2)|\mathsf{A}|^2/\zeta$$

can be made as small as desired with $\overline{\theta} \to 0$ (as long as $P[|\mathsf{A}|>0] = 1$). Thus, the right side of (73) is indeed the limit of the spectral efficiency achieved with power control. In contrast, without power control, the limit is given by (64) which is strictly lower than the controlled power capacity (73) achievable with any $|\mathsf{A}|$ with positive variance.

B. Decorrelator

We consider now a bank of decorrelators [4] followed by independent single-user decoders. At the output of each decorrelator there is no trace of multiaccess interference. Therefore, the capacity analysis is very simple and parallel to the unfaded case [1]

$$\mathsf{C}^{\mathsf{deco}}(\beta,\,\mathsf{SNR}) = \beta \log_2\left(1+\mathsf{SNR}(1-\beta)\right), \qquad \beta \le 1. \tag{75}$$

Fading only enters the picture through the modulation of the received power of the user of interest to each decorrelator. Thus, (75) generalizes to[8]

$$\mathcal{C}^{\mathsf{deco}}(\beta,\,\mathsf{SNR}) = \beta E\left[\log_2\left(1+\mathsf{SNR}(1-\beta)|\mathsf{A}|^2\right)\right], \qquad \beta \le 1. \tag{76}$$

If the fading coefficients for different users do not have the same distribution, then (as for the matched filter), the average in (76) is with respect to the mixture distribution.

If we denote the $\frac{E_b}{N_0}$ required by the decorrelator to achieve spectral efficiency $\mathcal{C}$, by $\frac{E_b^{\mathsf{deco}}}{N_0}(\mathcal{C})$, and the corresponding quantity in a single-user channel subject to the same fading distribution by $\frac{E_b^1}{N_0}(\mathcal{C})$, we obtain from (11) and (76) that

$$\frac{E_b^{\mathsf{deco}}}{N_0}(\mathcal{C}) = \frac{1}{1-\beta}\,\frac{E_b^1}{N_0}\left(\frac{\mathcal{C}}{\beta}\right). \tag{77}$$

Therefore, the curve of spectral efficiency versus $\frac{E_b}{N_0}$ achieved by the decorrelator is identical to the single-user case except that there is a horizontal displacement to the right by $-10\log_{10} \cdot (1-\beta)$ decibels, and a flattening in the vertical scale by a factor of β.

[8]If $P[|\mathsf{A}| = 0] > 0$, then it would be possible to extend the range of operation beyond $\beta = 1$, by monitoring the symbols in which individual users are silent. However, for the sake of clarity, we stick to $\beta < 1$ in this subsection.

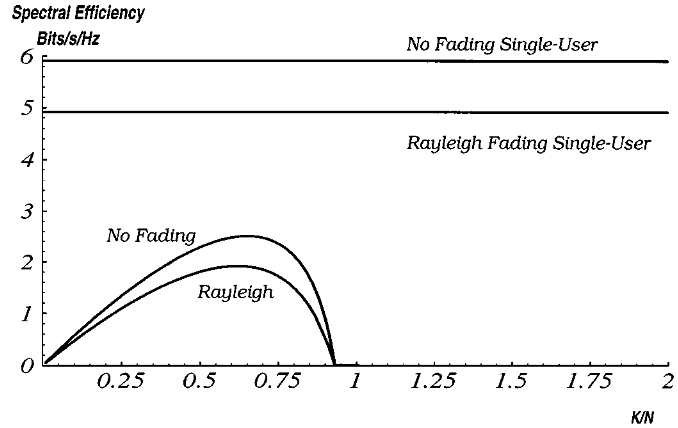

Fig. 8. Decorrelator spectral efficiency for $\frac{E_b}{N_0} = 10$ dB.

Now it follows immediately from (77) that the minimum $\frac{E_b}{N_0}$ required for reliable communication is

$$10\log_{10}\frac{E_b^{\mathsf{deco}}}{N_0}_{\min} = -1.59175 \text{ dB} + 10\log_{10}\frac{1}{1-\beta} \text{ dB} \tag{78}$$

regardless of the fading distribution. The slope in b/s/Hz per 3 dB achieved at that point is equal to (via (20))

$$\mathcal{S}_0^{\mathsf{deco}} = \frac{2\beta}{\kappa(|\mathsf{A}|)} \tag{79}$$

and the slope at high $\frac{E_b}{N_0}$ is (via (24))

$$\mathcal{S}_\infty^{\mathsf{deco}} = \beta P[|\mathsf{A}| > 0]. \tag{80}$$

The offset at high $\frac{E_b}{N_0}$ (in 3-dB units) is equal to (via (30))

$$\mathcal{L}_\infty^{\mathsf{deco}} = \log_2\frac{\beta}{1-\beta} - \frac{E[1\{|\mathsf{A}|>0\}\log_2|\mathsf{A}|^2]}{P[|\mathsf{A}|>0]}. \tag{81}$$

Fig. 8 shows the spectral efficiency as a function of β for a given $\frac{E_b}{N_0}$. As in the unfaded case, the $\beta \in [0,\,1]$ that maximizes spectral efficiency is in the interior of the interval, and depends on $\frac{E_b}{N_0}$ and the fading distribution.

Power Control: The optimum power control problem is very similar to the one we derived for the single-user matched filter. As in that case, it results in a water-filling solution

$$p[\theta] = \left[\frac{1}{\lambda} - \frac{1}{(1-\beta)\theta}\right]^+ \tag{82}$$

with λ chosen so as to satisfy

$$E\left[\left(\frac{1}{\lambda} - \frac{1}{(1-\beta)|\mathsf{A}|^2}\right) 1\{(1-\beta)|\mathsf{A}|^2 > \lambda\}\right] = \mathsf{SNR} \tag{83}$$

yielding capacity equal to

$$\mathcal{C}^{\mathsf{deco}}(\beta,\,\mathsf{SNR}) = E\left[\left[\log_2\left(\frac{(1-\beta)|\mathsf{A}|^2}{\lambda}\right)\right]^+\right]. \tag{84}$$

As in the single-user channel, the effect of optimum power control for the decorrelator is to extend the minimum required $\frac{E_b}{N_0}$ to $-\infty$ and to improve the low-$\frac{E_b}{N_0}$ spectral efficiency. This effect is more pronounced than in the single-user channel because of the increased minimum $\frac{E_b}{N_0}$ of the decorrelator. As in the single-user channel, at high $\frac{E_b}{N_0}$, power control offers negligible improvement. Overall, the conclusion is that power control with the decorrelator is not a sensible design for most scenarios, because in the only region where it helps appreciably (namely, low-$\frac{E_b}{N_0}$), it is preferable to use the lower complexity single-user matched filter with power control.

C. MMSE Linear Filter

We consider now a bank of MMSE linear filters followed by single-user decoders. The MMSE filter of a particular user is chosen to maximize the output signal-to-interference ratio seen by the information transmitted by that user [4]. The spectral efficiency of this scheme in the case of equal received powers not subject to fading admits the following expression obtained in [1]:

$$\mathsf{C}^{\mathsf{mmse}}(\beta, \mathsf{SNR}) = \beta \log_2 \left(1 + \mathsf{SNR} - \tfrac{1}{4}\mathcal{F}(\mathsf{SNR}, \beta)\right) \tag{85}$$

with

$$\mathcal{F}(x, z) \stackrel{\text{def}}{=} \left(\sqrt{x\left(1+\sqrt{z}\right)^2 + 1} - \sqrt{x\left(1-\sqrt{z}\right)^2 + 1}\right)^2. \tag{86}$$

In the absence of fading and equal received powers, the $\frac{E_b}{N_0}$ required by the MMSE receiver to achieve spectral efficiency C admits a closed-form expression due to Müller [27]

$$\frac{E_b^{\mathsf{mnf}}}{N_0}(C, \beta) = \frac{2^{C/\beta} - 1}{C} \frac{\beta}{1 - \beta + \beta 2^{-C/\beta}}. \tag{87}$$

In order to generalize (85) to the fading channel, we need to recall the result of Tse and Hanly [2] on the output signal-to-interference ratio of MMSE receivers with not necessarily equal received powers. Denote the (nonasymptotic) multiuser efficiency[9] [4] achieved by the kth-user MMSE filter by $\eta_k^{\mathsf{mmse}}(\mathsf{SNR})$. This is equal to the output SNR divided by the signal-to-noise at the output of a single-user matched filter in the absence of multiaccess interference. Using key results on the eigenvalue distribution of random matrices [28], [2] shows the following result (couched in the notation of the present paper):

$$\lim_{K \to \infty} \eta_k^{\mathsf{mmse}}(\mathsf{SNR}) = \eta(\mathsf{SNR}) \tag{88}$$

where convergence is almost surely, and η is the positive solution to

$$\eta + \beta E\left[\frac{\mathsf{SNR}|\mathsf{A}|^2 \eta}{\mathsf{SNR}|\mathsf{A}|^2 \eta + 1}\right] = 1. \tag{89}$$

We derive several properties of the function $\eta(\mathsf{SNR})$, which we find useful in the sequel.

Property 1: In the special case of no fading, $|\mathsf{A}| = 1$, the solution of (89) is

$$\eta(\mathsf{SNR}) = 1 - \frac{\mathcal{F}(\mathsf{SNR}, \beta)}{4\mathsf{SNR}}. \tag{90}$$

Property 2: Fading can only increase η. To see this note that the left side of (89) is monotonically increasing in η and apply Jensen's inequality to the expectation of the concave function of $|\mathsf{A}|^2$ therein.

Property 3: The behavior for low SNR is:

$$\eta(\mathsf{SNR}) = 1 - \beta E[|\mathsf{A}|^2]\mathsf{SNR} + \beta E^2[|\mathsf{A}|^2](\beta + \kappa(|\mathsf{A}|))\mathsf{SNR}^2 + o(\mathsf{SNR}^2). \tag{91}$$

[9]In contrast to the spectral efficiency, multiuser efficiency is not an information-theoretic quantity.

Property 4:

$$\lim_{\mathsf{SNR} \to \infty} \eta(\mathsf{SNR}) = [1 - \beta P[|\mathsf{A}| > 0]]^+. \tag{92}$$

Property 5: Let α be the solution to

$$E\left[\frac{1}{\alpha|\mathsf{A}|^2 + 1}\right] = \frac{\beta - 1}{\beta}. \tag{93}$$

Then,

$$\lim_{\mathsf{SNR} \to \infty} \mathsf{SNR}\, \eta(\mathsf{SNR}) = \begin{cases} \infty, & \beta P[|\mathsf{A}| > 0] \le 1 \\ \alpha, & \beta P[|\mathsf{A}| > 0] > 1. \end{cases} \tag{94}$$

Property 6:

$$\lim_{\mathsf{SNR} \to \infty} \frac{\mathsf{SNR}\, \dot{\eta}(\mathsf{SNR})}{\eta(\mathsf{SNR})} = \begin{cases} 0, & \beta P[|\mathsf{A}| > 0] < 1 \\ -1/2, & \beta P[|\mathsf{A}| > 0] = 1 \\ -1, & \beta P[|\mathsf{A}| > 0] > 1. \end{cases} \tag{95}$$

Property 7: Fix $\mathsf{SNR} > 0$, then the behavior for $\beta \to 0$ is

$$\eta(\mathsf{SNR}) = 1 - E\left[\frac{\mathsf{SNR}|\mathsf{A}|^2}{\mathsf{SNR}|\mathsf{A}|^2 + 1}\right]\beta + o(\beta). \tag{96}$$

Property 8: Fix $\mathsf{SNR} > 0$, then the behavior for $\beta \to \infty$ is

$$\eta(\mathsf{SNR}) = \frac{1}{1 + \beta\mathsf{SNR}} + o(\beta^{-1}). \tag{97}$$

Property 9: Fix $d \stackrel{\text{def}}{=} \beta\mathsf{SNR} \ge 0$, then the behavior for $\mathsf{SNR} \to 0$ is

$$\eta(\mathsf{SNR}) = \frac{1}{1+d} + \frac{d\kappa(|\mathsf{A}|)}{(1+d)^3}\mathsf{SNR} + o(\mathsf{SNR}). \tag{98}$$

As in the case discussed in Section III-A, the asymptotic normality of the multiaccess interference at the output of the MMSE transformation [23] and (88) yield that the asymptotic spectral efficiency is equal to

$$\mathcal{C}^{\mathsf{mmse}}(\beta, \mathsf{SNR}) = \beta E\left[\log_2(1 + \mathsf{SNR}|\mathsf{A}|^2 \eta(\mathsf{SNR}))\right] \tag{99}$$

with $\eta(\mathsf{SNR})$ given by (89). Note that in contrast to the decorrelator, the spectral efficiency is indeed impacted (through η) by the fact that the interferers are subject to fading.

Applying Jensen's inequality to the convex function $\log(1/x)$, $x \in (0, 1]$, we get a simple lower bound on (99)

$$\mathcal{C}^{\mathsf{mmse}}(\beta, \mathsf{SNR}) \ge -\beta \log_2\left(E\left[\frac{1}{1 + \mathsf{SNR}|\mathsf{A}|^2 \eta(\mathsf{SNR})}\right]\right) \tag{100}$$

$$= \beta \log_2\left(\frac{\beta}{\eta(\mathsf{SNR}) + \beta - 1}\right) \tag{101}$$

where (101) follows from (89). Note that the lower bound (101) depends on the fading distribution through $\eta(\mathsf{SNR})$. Note also that (100) holds with equality for deterministic $|\mathsf{A}|$.

It follows from (99) that the $\frac{E_b}{N_0}$ required by the MMSE receiver in the presence of fading can be written as a function of the $\frac{E_b}{N_0}$ required by the single-user channel subject to fading

$$\frac{E_b^{\mathsf{mmse}}}{N_0}(\mathcal{C}, \beta) = \frac{1}{\eta} \frac{E_b^1}{N_0}\left(\frac{\mathcal{C}}{\beta}\right) \tag{102}$$

where η depends implicitly on both β and $\mathcal{C}$ via (89) and (99). It is instructive to contrast (102) with the corresponding equation for the decorrelator (77).

Using (91) and (99) it can be concluded that the $\frac{E_b}{N_0}_{\min}$ necessary for reliable communication is the same as in the single-user unfaded channel (-1.59175 dB), and that the slope of the spectral efficiency therein is equal to 3 dB times

$$\mathcal{S}_0^{\mathsf{mmse}} = \frac{2\beta}{2\beta + \kappa(|\mathsf{A}|)} \tag{103}$$

which is the same result we obtained in (62) for the single-user matched filter. With (92)–(95), we can compute the slope at high $\frac{E_b}{N_0}$

$$\mathcal{S}_\infty^{\mathsf{mmse}} = \lim_{x\to\infty} x\dot{\mathcal{C}}^{\mathsf{mmse}}(x) \tag{104}$$

$$= \lim_{\mathsf{SNR}\to\infty} (1-\eta(\mathsf{SNR}))\left(1+\frac{\mathsf{SNR}\dot{\eta}(\mathsf{SNR})}{\eta(\mathsf{SNR})}\right) \tag{105}$$

$$= \begin{cases} \beta P[|\mathsf{A}|>0], & \beta P[|\mathsf{A}|>0]<1 \\ 1/2, & \beta P[|\mathsf{A}|>0]=1 \\ 0, & \beta P[|\mathsf{A}|>0]>1 \end{cases} \tag{106}$$

which, for $\beta < 1$, coincides with the result we obtained for the decorrelator in (80). It can also be shown that the asymptotic $\frac{E_b}{N_0}$ penalty satisfies

$$\mathcal{L}_\infty^{\mathsf{mmse}} = \begin{cases} \log_2 \frac{\beta}{1-\beta P[|\mathsf{A}|>0]} & \\ \quad -\frac{E[1\{|\mathsf{A}|>0\}\log_2 |\mathsf{A}|^2]}{P[|\mathsf{A}|>0]}, & \beta P[|\mathsf{A}|>0]<1 \\ -\frac{2E[1\{|\mathsf{A}|>0\}\log_2 |\mathsf{A}|^2]}{P[|\mathsf{A}|>0]}, & \beta P[|\mathsf{A}|>0]=1. \end{cases} \tag{107}$$

Note that the $\frac{E_b}{N_0}$ required in the case $\beta P[|\mathsf{A}|>0]=1$ is (in decibels) asymptotically twice that required in the single-user case (33), a result which was shown in the special case of the unfaded channel in [1, p. 636] and [27, eq. (4.76)]. If $\beta P[|\mathsf{A}|>0]>1$, then the MMSE receiver is interference-limited and the spectral efficiency goes to the following limit as $\frac{E_b}{N_0}\to\infty$:

$$\lim_{\mathsf{SNR}\to\infty} \beta E[\log_2(1+|\mathsf{A}|^2\eta\mathsf{SNR})] = \beta E[\log_2(1+\alpha|\mathsf{A}|^2)] \tag{108}$$

with α depending on both β and the distribution of $|\mathsf{A}|^2$ via (93).

As $\beta\to\infty$, the multiaccess interference is isotropic in all N dimensions and the MMSE receiver shows no advantage over the single-user matched filter (63).

Theorem III.1:

$$\lim_{\beta\to\infty} \mathcal{C}^{\mathsf{mmse}}\left(\beta, \frac{E_b}{N_0}\right) = \log_2 e - \frac{N_0}{E_b} \text{ b/s/Hz.} \tag{109}$$

Proof: Appendix B.

Fig. 9 compares the spectral efficiencies as a function of β achieved by the MMSE receiver with Rayleigh fading and without fading at $\frac{E_b}{N_0}=10$ dB. We can see the curious phenomenon that fading *increases* the spectral efficiency in the region of high β. To understand this, let us focus first on the unfaded channel. The curve is nonmonotonic with an optimum β^* that depends on $\frac{E_b}{N_0}$. If $\beta>\beta^*$, the system could achieve the same spectral efficiency as with β^*, by allowing only $\beta^* K/\beta$ users to transmit simultaneously. Thanks to the law of large numbers, this can be achieved by a randomized strategy whereby every user decides to transmit on a particular frame with probability β^*/β. The presence of fading makes a certain proportion of interferers appear so low-powered at the receiver, that the number of "effective" interferers seen by the MMSE receiver is reduced. The "interference population control" effect of fading more than compensates for its deleterious effect on the desired

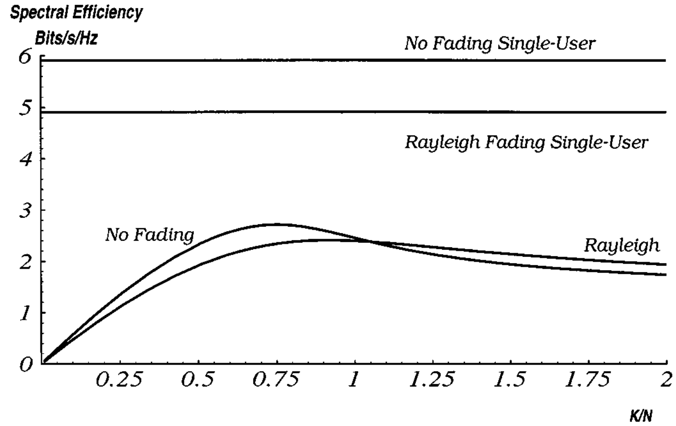

Fig. 9. MMSE spectral efficiency for $\frac{E_b}{N_0}=10$ dB.

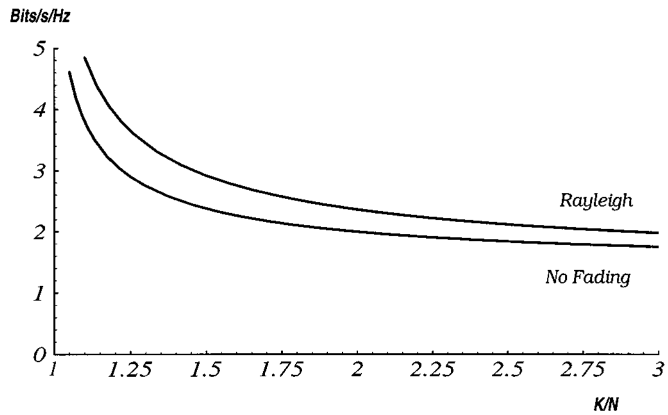

Fig. 10. MMSE spectral efficiency for $\frac{E_b}{N_0}\to\infty, \beta>1$.

user. For the decorrelator, fading is not beneficial, because the receiver treats all interferers in the same way regardless of whether their received power is infinitesimal or much greater than the desired user.

Fig. 10 compares the spectral efficiency attained by the MMSE receiver for $\frac{E_b}{N_0}\to\infty$ (108) with and without fading. The maximum relative improvement brought about by Rayleigh fading is 128% at $\beta=1.07$.

The improvement shown by the numerical results for Rayleigh fading in Figs. 9 and 10 suggests searching for the most and least favorable fading distributions in the high $\frac{E_b}{N_0}$ region and fixed β.

Theorem III.2: Fix β and an arbitrary fading distribution. Denote the spectral efficiency of the MMSE receiver by $\mathcal{C}^{\mathsf{mmse}}(\beta, \frac{E_b}{N_0})$ and $\mathsf{C}^{\mathsf{mmse}}(\beta, \frac{E_b}{N_0})$ with and without fading, respectively. Then for sufficiently large $\frac{E_b}{N_0}$

- if $\beta<1$, then $\mathcal{C}^{\mathsf{mmse}}(\beta, \frac{E_b}{N_0}) < \mathsf{C}^{\mathsf{mmse}}(\beta, \frac{E_b}{N_0})$;
- if $\beta>1$, then $\mathcal{C}^{\mathsf{mmse}}(\beta, \frac{E_b}{N_0}) > \mathsf{C}^{\mathsf{mmse}}(\beta, \frac{E_b}{N_0})$.

Proof: Appendix C.

For $\frac{E_b}{N_0}\to\infty$ the effect of any fading distribution without a point mass at 0, is a penalty equal to the single-user fading penalty, namely, $E[\log_2 |\mathsf{A}|^2]$ times 3 dB for $\beta<1$. For $\beta=1$, the required $\frac{E_b}{N_0}$ is doubled in decibels. For $\beta > 1$, the spectral efficiency improves from

$$\beta \log_2 \left(\frac{\beta}{\beta - 1}\right)$$

to (108).

As far as the most favorable fading distribution for high $\frac{E_b}{N_0}$ and fixed β is concerned, we see from (106) that we would like to choose

$$P[|\mathsf{A}| > 0] \uparrow \min\left\{1, \frac{1}{\beta}\right\} \tag{110}$$

thereby operating in the region $\beta P[|\mathsf{A}| > 0] < 1$. Having made that choice for $P[|\mathsf{A}| > 0]$, we see from (107) (using Jensen's inequality) that it is optimum to concentrate the strictly positive part of $|\mathsf{A}|$ on one point. Thus, we are lead to consider a two-mass distribution

$$|\mathsf{A}|^2 = \begin{cases} 0, & \text{with probability } 1 - 1/a \\ a, & \text{with probability } 1/a. \end{cases} \tag{111}$$

If we evaluate (89) and (99) with this distribution, we can find an explicit expression for the $\frac{E_b}{N_0}$ required to obtain a given spectral efficiency $\mathcal{C}$

$$\frac{E_b^{\mathsf{mmse}}}{N_0}(\mathcal{C}, \beta) = \frac{2^{a\mathcal{C}/\beta} - 1}{\mathcal{C}} \frac{\beta}{a - \beta + \beta 2^{-a\mathcal{C}/\beta}} \tag{112}$$

$$= \frac{E_b^{\mathsf{mnf}}}{N_0}(\mathcal{C}, \beta/a). \tag{113}$$

Therefore, we have obtained the same spectral efficiency $\frac{E_b}{N_0}$ curve as an unfaded channel (87) with a load equal to

$$\frac{\beta}{a} = \beta P[|\mathsf{A}| > 0].$$

Recall that the unfaded spectral efficiency is maximized by a certain β^* [1, Fig. 3], which is a function of $\frac{E_b}{N_0}$. Equivalently, for a given desired spectral efficiency, $\frac{E_b}{N_0}$ is minimized with an optimum choice of β. If the actual β is higher than β^*, then the two-mass fading distribution (111), with $a = \beta/\beta^*$, achieves the same performance as the unfaded channel with optimum coding–spreading tradeoff Fig. 11. On the other hand, for any given $\frac{E_b}{N_0}$, $\epsilon > 0$, and β, an arbitrarily large a can be found that makes $\beta P[|\mathsf{A}| > 0]$ so small as to make the spectral efficiency smaller than ϵ. Thus, the optimization problems we have posed with respect to fading distributions depend critically on whether the distribution is allowed to depend on $\frac{E_b}{N_0}$ or not.

Fig. 12 shows the spectral efficiency achievable with the optimum coding–spreading tradeoff. As we saw in (63), the optimum coding–spreading tradeoff for the single-user matched filter dictates $\beta \to \infty$. As we will see in Section IV, the same holds for the optimum receiver. To obtain the curve in Fig. 12 corresponding to the MMSE receiver, we numerically solve for the β that maximizes spectral efficiency as a function of $\frac{E_b}{N_0}$, in parallel to what we did in the unfaded case [1, Fig. 3]. As we would have expected from Fig. 9, Rayleigh fading raises the optimum value of β. Comparing Fig. 2 to Fig. 12 we see that Rayleigh fading introduces some degradation in the MMSE curve for large $\frac{E_b}{N_0}$. The following result gauges that degradation analytically.

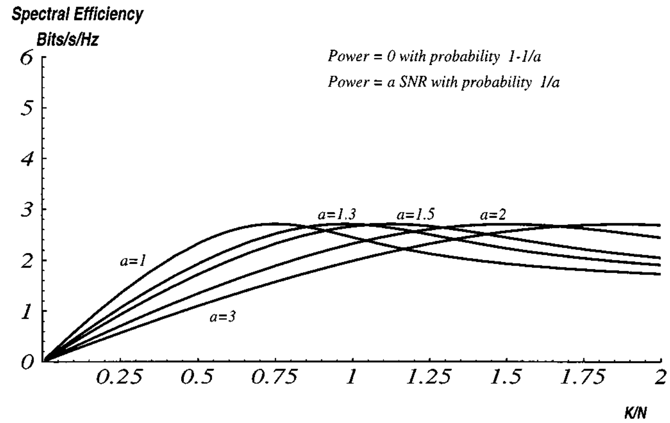

Fig. 11. MMSE spectral efficiency for $\frac{E_b}{N_0} = 10$ dB, with ON–OFF fading.

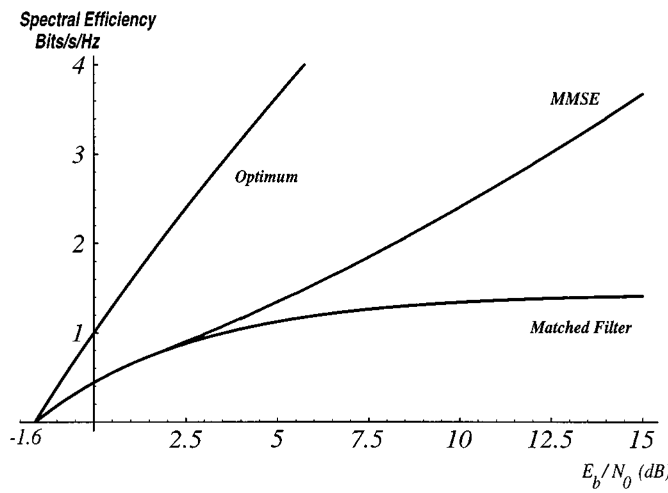

Fig. 12. Large-K spectral efficiencies with optimum K/N with Rayleigh fading.

Theorem III.3: Assume that $P[|\mathsf{A}| > 0] = 1$. The high-SNR behavior of the MMSE receiver and the decorrelator with optimum coding–spreading tradeoff is

$$10\log_{10}\frac{E_b}{N_0}(\mathcal{C}) = 10\log_{10}\frac{E_b}{N_0}_{\min} + \left(\mathcal{C} - E[\log_2 |\mathsf{A}|^2]\right) 10\log_{10} 2 + o(1) \tag{114}$$

where $\frac{E_b}{N_0}_{\min} = -1.59175$ dB.

Proof: Appendix D.

It is interesting to see that the asymptotic behavior in (114) is the only one found in this paper that does not contain a logarithmic component in $\mathcal{C}$. The (asymptotic) effect of fading is the $\frac{E_b}{N_0}$ penalty $E[\log_2 |\mathsf{A}|^2]$, which is the same experienced by the single-user channel (33).

It has been observed in [27] (unfaded channel) and in [29] that the optimum coding–spreading ratio for the MMSE receiver is equal to $\beta = \infty$ for sufficiently low $\frac{E_b}{N_0}$. The following result (shown following an approach different from that in [29]) formalizes this fact for any fading distribution with finite fourth moment.

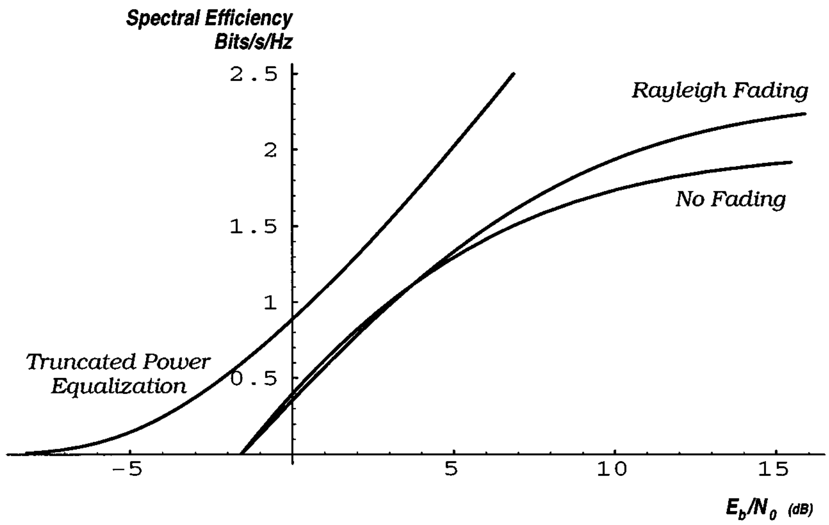

Fig. 13. MMSE spectral efficiency for $\beta = 2$.

Theorem III.4: Suppose that $E[|\mathsf{A}|^4] < \infty$. The optimum coding–spreading ratio for the MMSE receiver is finite if and only if $\frac{E_b}{N_0} > 2\frac{E_b}{N_0}_{\min} = 1.41855$ dB.

Proof: Appendix E.

Power Control: The optimum power control problem for the MMSE receiver is somewhat more complex than the one we solved for the single-user matched filter. From (89) and (99) we have

$$\overline{C}^{\mathsf{mmse}}(\beta, \mathsf{SNR}) = \max_{\eta} \max_{\substack{p: \\ \eta+\beta E[1/(1+1/(p(|\mathsf{A}|^2)\eta|\mathsf{A}|^2))]=1 \\ E[p(|\mathsf{A}|^2)]=\mathsf{SNR}}} \cdot E\left[\beta \log_2\left(1+p(|\mathsf{A}|^2)|\mathsf{A}|^2\eta\right)\right]. \quad (115)$$

The solution to the inner maximization problem is given by the optimum power control law

$$p(\theta) = \frac{1}{\lambda_1}\left[1+\sqrt{1-\frac{\lambda_1\lambda_2}{\theta\eta}}-\frac{\lambda_1}{\theta\eta}\right]^+ \quad (116)$$

where λ_1 and λ_2 chosen so as to satisfy both constraints. Then, the outer (scalar) maximization can be solved numerically, in general.

When β is greater than the optimum coding–spreading tradeoff, randomization of the power control law may be beneficial, as illustrated by the special cases of no fading and the two-level fading distribution in (111). In those cases, allowing transmitters who face the same nonzero fade coefficient the degree of freedom to randomly decide whether to transmit or not has a population control effect which allows an optimum choice of β.

For high SNR, we saw in (73) that the asymptotic spectral efficiency of the single-user matched filter increases from $\beta E[\log_2(1+\frac{|\mathsf{A}|^2}{\beta})]$ without power control to $\beta E[\log_2(1+\frac{1}{\beta})]$ with optimal power control, which is also the asymptotic capacity in the absence of fading. For the unfaded channel, [1] showed that the spectral efficiency of the MMSE receiver goes to

$$\beta \log_2 \frac{\beta}{\beta-1}$$

as $\frac{E_b}{N_0} \to \infty$ and $\beta > 1$. The spectral efficiency of the MMSE receiver with optimal power control grows without bound as $\frac{E_b}{N_0} \to \infty$ even if $\beta > 1$. We will show this using the suboptimum *truncated power equalization* control law (74)

$$p(|\mathsf{A}|^2) = \begin{cases} \frac{\nu\mathsf{SNR}}{|\mathsf{A}|^2}, & \text{if } |\mathsf{A}|^2 > \overline{\theta} \\ 0, & \text{if } |\mathsf{A}|^2 \le \overline{\theta} \end{cases} \quad (117)$$

with

$$\nu = \frac{1}{E[|\mathsf{A}|^{-2}1\{|\mathsf{A}|^2 > \overline{\theta}\}]}. \quad (118)$$

We are led to consider (117) from the high-$\frac{E_b}{N_0}$ optimality of the two-mass fading distribution we saw above. For brevity, denote

$$q = P[|\mathsf{A}|^2 > \overline{\theta}]. \quad (119)$$

The ensuing capacity is given by

$$\overline{C}^{\mathsf{mmse}}(\beta, \mathsf{SNR}) = \beta q \log_2\left(1+\nu\mathsf{SNR}\eta\right) \quad (120)$$

with (89) now taking the form

$$\eta + \beta q \frac{\nu\mathsf{SNR}\eta}{1+\nu\mathsf{SNR}\eta} = 1. \quad (121)$$

This leads to a closed-form solution for the $\frac{E_b}{N_0}$ required to achieve spectral efficiency $\mathcal{C}$

$$\frac{E_b^{\mathsf{mmse}}}{N_0}(\mathcal{C}, \beta) = \frac{1}{\nu q}\,\frac{E_b^{\mathsf{mnf}}}{N_0}(\mathcal{C}, \beta q), \quad (122)$$

with the function on the right side defined in (87). For given β and desired $\mathcal{C}$, the threshold $\overline{\theta}$ can be chosen to minimize (122). Fig. 13 shows the spectral efficiency achieved by the MMSE receiver for $\beta = 2$ with and without Rayleigh fading. The unfaded channel has a very slight edge for $\frac{E_b}{N_0}$ below about 4 dB, whereas for $\frac{E_b}{N_0} \to \infty$, the unfaded and Rayleigh channels achieve 2 and 2.36 b/s/Hz, respectively. Truncated power equalization in the presence of Rayleigh fading achieves a substantial improvement in all $\frac{E_b}{N_0}$ ranges; in particular, the spectral efficiency grows without bound as $\frac{E_b}{N_0} \to \infty$. In general, since the optimum load behaves as $\beta^* \uparrow 1$ as $\frac{E_b}{N_0} \to \infty$, we would like to select the threshold $\overline{\theta}$ so that $q \uparrow 1/\beta$. This is possible for any continuous fading distribution that satisfies[10]

$$E[|\mathsf{A}|^{-2}] > \frac{1}{\beta}. \tag{123}$$

Whereas at high $\frac{E_b}{N_0}$, power control had negligible effect for the single-user channel and a quantifiable, but bounded, effect on the matched-filter spectral efficiency, for the MMSE receiver with $\beta > 1$ the effect is much more profound, by making the spectral efficiency unbounded with $\frac{E_b}{N_0}$. We can attribute this behavior to the strong "population control" effect of optimum power control (or suboptimum truncated power equalization) which drives the effective number of active users below the spreading gain.

IV. Optimum Receiver

We now turn our attention to the derivation of the capacity of the CDMA channel subject to flat fading without putting any constraints on the receiver structure.

The capacity of the unfaded channel admits the following closed-form expression found in [1]:

$$\begin{aligned}\mathsf{C}^{\mathsf{opt}}(\beta, \mathsf{SNR}) =& \beta \log_2\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR}, \beta)\right)\\ &+\log_2\left(1+\mathsf{SNR}\beta-\frac{1}{4}\mathcal{F}(\mathsf{SNR}, \beta)\right)-\frac{\log_2 e}{4\mathsf{SNR}}\mathcal{F}(\mathsf{SNR}, \beta).\end{aligned}$$

For any given matrix of signature vectors $\boldsymbol{S}$, K, N, and received fading coefficients

$$\boldsymbol{A} = \operatorname{diag}\{\mathsf{A}_1, \ldots, \mathsf{A}_K\} \tag{124}$$

the total capacity of the CDMA multiaccess channel was found in [30]

$$\mathsf{C}^{\mathsf{opt}}(K, N, \boldsymbol{A}, \boldsymbol{S}, \mathsf{SNR}) = \frac{1}{N}\log\det\left[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{S}\boldsymbol{A}\boldsymbol{A}^H\boldsymbol{S}^H\right]. \tag{125}$$

Applying Fan's theorem on the concavity of the log determinant [31] to (125) we conclude that, unlike for the single-user matched filter and the MMSE receiver, fading cannot improve the capacity of CDMA for the optimum receiver. As we will see, this conclusion will cease to hold when transmitter power control is allowed.

We are interested in finding the limit as $K \to \infty$ $(K/N = \beta)$ of (125) when the entries of $\boldsymbol{S}$ are independent and identically distributed and the diagonal elements of $\boldsymbol{A}$ are also independent and identically distributed. The limit is a deterministic quantity, denoted by $\mathcal{C}^{\mathsf{opt}}(\beta, \mathsf{SNR})$. The following theorem is the main result in the paper.

Theorem IV.1: Fix β and SNR. Let the efficiency of the MMSE receiver be denoted by η, which satisfies (89). Then the optimum capacity is given by

$$\mathcal{C}^{\mathsf{opt}}(\beta, \mathsf{SNR}) = \mathcal{C}^{\mathsf{mmse}}(\beta, \mathsf{SNR}) + \log_2\frac{1}{\eta} + (\eta-1)\log_2 e. \tag{126}$$

Proof: We saw in (91) that $\eta(0) = 1$. Furthermore,

$$\mathcal{C}^{\mathsf{opt}}(\beta, 0) = \mathcal{C}^{\mathsf{mmse}}(\beta, 0) = 0. \tag{127}$$

Therefore, the claim holds for $\mathsf{SNR} = 0$. It follows that proving the claim is equivalent to showing that, for fixed β, the derivatives with respect to SNR of both sides are equal. Using the notation we introduced in Section II, the derivatives with respect to SNR of the capacities in nats will be denoted by $\dot{\mathcal{C}}$.

We first show that the derivative of the capacity of the optimum receiver satisfies

$$\dot{\mathcal{C}}^{\mathsf{opt}} = \frac{1-\eta}{\mathsf{SNR}}. \tag{128}$$

Let $\sigma_1(\boldsymbol{SA}), \ldots, \sigma_N(\boldsymbol{SA})$ denote the singular values of the matrix $\boldsymbol{SA}$. Using (125) we have

$$\begin{aligned}\dot{\mathcal{C}}^{\mathsf{opt}} &= \frac{d}{d\mathsf{SNR}}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\log_e\left(1+\mathsf{SNR}\sigma_i^2(\boldsymbol{SA})\right) && (129)\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{\sigma_i^2(\boldsymbol{SA})}{1+\mathsf{SNR}\sigma_i^2(\boldsymbol{SA})} && (130)\\ &= E\left[\frac{\sigma^2(\boldsymbol{SA})}{1+\mathsf{SNR}\sigma^2(\boldsymbol{SA})}\right] && (131)\\ &= \frac{1}{\mathsf{SNR}}\left(1-\frac{1}{\mathsf{SNR}}E\left[\frac{1}{\frac{1}{\mathsf{SNR}}+\sigma^2(\boldsymbol{SA})}\right]\right) && (132)\\ &= \frac{1}{\mathsf{SNR}}\left(1-\frac{\mu}{\mathsf{SNR}}\right) && (133)\end{aligned}$$

where (131) follows from the law of large numbers, and (133) follows from the fundamental random eigenvalue results of [28] with μ equal to the positive solution of

$$\frac{1}{\mu} = \frac{1}{\mathsf{SNR}} + \beta E\left[\frac{|\mathsf{A}|^2}{1+\mu|\mathsf{A}|^2}\right]. \tag{134}$$

Upon comparison to (89), we see that

$$\eta = \frac{\mu}{\mathsf{SNR}} \tag{135}$$

and (133) becomes the desired equation (128).

Taking the derivative of (99) with respect to SNR, we get

$$\dot{\mathcal{C}}^{\mathsf{mmse}} = \beta E\left[\frac{|\mathsf{A}|^2\eta(\mathsf{SNR}) + \mathsf{SNR}|\mathsf{A}|^2\dot{\eta}(\mathsf{SNR})}{1+\mathsf{SNR}|\mathsf{A}|^2\eta(\mathsf{SNR})}\right]. \tag{136}$$

Using (128) and (136), the difference between the derivatives of both capacities is

$$\begin{aligned}\dot{\mathcal{C}}^{\mathsf{opt}} - \dot{\mathcal{C}}^{\mathsf{mmse}} &= \frac{1-\eta}{\mathsf{SNR}} - \beta(\eta+\mathsf{SNR}\dot{\eta})E\left[\frac{|\mathsf{A}|^2}{1+\mathsf{SNR}|\mathsf{A}|^2\eta}\right] && (137)\\ &= \frac{1-\eta}{\mathsf{SNR}} + \frac{\eta+\mathsf{SNR}\dot{\eta}}{\mathsf{SNR}}\left(1-\frac{1}{\eta}\right) && (138)\\ &= \left(1-\frac{1}{\eta}\right)\dot{\eta} && (139)\\ &= \frac{d}{d\mathsf{SNR}}\left\{\log_e\frac{1}{\eta} + (\eta-1)\right\} && (140)\end{aligned}$$

where (138) follows from (89). Thus, we have shown that the derivatives of (126) coincide. □

From the foregoing proof we can verify independently the asymptotic version of the assertion we made above on the fact that, unlike the MMSE receiver, the performance of the optimum receiver cannot be helped by the presence of fading. To see this, simply recall that the derivative of the capacity is monotonically decreasing with η (128), and that η increases with a nonzero-variance fading distribution.

[10] The Rayleigh distribution satisfies (123) for any β.

For completeness, we mention a less appealing representation of the optimum capacity shown in [32]

$$\mathcal{C}^{\mathsf{opt}}(\beta, \mathsf{SNR}) = \int_0^{\mathsf{SNR}} \frac{\beta}{x}\left(1 - \frac{1}{x}G_\lambda\left(\frac{-1}{x}\right)\right) dx \tag{141}$$

where $G_\lambda(z)$ denotes the Stieltjes transform of the distribution F_λ of the singular values of $\boldsymbol{SA}$

$$G_\lambda(z) = \int_0^\infty \frac{dF_\lambda(t)}{t-z} \tag{142}$$

and satisfies the fundamental functional equation [33]

$$G_\lambda(z) = E\left[\frac{1}{|\mathsf{A}|^2(1-\beta-\beta z\, G_\lambda(z)) - z}\right]. \tag{143}$$

The expression found in (126) allows us to readily analyze the behavior of the capacity of the optimum receiver for both low and high $\frac{E_b}{N_0}$. Since $\eta(\mathsf{SNR}) \to 1$ as $\mathsf{SNR} \to 0$, the minimum $\frac{E_b}{N_0}$ required for reliable communication is the same as that found for linear receivers, namely, -1.59175 dB. The slope at that point in b/s/Hz per 3 dB is

$$\begin{aligned}\mathcal{S}_0^{\mathsf{opt}} &= \frac{2\dot{\mathcal{C}}^{\mathsf{opt}}(0)^2}{-\ddot{\mathcal{C}}^{\mathsf{opt}}(0)} &(144)\\ &= \frac{2\dot{\mathcal{C}}^{\mathsf{mmse}}(0)^2}{-\ddot{\mathcal{C}}^{\mathsf{mmse}}(0) - \beta^2} &(145)\\ &= \frac{2\beta^2}{2\beta^2 + \beta\kappa(|\mathsf{A}|) - \beta^2} &(146)\\ &= \frac{2\beta}{\beta + \kappa(|\mathsf{A}|)}. &(147)\end{aligned}$$

Comparing this value to the results obtained for both single-user matched filter (62) and MMSE receiver (103), we see that the improvement in spectral efficiency thanks to optimum nonlinear processing approaches a factor of 2 as $\beta \to \infty$, which is the most favorable value of β. As we noted in [1], even in the region where the thermal noise dominates, taking into account the structure of the multiuser interference through nonlinear processing can result in a dramatic improvement.

At high $\frac{E_b}{N_0}$, according to (126), the slope is equal to (in b/s/Hz per 3 dB)

$$\begin{aligned}\mathcal{S}_\infty^{\mathsf{opt}} &= \lim_{x\to\infty} x\dot{\mathcal{C}}^{\mathsf{opt}}(x) &(148)\\ &= \lim_{x\to\infty} x\dot{\mathcal{C}}^{\mathsf{mmse}}(x) + \lim_{x\to\infty}\frac{x\dot{\eta}(x)}{\eta(x)}(\eta(x)-1) &(149)\\ &= \min\{1,\, \beta P[|\mathsf{A}|>0]\} &(150)\end{aligned}$$

where (150) follows from (92), (95), and (106). For comparison purposes, recall that in the single-user unfaded channel the slope of the capacity at high $\frac{E_b}{N_0}$ is equal to 1 b/s/Hz per 3 dB.

The $\frac{E_b}{N_0}$ penalty is given by

$$\mathcal{L}_\infty^{\mathsf{opt}} = \log_2 \beta + \lim_{x\to\infty}\left(\log_2 x - \frac{\mathcal{C}^{\mathsf{opt}}(x)}{\mathcal{S}_\infty^{\mathsf{opt}}}\right). \tag{151}$$

Let us analyze (151) in three separate cases.

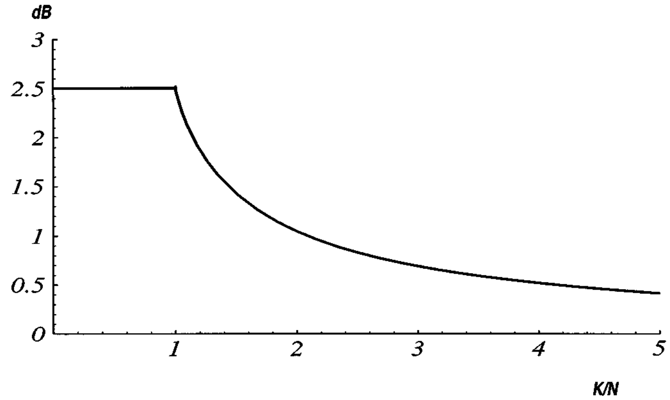

Fig. 14. Optimum receiver: asymptotic $\frac{E_b}{N_0}$ loss due to Rayleigh fading.

- $\beta P[|\mathsf{A}|>0] < 1$. Then

  $$\mathcal{S}_\infty^{\mathsf{opt}} = \mathcal{S}_\infty^{\mathsf{mmse}} = \beta P[|\mathsf{A}|>0] = 1 - \lim_{x\to\infty}\eta(x).$$

  Then using (126) we obtain

$$\begin{aligned}\mathcal{L}_\infty^{\mathsf{opt}} &= \mathcal{L}_\infty^{\mathsf{mmse}} + \log_2 e\\ &\quad + \frac{1}{\beta P[|\mathsf{A}|>0]}\log_2(1-\beta P[|\mathsf{A}|>0]) &(152)\\ &= \log_2 e + \log_2\beta\\ &\quad + \left(\frac{1}{\beta P[|\mathsf{A}|>0]} - 1\right)\log_2(1-\beta P[|\mathsf{A}|>0])\\ &\quad - \frac{E[1\{|\mathsf{A}|>0\}\log_2|\mathsf{A}|^2]}{P[|\mathsf{A}|>0]}. &(153)\end{aligned}$$

- $\beta P[|\mathsf{A}|>0] = 1$. In this case, $\eta(x) \to 0$ and $x\eta(x) \to \infty$ as $x \to \infty$. Thus, if Z denotes the random variable $|\mathsf{A}|^2$ restricted to $(0, \infty)$, then

$$\begin{aligned}\mathcal{L}_\infty^{\mathsf{opt}} &= \log_2\beta + \lim_{x\to\infty}\left(\log_2 x - \mathcal{C}^{\mathsf{opt}}(x)\right) &(154)\\ &= \log_2\beta + \log_2 e\\ &\quad - \lim_{x\to\infty} E\left[\log_2\left(\frac{1}{x\eta(x)} + Z\right)\right] &(155)\\ &= \log_2\beta + \log_2 e - \frac{E[1\{|\mathsf{A}|>0\}\log_2|\mathsf{A}|^2]}{P[|\mathsf{A}|>0]}. &(156)\end{aligned}$$

- $\beta P[|\mathsf{A}|>0] > 1$. In this case,

  $$\mathcal{C}^{\mathsf{mmse}}(x) \to \beta E\log_2(1+\alpha|\mathsf{A}|^2)$$

  and $\eta(x) \to 0$ as $x \to \infty$. Thus,

$$\begin{aligned}\mathcal{L}_\infty^{\mathsf{opt}} &= \log_2\beta + \lim_{x\to\infty}\left(\log_2 x - \mathcal{C}^{\mathsf{opt}}(x)\right) &(157)\\ &= \log_2\beta + \log_2 e - \beta E\log_2(1+\alpha|\mathsf{A}|^2)\\ &\quad + \lim_{x\to\infty}\log_2 x\eta(x) &(158)\\ &= \log_2 \alpha\beta e - \beta E\log_2(1+\alpha|\mathsf{A}|^2) &(159)\end{aligned}$$

  where α is the solution to (93).

Fig. 14 shows the high-$\frac{E_b}{N_0}$ fading penalty plotting the difference between the values taken by $\mathcal{L}_\infty^{\mathsf{opt}}10\log_{10}2$ with and without Rayleigh fading. For $\beta \le 1$, the penalty (153), (156) is

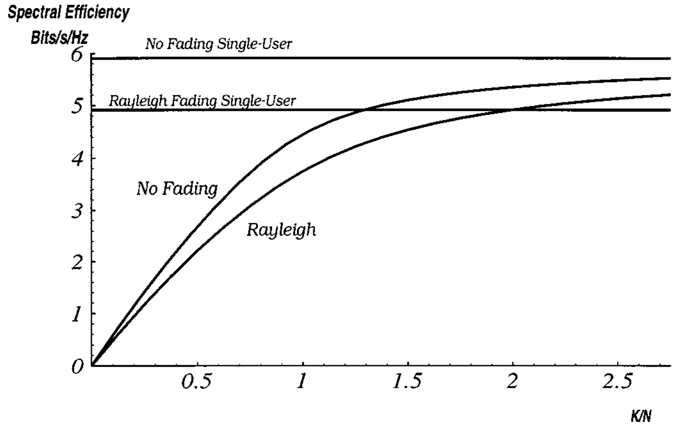

Fig. 15. Optimum spectral efficiency for $\frac{E_b}{N_0} = 10$ dB.

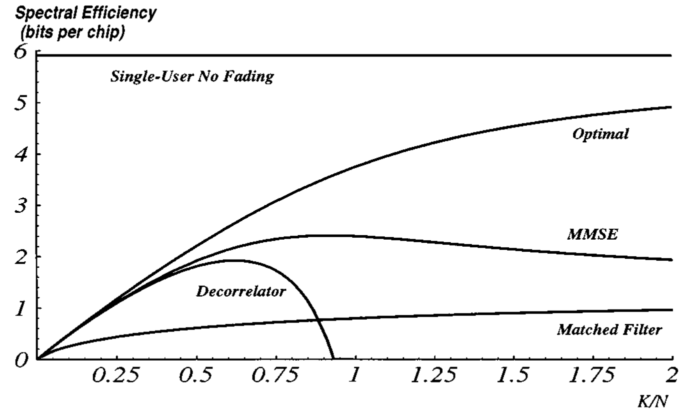

Fig. 16. Large-K spectral efficiencies for $\frac{E_b}{N_0} = 10$ dB and Rayleigh fading.

the same as in the single-user channel. For $\beta > 1$, the penalty vanishes as $\beta \to \infty$.

In addition to (126), another useful relationship between the optimum and MMSE fading capacities is

$$\mathcal{C}^{\text{opt}}(\beta, \mathsf{SNR}) = \int_0^\beta \frac{\mathcal{C}^{\text{mmse}}(z, \mathsf{SNR})}{z}\, dz \tag{160}$$

which stems from the asymptotic optimality (as the code block length grows without bound) of a successive cancellation nonlinear receiver where each stage consists of an MMSE transformation which takes into account only those users not yet demodulated [34]. Equation (160) can be easily shown following the same approach used in [1] to show it for the equal received-power channel.

We see from (160) that $\mathcal{C}^{\text{opt}}(\beta, \mathsf{SNR})$ is monotonically increasing as a function of β. In fact, just as for the matched filter and the MMSE receiver, the effect of fading vanishes as $\beta \to \infty$. Using (219), (220), and (126) we can readily show that the limiting capacity satisfies

$$\mathcal{C}^{\text{opt}} = \log_2 \frac{1}{\eta} \tag{161}$$

$$= \log_2 \left(1 + \frac{E_b}{N_0}\, \mathcal{C}^{\text{opt}}\right). \tag{162}$$

Thus, the spectral efficiency—$\frac{E_b}{N_0}$ curve has a remarkable limiting behavior

$$\lim_{\beta\to\infty} \frac{E_b^{\text{opt}}}{N_0}(\mathcal{C}, \beta) = \frac{2^{\mathcal{C}} - 1}{\mathcal{C}} \tag{163}$$

which we immediately recognize as the curve of the single-user additive white Gaussian noise channel. Therefore, as the number of users per dimension grows, the effect of fading vanishes, and the channel can be used as efficiently as if all the power were concentrated in one user. Thus, for sufficiently large β, the optimum receiver manages to wipe out any penalty due to random spreading, multiaccess, and fading.

Note also that (163) and (160) can be readily used to verify that the MMSE capacity with fading can lie neither below nor above the unfaded capacity for all β, as we saw in Section III.

Fig. 15 shows the behavior of the spectral efficiency of the optimum receiver with and without Rayleigh fading. Interestingly, the Rayleigh multiaccess channel achieves, for sufficiently high β, better capacity than the single-user Rayleigh channel. This is another manifestation of the phenomenon of multiuser diversity we encountered before: splitting the power among many users subject to independent fading is beneficial. As predicted by (163), the multiaccess capacity with and without fading converge as β grows to the ultimate limit, i.e., the unfaded single-user capacity. Note that the values of β for which 90% of that limit is achieved are quite moderate.

Fig. 16 compares the spectral efficiency versus β of the various receivers considered in this paper for a fixed $\frac{E_b}{N_0}$ in the presence of Rayleigh fading (without power control). Comparing this figure with Fig. 1, we see that the optimum receiver needs a higher β in the presence of fading to achieve the same fraction of the single-user no-spreading no-fading capacity. The relative performance of the linear receivers is very similar to the unfaded case. The sensitivity of MMSE spectral efficiency with β is seen to decrease slightly in the presence of fading, increasing the maximum β for which MMSE processing offers worthwhile performance gains over the matched filter.

The reader is referred back to Fig. 12 for a comparison of the spectral efficiencies achieved with the optimum, MMSE, and matched-filter receivers with optimum coding–spreading ratios. For both the optimum receiver and the single-user matched filter spectral efficiency increases monotonically with β. Thus, those curves correspond to $\beta \to \infty$ and are the same as those achieved in the unfaded channel shown in Fig. 2.

Power Control: Let us now turn to the derivation of the power control policy that maximizes capacity for the optimum receiver.

Theorem IV.2: Fix $\beta > 0$, $\mathsf{SNR} > 0$, and the fading distribution. The power control policy that maximizes $\mathcal{C}^{\text{opt}}(\beta, \mathsf{SNR})$ is the water-filling law

$$p(|\mathsf{A}|^2) = (1 + \beta \overline{g} \mathsf{SNR}) \left[\frac{1}{\overline{g}} - \frac{1}{|\mathsf{A}|^2}\right]^+ \tag{164}$$

where $\overline{g}$ is the positive solution of the equation

$$E\left[\left(\frac{1}{g} - \frac{1}{|\mathsf{A}|^2}\right)^+\right] = \frac{\mathsf{SNR}}{1 + \beta g \mathsf{SNR}}. \tag{165}$$

Proof: Appendix G.

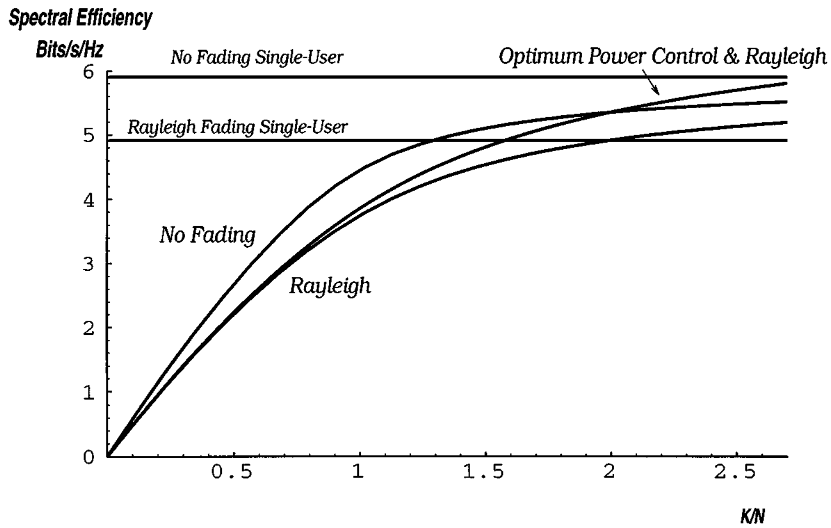

Fig. 17. The effect of optimum power control at $\frac{E_b}{N_0} = 10$ dB.

The functional form of the claimed optimal power control may strike the alert reader as suspect. The water-filling law failed to be the solution for either the matched-filter or the MMSE receivers, because although the capacity was given in terms of a single-user-like capacity, the power control law entered the expression not only directly, but indirectly through the received power (for the matched filter) and the SNR efficiency achieved by the MMSE receiver. Analogously, the power control law also affects the optimum capacity through the MMSE efficiency. However, (164) is indeed the solution thanks to the fortuitous fact (see Appendix G)

$$\begin{aligned}\mathcal{C}^{\mathsf{opt}}(\beta, \mathsf{SNR}) &= \beta E\left[\log_2\left(1+\mathsf{SNR}|\mathsf{A}|^2\eta\right)\right] \\ &\quad + \log_2\frac{1}{\eta} + (\eta-1)\log_2 e \qquad (166)\\ &= \min_{0\le x\le 1} \beta E\left[\log_2(1+\mathsf{SNR}|\mathsf{A}|^2 x)\right] \\ &\quad + \log_2\frac{1}{x} + (x-1)\log_2 e. \qquad (167)\end{aligned}$$

Using different methods, an equivalent power control has been shown to be optimal recently in [14] even within a larger class of centralized laws that are allowed to depend on the instantaneous signature sequences.

Using the expression found in (251) for the spectral efficiency attained by the optimum receiver with power control, we examine the high-$\frac{E_b}{N_0}$ behavior for fixed arbitrary β. It can be shown that the slope of the spectral efficiency is the same we obtained without power control (150), namely,

$$\mathcal{S}_\infty^{\mathsf{opc}} = \min\{1, \beta P[|\mathsf{A}| > 0]\}. \qquad (168)$$

The $\frac{E_b}{N_0}$ penalty is given by

- $\beta P[|\mathsf{A}| > 0] \le 1$

$$\mathcal{L}_\infty^{\mathsf{opc}} = \mathcal{L}_\infty^{\mathsf{opt}} + \log_2 P[|\mathsf{A}| > 0]. \qquad (169)$$

- $\beta P[|\mathsf{A}| > 0] > 1$

$$\mathcal{L}_\infty^{\mathsf{opc}} = \log_2\frac{e}{g_\beta} - \beta E\left[\left(\log_2\frac{|\mathsf{A}|^2}{g_\beta}\right)^+\right] \qquad (170)$$

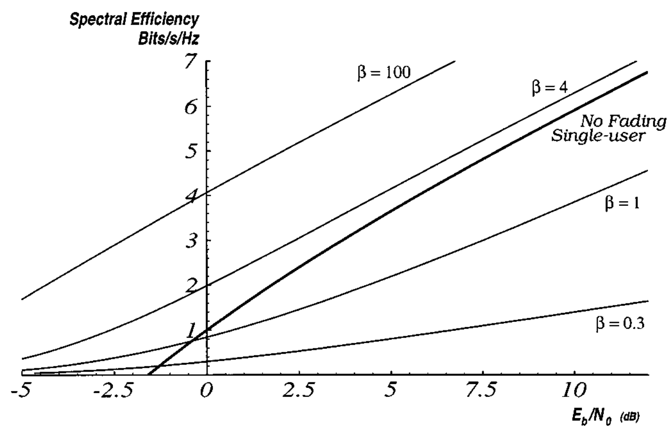

Fig. 18. Optimum power control with Rayleigh fading.

where g_β is the solution to

$$E\left[\left(1-\frac{g_\beta}{|\mathsf{A}|^2}\right)^+\right] = \frac{1}{\beta}. \qquad (171)$$

Fig. 17 shows the improvement attained by optimum power control as a function of β for fixed $\frac{E_b}{N_0}$. Regardless of the value of $\frac{E_b}{N_0}$, the spectral efficiency is monotonically increasing with β.

Fig. 18 shows the spectral efficiency attained by the optimum receiver with optimum power control and with Rayleigh fading for various values of β. Fig. 18 illustrates that for large values of β, the improvement over the capacity of the single-user additive white Gaussian noise channel is remarkable. In fact, with any fixed $\frac{E_b}{N_0}$, the spectral efficiency goes to ∞ as $\beta \to \infty$:

Theorem IV.3: If the fading distribution is unbounded, then the $\frac{E_b}{N_0}$ required to achieve spectral efficiency $\mathcal{C}$ with an optimum receiver and optimum power control (164) satisfies

$$\lim_{\beta\to\infty} \frac{E_b^{\mathsf{opc}}}{N_0}(\mathcal{C}, \beta) = 0 \qquad (172)$$

for any $\mathcal{C}$.

Proof: Appendix H. □

To gain more insight into how the phenomenal performance of Theorem IV.3 is attained, let us consider the special case of Rayleigh fading. The asymptotic behavior of the exponential integral function [22, eq. (8.215)] implies that the left side of (165) behaves as

$$\lim_{g\to\infty} \frac{g^2 E\left[\left(\frac{1}{g} - \frac{1}{|\mathsf{A}|^2}\right)^+\right]}{P[|\mathsf{A}|^2 > g]} = 1. \tag{173}$$

Also, according to the proof of Theorem IV.3, as $\beta \to \infty$, $\mathsf{SNR} = \delta/\beta$, the solution to (165) satisfies $g \to \infty$. Using (173) in (165) and $\beta\mathsf{SNR}g \to \infty$, we see that

$$\overline{\beta} \stackrel{\text{def}}{=} \beta P[|\mathsf{A}|^2 > g] \tag{174}$$
$$= \beta e^{-g} \tag{175}$$
$$= g. \tag{176}$$

Thus,

$$\beta = \overline{\beta} e^{\overline{\beta}}. \tag{177}$$

The parameter $\overline{\beta}$ can be interpreted as the effective load, normalizing by the spreading factor, that is, not the total number of users but the average number of users whose power control allows them to transmit nonzero power at any given time. Thus, the number of users transmitting simultaneously (those for which the fade level is exceptionally good) grows only logarithmically with K.

Recently, [14] shows that in the present model (random spreading, $K \to \infty$ and $K/N = \beta$) no advantage is realizable with a centralized power control policy (originally examined in [25] in the unspread case) where the received fading powers of all users are available to each of the transmitting users. The same trend is evidenced for optimal unspread systems [35] but in that case this is achieved with different strategies. While for the optimal centralized control only one user is active at a time [25], in case of decentralized power control either a logarithmic portion of the users are active [35], or a linear portion in case the fading takes on a finite number of values [36]. The conclusion in Theorem IV.3 is in accordance with the behavior of decentralized power control with no spreading ($N = 1$) found in [35]. It was shown in [35] that for large K, the capacity behaves as $\log(1 + K\mathsf{SNR}\log K)$, which corresponds to a decay in $\frac{E_b}{N_0}$ with K of

$$\frac{2^{\mathcal{C}} - 1}{\mathcal{C}\log K} + o(1).$$

V. Multiantenna Receiver

A. Model

So far, this paper has dealt exclusively with the scalar channel in (2). Now let us consider a more general model in which there are L receiving elements

$$\mathsf{y}_\ell(t) = \sum_{k=1}^{K}\sum_{i=1}^{n} \mathsf{A}_{k\ell}[i] b_k[i]\,\mathsf{s}_k(t - iT) + \mathsf{n}_\ell(t), \quad \ell = 1, \ldots, L. \tag{178}$$

As in (2), the real and imaginary components of the complex random process n_ℓ are independent white Gaussian each with double-sided power spectral density equal to $1/2$. Moreover, the processes n_ℓ are mutually independent. Note that now there is a fading coefficient $\mathsf{A}_{k\ell}$ for every pair (user, antenna element). Not only do we assume that the fading coefficients are independent from user to user but also from antenna element to antenna element. Moreover, we assume, for simplicity, that all coefficients are equally distributed. To make a fair comparison with previous results, we maintain the constraints (3) but instead of (4) now we assume that

$$E[|\mathsf{A}_{k\ell}|^2] = \frac{1}{L}. \tag{179}$$

Instead of working with (178), we can consider the equivalent discrete-time model of NL observables, where each antenna output is projected along the orthonormal waveforms $\{\psi_i(t),\, i = 1, \ldots, N\}$

$$\boldsymbol{r} = \begin{bmatrix} \boldsymbol{SA}_1 \\ \cdots \\ \boldsymbol{SA}_L \end{bmatrix} \boldsymbol{b} + \boldsymbol{m} \tag{180}$$

where $\boldsymbol{r}$ and $\boldsymbol{m}$ have NL complex components and

$$\boldsymbol{A}_{\boldsymbol{\ell}} = \operatorname{diag}\{\mathsf{A}_{1\ell}, \ldots, \mathsf{A}_{K\ell}\}. \tag{181}$$

B. Matched Filter Receiver

Consider the vector of K observables

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{SA}_1 \\ \cdots \\ \boldsymbol{SA}_L \end{bmatrix}^H \boldsymbol{r}. \tag{182}$$

The matched filter receiver simply feeds z_k to a single-user decoder for user k. Note that

$$\begin{bmatrix} \boldsymbol{SA}_1 \\ \cdots \\ \boldsymbol{SA}_L \end{bmatrix}^H = \left[\boldsymbol{A}_1^*\boldsymbol{S}^T \vdots \cdots \vdots \boldsymbol{A}_L^*\boldsymbol{S}^T\right] \tag{183}$$

whose kth row is only a function of the fading coefficients of the kth user. Thus, this receiver only needs to know the fading coefficients of the user or users of interest. Note that for the kth user, (182) is equivalent to matched-filtering each antenna output with s_k and combining all such outputs with a maximal-ratio combiner that ignores the presence of the multiaccess interference.

The $K \times K$ crosscorrelation matrix

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{SA}_1 \\ \cdots \\ \boldsymbol{SA}_L \end{bmatrix}^H \begin{bmatrix} \boldsymbol{SA}_1 \\ \cdots \\ \boldsymbol{SA}_L \end{bmatrix} \tag{184}$$

has entries equal to

$$R_{ij} = \rho_{ij} \sum_{\ell=1}^{L} \mathsf{A}_{i\ell}^* \mathsf{A}_{j\ell} \tag{185}$$

where, as usual, ρ_{ij} denotes the cross correlation between the signature waveforms. Thus, the equivalent cross-correlation matrix is the componentwise (Hadamard) product of the time-domain cross-correlation matrix with the space-domain cross-correlation matrix.

The output of the first-user linear transformation is equal to

$$\mathsf{y}_1 = R_{11} b_1 + \sum_{k=2}^{K} R_{1k} b_k + \mathsf{n}_1. \tag{186}$$

Conditioning on the fading amplitudes of user 1, n_1 has variance

$$R_{11} = \sum_{\ell=1}^{L} |\mathsf{A}_{1\ell}|^2 \tag{187}$$

and

$$E[|R_{1k}|^2] = \frac{R_{11}}{NL}. \tag{188}$$

Since $E[|b_k|^2] = \mathsf{SNR}$, according to the law of large numbers, the output SNR behaves as

$$\frac{R_{11}^2 \mathsf{SNR}}{R_{11} + \mathsf{SNR} \sum_{k=2}^{K} |R_{1k}|^2} \rightarrow \frac{\mathsf{SNR} \sum_{\ell=1}^{L} |\mathsf{A}_{1\ell}|^2}{1 + \mathsf{SNR}\beta/L} \tag{189}$$

a conclusion that was reached in [12]. Using (189) in lieu of

$$\frac{\mathsf{SNR}|\mathsf{A}|^2}{1 + \mathsf{SNR}\beta}$$

in (57), we see that the effect on capacity of the L-diversity reception at the receiver is threefold.

1) The expectation is computed with respect to the distribution of

$$|\overline{\mathsf{A}}|^2 = \sum_{\ell=1}^{L} |\mathsf{A}_{1\ell}|^2 \tag{190}$$

which corresponds to replacing the distribution of $|\mathsf{A}|^2$ by its L-fold convolution suitably normalized to preserve unit expected value. Note that the variance of $|\overline{\mathsf{A}}|^2$ is equal to the variance of $|\mathsf{A}|^2$ divided by L.

2) Substitute β by β/L, or equivalently, N by NL.
3) Multiply $\mathcal{C}(\beta, \mathsf{SNR})$ by a factor of L.

In view of (10), the impact on the spectral efficiency-$\frac{E_b}{N_0}$ curve is described by items 1) and 2) and by the multiplication of $\mathcal{C}(\beta, \frac{E_b}{N_0})$ by a factor of L.

It is easy to verify that L-diversity decreases kurtosis as

$$\kappa(|\overline{\mathsf{A}}|) = \frac{L - 1 + \kappa(|\mathsf{A}|)}{L}. \tag{191}$$

Moderately large values of L bring the kurtosis close to its minimum value of 1. Whereas the minimum $\frac{E_b}{N_0}$ required for reliable communication does not change with L, the slope of the spectral efficiency at that point becomes

$$\mathcal{S}_{0[L]}^{\mathsf{sumf}} = \frac{2\beta L}{2\beta + \kappa(|\mathsf{A}|) + L - 1} \tag{192}$$

which is equal to (62) if $L = 1$. The value in (192) increases monotonically with L with an asymptotic value equal to 2β. Therefore, if the kurtosis of the individual antenna fading is close to 1, and β is small, L-antenna diversity is of little use in the regime of low $\frac{E_b}{N_0}$. On the other hand, for large β, the benefits of L-antenna diversity may be substantial since $\mathcal{S}_{0[1]}^{\mathsf{sumf}} < 1$.

As $\beta \rightarrow \infty$, L-diversity has the effect of multiplying the spectral efficiency by L

$$\lim_{\beta\rightarrow\infty} \mathcal{C}_{[L]}^{\mathsf{sumf}}\left(\beta, \frac{E_b}{N_0}\right) = L\log_2 e - L\frac{N_0}{E_b} \text{ b/s/Hz}. \tag{193}$$

C. Decentralized Decorrelators

In a decentralized setting, where each individual receiver only knows the received fading coefficients of one user (and no adaptive filtering is used), it makes sense to consider a structure where L independent decorrelators act on each of the antenna outputs. The observable at the output of the ℓth decorrelator can be written asymptotically as

$$\mathsf{d}_{k\ell}[i] = \mathsf{A}_{k\ell}\sqrt{1-\beta}\, b_k[i] + \mathsf{n}_{k\ell}[i] \tag{194}$$

where the noise random variables are independent Gaussian with unit variance. The outputs $\mathsf{d}_{k\ell}$ of those decorrelators are then maximal-ratio combined using the L fading coefficients of the desired user. This results in the following generalization of (76):

$$\mathcal{C}_{[L]}^{\mathsf{deco}}(\beta, \mathsf{SNR}) = \beta E\left[\log_2\left(1 + \mathsf{SNR}(1-\beta)|\overline{\mathsf{A}}|^2\right)\right], \qquad \beta \le 1. \tag{195}$$

Thus, the beneficial effect of diversity on this receiver is purely based on reducing the Jensen penalty by removing some of the randomness in the received fading.

D. MMSE Receiver

The main (and highly nontrivial) result of [12] is that the effect of the L-diversity reception on the SNR for the MMSE receiver is identical to what we saw for the single-user matched filter, namely, the substitution of the distribution of $|\mathsf{A}|^2$ by $|\overline{\mathsf{A}}|^2$ and the substitution of β by β/L. Accordingly, the capacity of the L-antenna linear MMSE receiver is given by

$$\mathcal{C}_{[L]}^{\mathsf{mmse}}(\beta, \mathsf{SNR}) = L\,\frac{\beta}{L}\, E\left[\log_2\left(1 + \mathsf{SNR}|\overline{\mathsf{A}}|^2\eta(\mathsf{SNR})\right)\right] \tag{196}$$

with $\eta(\mathsf{SNR})$ given by the solution to

$$\eta + \frac{\beta}{L} E\left[\frac{\mathsf{SNR}|\overline{\mathsf{A}}|^2\eta}{\mathsf{SNR}|\overline{\mathsf{A}}|^2\eta + 1}\right] = 1 \tag{197}$$

and the expectations in (196) and (197) taken with respect to (190).

Note that as $L \rightarrow \infty$, the solution to (197) converges to 1, and $|\overline{\mathsf{A}}| \rightarrow 1$. Thus, (196) converges to $\beta\log_2(1 + \mathsf{SNR})$. According to (103), the impact of L-antenna diversity on the low $\frac{E_b}{N_0}$ behavior of the MMSE receiver is identical to the impact we discussed above for the matched filter. Analogously, using (109) we obtain that the large-β behavior of the MMSE receiver is identical to (193).

Regarding the generalization of the high-$\frac{E_b}{N_0}$ slope found in (106), note that

$$P[|\overline{\mathsf{A}}| > 0] = 1 - (1 - P[|\mathsf{A}| > 0])^L \tag{198}$$

and that if $P[|\mathsf{A}| > 0] = 1$, the high-$\frac{E_b}{N_0}$ slope becomes

$$\mathcal{S}^{\mathsf{mmse}}_{\infty[L]} = \begin{cases} \beta, & \beta < L \\ L/2, & \beta = L \\ 0, & \beta > L. \end{cases} \tag{199}$$

So as long as $\beta \le L$, antenna diversity can increase spectral efficiency considerably in the high-$\frac{E_b}{N_0}$ regime.

E. Centralized Decorrelator

If a central receiver demodulates all users and has knowledge of all received fading coefficients, then one option is the centralized decorrelator which feeds the K (multiaccess interference-free) components of the vector $\boldsymbol{R}^{-1}\boldsymbol{z}$ to individual single-user decoders. This corresponds to the limiting $\mathsf{SNR} \to \infty$ version of the MMSE detector considered in the previous subsection. Using the previous results on the eigenvalue distribution of $\boldsymbol{R}$, we see that in this case

$$\mathcal{C}^{\mathsf{deco}}_{[L]}(\beta, \mathsf{SNR}) = \beta E\left[\log_2\left(1 + \mathsf{SNR}\left(1 - \frac{\beta}{L}\right)|\overline{\mathsf{A}}|^2\right)\right], \quad \beta \le L. \tag{200}$$

Notice that doubling L, we can double β, achieving a multiplication of capacity by a factor of 2 and a reduction in the Jensen penalty.

F. Optimum Receiver

For an L-antenna receiver, Theorem IV.1 holds verbatim provided that $\mathcal{C}^{\mathsf{mmse}}$ and η therein are evaluated as we saw for the multiantenna MMSE receiver. To see this, recall from (125) that, for fixed K and N, the optimum receiver capacity depends on the nonzero singular values of $\boldsymbol{SA}$, which are equal to the square root of the nonzero eigenvalues of the unnormalized $K \times K$ cross-correlation matrix $\boldsymbol{A}^H\boldsymbol{S}^H\boldsymbol{SA}$. Alternatively, we can use the original expression found in [30] to reach the same conclusion. In the case of L-diversity reception, the optimum receiver can use the observables in (182) since they are sufficient statistics. This implies that the capacity can be represented in the same way as for $L = 1$ with the appropriate change in the cross-correlation matrix. Namely, the square root of the nonzero eigenvalues of the unnormalized cross-correlation matrix are now the nonzero singular values of

$$\begin{bmatrix} \boldsymbol{SA}_1 \\ \cdots \\ \boldsymbol{SA}_L \end{bmatrix}.$$

The key result of [12] is that the distribution of those singular values is the same as those of $\boldsymbol{SA}$ provided that β is substituted by β/L and the fading distribution $|\mathsf{A}|$ is replaced by $|\overline{\mathsf{A}}|$.

Now, however, we can no longer claim that the effect of L-diversity is summarized by the three changes that we saw in the single-user matched filter and the MMSE receiver. Redoing the expression for the slope at zero spectral efficiency we obtain

$$\mathcal{S}^{\mathsf{opt}}_{0[L]} = \frac{2\beta L}{L + 2\beta + \kappa(|\mathsf{A}|) - 1 - \beta/L} \tag{201}$$

which as L grows gives a negligible improvement over the linear processing slope. Thus, with enough antennas in the low-$\frac{E_b}{N_0}$ regime any increase in complexity due to nonlinear processing is hardly justified.

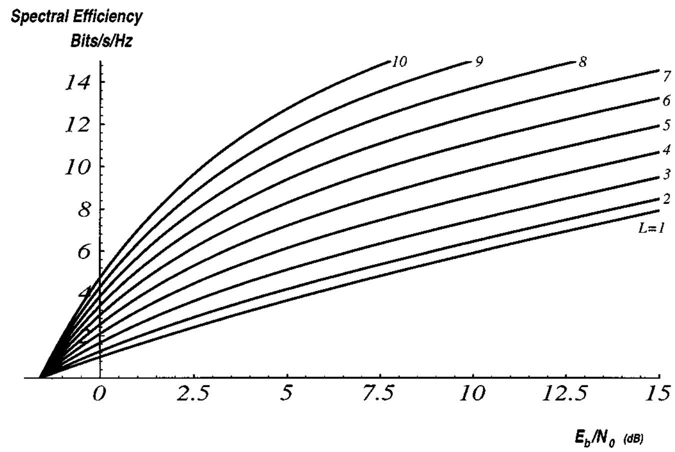

Fig. 19. Large-β spectral efficiencies parametrized by the number of antennas L.

For high $\frac{E_b}{N_0}$, the increase in slope due to nonlinear processing is given by

$$\mathcal{S}^{\mathsf{opt}}_{\infty[L]} - \mathcal{S}^{\mathsf{mmse}}_{\infty[L]} = \begin{cases} 0, & \beta P[|\overline{\mathsf{A}}| > 0] < L \\ 1/2, & \beta P[|\overline{\mathsf{A}}| > 0] = L \\ 1, & \beta P[|\overline{\mathsf{A}}| > 0] > L. \end{cases} \tag{202}$$

Again, for fixed β, nonlinear processing is not particularly helpful if L is sufficiently high.

In the regime of high load β, we saw in (163) that the optimum receiver achieves single-user additive white Gaussian noise performance, regardless of the fading distribution. This performance can be further improved using antenna diversity. To assess this improvement, first note that with L antennas, the asymptotic behavior of the MMSE efficiency (97) becomes

$$\eta = \frac{1}{1 + \mathsf{SNR}\beta/L} + o(\beta^{-1}). \tag{203}$$

Furthermore, the first term in the Taylor series of the L-diversity MMSE capacity becomes

$$\mathcal{C}^{\mathsf{mmse}}_{[L]} = L(1 - \eta)\log_2 e + o(\beta^{-1}). \tag{204}$$

Therefore, as $\beta \to \infty$, the optimum capacity satisfies

$$\mathcal{C}^{\mathsf{opt}}_{[L]} = (L - 1)(1 - \eta)\log_2 e + \log_2 \frac{1}{\eta} + o(\beta^{-1}). \tag{205}$$

Replacing $\mathsf{SNR}\beta$ by $\frac{E_b}{N_0}\mathcal{C}^{\mathsf{opt}}_{[L]}$, (203) and (205) give a parametric expression for the function

$$\lim_{\beta\to\infty} \mathcal{C}^{\mathsf{opt}}_{[L]}\left(\beta, \frac{E_b}{N_0}\right).$$

We can see from Fig. 19 that for typical $\frac{E_b}{N_0}$ of interest each additional antenna (beyond 2) buys roughly 1 b/s/Hz.

VI. Discussion and Conclusion

Conceptually, the main contribution of this paper is Theorem IV.1 which characterizes the “nonlinear gain” in capacity as a function of the SNR-performance of the MMSE receiver.

We believe that the full import of this result on analysis and design is yet to be unveiled.

The analysis of the spectral efficiency of various receivers in the presence of fading has revealed a number of interesting new findings of engineering interest that we compile in this section. We emphasize that comparisons are always made on the basis of equal $\frac{E_b}{N_0}$. Comparisons on the basis of equal per-symbol SNR may lead, in some cases, to different conclusions. Also, the performance measure is total capacity per second per hertz. Other performance measures such as individual-user outage probability may also lead to different conclusions.

A. Single-User Channel

Even in the context of the single-user channel we have found several new observations.

- As is well known, without transmitter power control, fading can only degrade single-user capacity because of the concavity of the capacity-cost function. Intuitively, the more "random" the fading distribution, the higher the capacity penalty. We have identified the kurtosis of the fading distribution as the peakedness or randomness measure that determines the capacity penalty in the region of low spectral efficiency.
- From an analysis of capacity versus per-symbol SNR [21], one could be led to conclude that the capacity improvement afforded by optimum power control is not significant at low SNR. However, our analysis as a function of $\frac{E_b}{N_0}$ reveals that an unbounded fading distribution allows reliable transmission no matter how low $\frac{E_b}{N_0}$. Equivalently, for any background noise level, the required joules per bit is as low as desired. This can be achieved by a transmitter which concentrates its power at those rare instances at which channel gain is exceedingly large.
- Even if the transmitter has only partial knowledge of the fade level, unbounded slow fading allows reliable transmission at arbitrarily low $\frac{E_b}{N_0}$ (Appendix A).

B. Low-SNR Behavior

The spectral efficiency behaves as an affine function of $\frac{E_b}{N_0}$ in the region of high background Gaussian noise. In this region and in the absence of power control we have shown the following.

- The minimum $\frac{E_b}{N_0}$ required for reliable communication is equal to -1.59175 dB regardless of the fading distribution, the number of users, and spreading gain if the matched filter, the MMSE receiver, or the optimum receiver are employed. In contrast, the decorrelator suffers an $\frac{E_b}{N_0}{}_{\min}$ penalty of $10\log_{10}(1-\beta)$ decibels as it is not designed to cope with the dominant Gaussian noise.
- As in the single-user case, kurtosis $\kappa(|\mathsf{A}|)$ encapsulates the effect of fading at low SNR for all the CDMA receivers considered. Comparing the values of b/s/Hz per 3 dB achievable by linear (matched filter and MMSE) and optimum receivers

$$\mathcal{S}_0^{\mathsf{sumf}} = \mathcal{S}_0^{\mathsf{mmse}} = \frac{2\beta}{2\beta + \kappa(|\mathsf{A}|)} \tag{206}$$

$$\mathcal{S}_0^{\mathsf{opt}} = \frac{2\beta}{\beta + \kappa(|\mathsf{A}|)}. \tag{207}$$

We conclude that regardless of the fading distribution:

— for small β the advantage of nonlinear processing in low SNR is marginal;

— Optimum nonlinear processing doubles the achievable spectral efficiency using the optimum coding/spreading ratio for each receiver.

C. High-SNR Behavior

- Regardless of the fading distribution and the value of β, the spectral efficiency of the single-user matched filter converges to a finite limit as SNR goes to infinity.
- If the effective number of active users (seen at the receiver) per chip (i.e., $\beta P[|\mathsf{A} > |0])$ is less than one, the slope of the spectral efficiencies of the decorrelator, the MMSE, and the optimum receiver are all equal to $\beta P[|\mathsf{A}| > 0]$ b/s/Hz/(3 dB). If $\beta P[|\mathsf{A}| > 0] > 1$, then the linear receivers are also interference limited, whereas the optimum receiver achieves the same slope as a single-user system.
- With the optimum coding/spreading ratio the $\frac{E_b}{N_0}$ required by the MMSE receiver (or the decorrelator) to achieve a given spectral efficiency $\mathcal{C}$ is the same as in a single-user system subject to the same fading plus a penalty of $10\log_{10}(\mathcal{C})$ decibels.

D. Multiuser Diversity

Without fading, the single-user spectral efficiency is always larger than or equal to that of the multiuser channel whose aggregate power is equal to the power of the single user. Regardless of whether spreading is used, fading changes the picture. To the usual diversity mechanisms in time, frequency, and space, used to combat fading, we have identified the beneficial effects of allocating power among several transmitters subject to independent fading.

- *Low-SNR Benefit.* Under certain conditions on the fading distribution, β and $\frac{E_b}{N_0}$, the spectral efficiency is higher in the multiuser channel than in a single-user channel with the same energy per information bit and the same fading distribution. Surprisingly, the multiuser diversity gain (afforded by the independence of the fades experienced by the transmitters) can even offset the performance loss incurred by completely neglecting multiuser interference at the receiver. Key to this gain is the ability to operate at a much lower energy per symbol, which robustifies performance against the fading distribution.
- *Population Control Effect.* For linear receivers, fading has a triple effect: i) it inflicts a penalty by making the received amplitude of the desired user random, ii) it boosts the power of some of the interferers, and iii) it has a positive effect by weakening some of the interferers.

— For the matched filter, the influence of the specific fading distribution is only seen through i).

— For the decorrelator, only the first effect holds, as its performance is independent of the positive fade levels affecting the interferers. The decorrelator performance is equal to single-user performance with a penalty in $\frac{E_b}{N_0}$ of a factor of $1-\beta$ (receiver noise enhancement) and a factor of β penalty in capacity due to the nonorthogonal multiplexing with more dimensions than users.

— For the MMSE filter, effect iii) dominates the others if β exceeds the unfaded optimum coding/spreading ratio and $\frac{E_b}{N_0}$ is large enough; the "population control" effect of fading reduces the number of effective interferers seen by each MMSE filter. Among all the receivers we analyzed, only the MMSE receiver may exhibit an increase in spectral efficiency due to fading (without power control). Furthermore, η, the SNR-efficiency of the MMSE filter is always increased by fading.

- *Large Diversity.* In parallel with classical single-user results [16], we have shown in each of the cases considered that infinite diversity performance is equivalent to unfaded performance. Indeed, in the absence of power control, as $\beta \to \infty$, the spectral efficiency of the matched filter, the MMSE receiver, and the optimum receiver converge to their respective unfaded performances. Fortunately, to achieve 90% of the unfaded spectral efficiency, quite moderate β suffices. For sufficiently large β, the negligible improvement the MMSE receiver over the matched filter does not warrant the associated increase in complexity. For the optimum receiver, not only does the fading penalty vanish asymptotically but the spectral efficiency converges to that of the single-user unfaded channel. So for large β, the optimum receiver eliminates the penalties due to random spreading, uncoordinated multiaccess, and fading. This implies that for sufficiently large β, the optimum multiuser spectral efficiency exceeds the single-user spectral efficiency with the same $\frac{E_b}{N_0}$ and the same fading distribution.

E. Power Control

We have studied the potential performance improvements achievable by letting the transmitters adjust their powers as a function of the channel fade levels they experience (without knowledge of other transmitters' fading conditions). From the standpoint of Shannon capacity, the strategy of power equalization (or equal power control) used in CDMA commercial cellular systems is notoriously poor in terms of power efficiency, even worse than no power control. Our emphasis has been on the design and analysis of optimum power control laws tailored to the receiver used. The various laws we have obtained retain the structural property of single-user optimum power control: no power is sent when the channel attenuation exceeds a certain threshold.

At low SNR, an unbounded fading distribution coupled with optimum power control enables reliable communication no matter how low $\frac{E_b}{N_0}$. This is even the case if the fade levels are known only roughly at the transmitters. This behavior is common to both the single-user and the multiuser channels and for all the receivers considered.

The single-user results do not offer much encouragement for the prospects of power control above the very low SNR region. In contrast, in CDMA optimum power control (or even suboptimum power control such as truncated channel equalization) offers performance gains even at high $\frac{E_b}{N_0}$. The nature of those gains turns out to depend heavily on the receiver.

- *Matched Filter:* For any β, at high $\frac{E_b}{N_0}$, truncated power equalization is asymptotically optimum and it wipes out the fading penalty. Thus, the (interference-limited) limiting value of capacity for $\frac{E_b}{N_0} \to \infty$ is raised by eliminating the Jensen penalty.
- *Decorrelator:* Power control is not a sensible design choice here. As for single-user channels, at high $\frac{E_b}{N_0}$, it hardly offers any improvement. At low $\frac{E_b}{N_0}$, the matched filter with power control offers better performance and lower complexity.
- *MMSE Receiver:* Power control helps the MMSE receiver tremendously, particularly when $\beta > 1$ and $\frac{E_b}{N_0}$ is high. Whereas without power control, the spectral efficiency is bounded if $\beta > 1$, optimum power control (or truncated power equalization) has a "population control" effect that makes the receiver operate with an effective number of interferers that is lower than the spreading gain, a scenario in which the MMSE spectral efficiency is unbounded with $\frac{E_b}{N_0}$.
- *Optimum Receiver:* Curiously, the structure of the optimum power control law is simpler than those of the linear receivers. With sufficiently large β, the potential gains achievable by optimum power control and an optimum receiver are staggering. It is possible to outperform even the single-user channel without fading. The strategy for large β is to allow transmissions by only those users whose channel gains are exceedingly favorable (no more than $\log K$ in Rayleigh fading), but with very small per-symbol SNR. Still, for sufficiently large β, the number of simultaneous transmitters can be much larger than the spreading gain. With this scheme and any unbounded fading distribution, the surprising result is that it is possible to achieve any desired spectral efficiency at any $\frac{E_b}{N_0}$ provided β is allowed to be large enough. The increase of capacity with β is extremely slow, however. Overly ambitious b/s/Hz-$\frac{E_b}{N_0}$ pairs come, as we would expect, at a huge complexity cost.

In our development of power control laws, we have restricted attention to identical power control laws for all users. Because the users are affected by identically distributed fading, it can be seen that allowing a randomized identical power control law gives the same generality as allowing users to employ nonidentical deterministic power control laws. However, randomization does not help when the receiver is the matched filter or the optimum receiver. For the decorrelator and MMSE receivers randomization may help, for example, when the fading distribution is discrete.

F. Multiantenna Receivers

If the receiver has L antennas subject to independent fading and it uses a bank of matched filters or MMSE receivers, the performance improvement is summarized by i) a change in the fading distribution which becomes more "deterministic" by L-fold convolution, ii) the multiplication by a factor of L of capacity as a function of SNR, and iii) the multiplication of spreading gain by a factor of L. For small β and low SNR, the improvement by L antenna diversity for either matched filter or MMSE is very slight unless the fading distribution has very large kurtosis. Conversely, for large β and any $\frac{E_b}{N_0}$, the spectral efficiency is multiplied by a factor of L for both receivers.

With enough antennas nonlinear processing does not improve performance appreciably for low $\frac{E_b}{N_0}$. On the other hand, at moderately large $\frac{E_b}{N_0}$ and large β, each additional antenna (beyond two) buys 1 b/s/Hz.

Appendix A
Partially Known Fading

We now derive the optimum power control law for the scenario of partially known fading at the transmitter. The role of $|\mathsf{A}[i]|^2\mathsf{SNR}$ is now taken by

$$cR^2[i]\theta[i]p(\theta[i])$$

where c is a normalization constant, $R^2[i]$ are unit-mean exponential random variables unknown to the transmitter, and $\theta[i]$ are the magnitude squared of the slowly-varying fading coefficients which are trackable at the transmitter and have an arbitrary distribution with nonzero variance. The objective is to maximize the expectation

$$E\left[\log_2\left(1+cR^2\theta p(\theta)\right)\right]$$

with respect to the nonnegative functions p that satisfy

$$E[p(\theta)] = \mathsf{SNR}. \tag{208}$$

Using the Lagrange multiplier λ chosen to satisfy (208), the solution is given by the power control law

$$p(\theta) = \frac{1}{c\theta}\,V\left(\frac{\lambda}{c\theta}\right) \tag{209}$$

where $V(x)=0$ if $x>1$, and for $x\le 1$, it is equal to $(\dot{\Lambda})^{-1}$, the inverse function of the confluent hypergeometric function

$$\dot{\Lambda}(x) = \int_0^\infty \frac{t}{1+xt}\,e^{-t}\,dt \tag{210}$$

$$= \frac{x-\Lambda(x)}{x^2} \tag{211}$$

with Λ defined in (36). Denoting the composition of the functions Λ and V by $\Lambda\circ V$, the capacity as a function of λ chosen to satisfy (208) is equal to

$$\mathcal{C}(\mathsf{SNR}) = E\left[\Lambda\circ V\left(\frac{\lambda}{c\theta}\right)\right]. \tag{212}$$

As a function of λ, we have

$$\frac{E_b}{N_0} = \frac{\mathsf{SNR}}{\mathcal{C}(\mathsf{SNR})} \tag{213}$$

$$= \frac{E\left[\frac{1}{c\theta}V\left(\frac{\lambda}{c\theta}\right)\right]}{E\left[\Lambda\circ V\left(\frac{\lambda}{c\theta}\right)\right]} \tag{214}$$

$$\le \frac{1}{\lambda}. \tag{215}$$

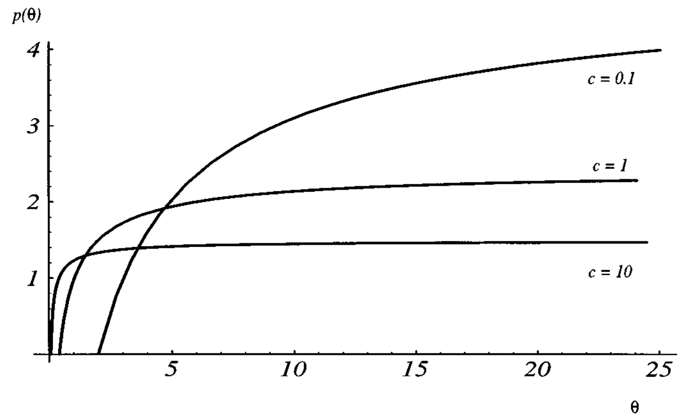

Fig. 20. Power control policy with partially known fading (linear scale).

To check inequality (215), note that by concavity of Λ

$$x\dot{\Lambda}(x) \le \Lambda(x) \tag{216}$$

which is equivalent (taking $V(z)=x$) to

$$zV(z) \le \Lambda\circ V(z) \tag{217}$$

from which (215) follows upon multiplying and dividing its left side by λ.

If the distribution of θ is unbounded, then λ grows without bound as $\mathsf{SNR}\downarrow 0$. Therefore, according to (215), no matter how small $\frac{E_b}{N_0}$, reliable communication is possible. Note that this conclusion holds regardless of the relative strengths of the known and unknown fading components.

Fig. 20 shows the function $p(\theta)$ for θ following the log-normal distribution with $\sigma_p = 10$ dB and $\mathsf{SNR}=1$. As the relative power of the unknown Rayleigh fading component decreases (lower c) the threshold of the power control law increases. The curve for $c = 1$ corresponds to a Rayleigh component 11.5 dB below the known log-normal component.

Appendix B

Proof of Theorem III.1: To show (109), which was proven in [1] in the special case of the unfaded channel, note that to achieve capacity $\mathcal{C}$ with $\frac{E_b}{N_0}$, the SNR must go to 0 as

$$\mathsf{SNR} = \frac{E_b}{N_0}\frac{\mathcal{C}}{\beta} \tag{218}$$

so according to (97)

$$\eta = \frac{1}{1+\mathcal{C}\,\frac{E_b}{N_0}} + o(\beta^{-1}). \tag{219}$$

The first term in the Taylor series of (99) together with (89) yields

$$\mathcal{C}^{\mathsf{mmse}} = (1-\eta)\log_2 e + o(\beta^{-1}). \tag{220}$$

Then (109) follows upon substitution of (219) into (220).

Appendix C

Proof of Theorem III.2: If $\beta<1$, then $\mathcal{S}_\infty^{\mathsf{mmse}}$ is maximized by choosing $P[|\mathsf{A}|>0]=1$ (cf. (106)). Once that choice is made, the asymptotic penalty $\mathcal{L}_\infty^{\mathsf{mmse}}$ is minimized by letting $|\mathsf{A}|$ be deterministic (107).

If $\beta > 1$, to show that deterministic $|\mathsf{A}|$ is the least favorable fading distribution, let us assume that the we are operating in the region $\beta P[|\mathsf{A}| > 0] > 1$. If we are not, then (107) tells us that the capacity grows without bound with $\frac{E_b}{N_0}$ which is not the case for the unfaded channel. Denote $|\mathsf{A}|$ restricted to $1\{|\mathsf{A}| > 0\}$ by Z. Then

$$\begin{aligned}\lim_{\frac{E_b}{N_0}\to\infty} & \mathcal{C}^{\mathsf{mmse}}\left(\beta, \frac{E_b}{N_0}\right) \\ &= -\beta E\left[\log_2\left(\frac{1}{\alpha|\mathsf{A}|^2+1}\right)\right] && (221)\\ &= -\beta P[|\mathsf{A}|>0] E\left[\log_2\left(\frac{1}{\alpha Z+1}\right)\right] && (222)\\ &\geq -\beta P[|\mathsf{A}|>0]\log_2\left(E\left[\frac{1}{\alpha Z+1}\right]\right) && (223)\\ &= \beta P[|\mathsf{A}|>0]\log_2\left(\frac{\beta P[|\mathsf{A}|>0]}{\beta P[|\mathsf{A}|>0]-1}\right) && (224)\\ &\geq \beta\log_2\left(\frac{\beta}{\beta-1}\right) && (225)\end{aligned}$$

where (94) results in (221), Jensen's inequality led to (223), (224) is obtained from (93), and (225) is a consequence of the monotonicity of $x\log(x/(x-1))$, $x > 1$. Both inequalities (223) and (225) become tight if and only if $|\mathsf{A}| = 1$, i.e., in the absence of fading.

APPENDIX D

Proof of Theorem III.3: According to the results in Section III, the high-$\mathcal{C}$ behavior of both the decorrelator and the MMSE capacities if $\beta < 1$ is given by

$$\begin{aligned}10\log_{10}\frac{E_b}{N_0}(\mathcal{C}) &= \frac{\mathcal{C}}{\beta}10\log_{10}2 - 10\log_{10}(\mathcal{C})\\ &+\left(\log_2\frac{\beta}{1-\beta} - E[\log_2|\mathsf{A}|^2]\right)10\log_{10}2 + o(1). \quad (226)\end{aligned}$$

Since, asymptotically, the function $\frac{E_b}{N_0}(\mathcal{C})$ can only be higher with $\beta \geq 1$, we restrict attention to $\beta < 1$ and, thus, (226). The value of β that minimizes the right side of (226) is

$$\beta^* = \frac{\mathcal{C}}{\log_2 e + \mathcal{C}}. \quad (227)$$

Substituting (227) into (226) we get

$$\begin{aligned}10\log_{10}\frac{E_b}{N_0}(\mathcal{C}) &= (\mathcal{C} - \log_2(\log_2 e)\\ &\quad - E[\log_2|\mathsf{A}|^2])\,10\log_{10}2 + o(1) && (228)\\ &= 10\log_{10}\frac{E_b}{N_0}_{\min}\\ &\quad + \left(\mathcal{C} - E[\log_2|\mathsf{A}|^2]\right)10\log_{10}2 + o(1). && (229)\end{aligned}$$

To rigorously justify that (228) is indeed the asymptotic behavior achievable with optimum β, it is actually necessary to do a bit more work. For example, consider a family of curves parametrized by β such that the asymptotic slope is 0 for any value of β, but such that the maximum over β has a finite slope. So, in principle, we have not ruled out that we could do better than (114). To verify that indeed (114) is the answer, it is enough to note that for any $\epsilon > 0$ and all sufficiently large SNR

$$\begin{aligned}\mathcal{C}^{\mathsf{mmse}} \leq \beta\log_2(1+\epsilon) &+ \beta\log_2(\mathsf{SNR})\\ &+\beta E[\log_2|\mathsf{A}|^2] + \beta\log_2(1-\beta). \quad (230)\end{aligned}$$

This leads to

$$\frac{E_b}{N_0}(\mathcal{C}) \geq \frac{\beta 2^{\frac{\mathcal{C}}{\beta} - E[\log_2|\mathsf{A}|^2]}}{(1+\epsilon)(1-\beta)\mathcal{C}} \quad (231)$$

for all sufficiently large $\mathcal{C}$. But (231) is also minimized by (227), and since ϵ is arbitrarily small, the optimized lower bound coincides with (114).

APPENDIX E

Proof of Theorem III.4: To show the desired result, note that we can tell whether the optimum β is finite or not for a given $\frac{E_b}{N_0}$, by examining the behavior of the spectral efficiency as $\beta \to \infty$, while SNR is adjusted to comply with the given $\frac{E_b}{N_0}$. To that end, we fix an arbitrary $d = \beta\mathsf{SNR}$, and we investigate the value of d for which the derivative of the spectral efficiency with respect to $1/\beta$, or equivalently, SNR, becomes 0 at $\mathsf{SNR} = 0$

$$\frac{\partial}{\partial\mathsf{SNR}}\frac{d}{\mathsf{SNR}}E\left[\log_2\left(1+\mathsf{SNR}|\mathsf{A}|^2\eta(\mathsf{SNR})\right)\right]\Big|_{\mathsf{SNR}=0} = 0. \quad (232)$$

After some algebra, it can be shown that (232) requires that

$$\dot{\eta}(0) = \tfrac{1}{2}\eta^2(0)E[|\mathsf{A}|^4] \quad (233)$$

where the function $\eta(\mathsf{SNR})$ is analyzed for fixed $\beta\mathsf{SNR}$ in (98). Using the values of $\eta(0)$ and $\dot{\eta}(0)$ found therein, (233) leads to

$$d = 1. \quad (234)$$

Now, recall (Theorem III.1) that as $\beta \to \infty$, the spectral efficiency goes to $\mathcal{C} = \log_2 e - \frac{N_0}{E_b}$. Therefore,

$$1 = \beta\mathsf{SNR} = \mathcal{C}\frac{E_b}{N_0} \quad (235)$$

results in

$$\frac{E_b}{N_0} = 2\log_e 2. \quad (236)$$

□

APPENDIX F
OUTAGE PROBABILITES

Because of the infinite-user model considered, the phenomenon of "user-ergodization" is a form of statistical multiplexing that ensures convergence to the deterministic total capacity even if the fading processes affecting individual users are not ergodic. However, from the viewpoint of the rate achievable by every individual user, the presence of nonergodic, or even ergodic but slow, fading variations, leads to the consideration of outage probability [37]. This can be characterized by the probability that a particular user can convey a reliable rate equal to a fraction r of its average rate (ergodic capacity). That probability can be written in terms of the fading cumulative distribution function. In the single-user channel [37], the result follows by interpreting the rate per user

$$R(\mathsf{SNR}) = \log\left(1 + \mathsf{SNR}|\mathsf{A}|^2\right) \tag{237}$$

as a random variable (which is a function of the random variable $|\mathsf{A}|^2$). Then

$$\begin{aligned} P\left[R(\mathsf{SNR}) \le rE[R(\mathsf{SNR})]\right] &= P\left[|\mathsf{A}|^2 \le \frac{2^{rE\log_2(1+\mathsf{SNR}|\mathsf{A}|^2)}-1}{\mathsf{SNR}}\right]. \end{aligned} \tag{238}$$

Using the results presented in this appendix, it is easy to obtain the capacity versus outage function for the three linear detectors considered. As in (238), this is again given in terms of the probability distribution function of the fading power $|\mathsf{A}|^2$. The expressions in (57), (76), and (99) result in the following expressions:

$$\begin{aligned} &P\left[R^{\mathsf{sumf}}(\beta,\,\mathsf{SNR}) \le rER^{\mathsf{sumf}}(\beta,\,\mathsf{SNR})\right] \\ &\quad = P\left[|\mathsf{A}|^2 \le \left(2^{rE\log_2\left(1+\frac{\mathsf{SNR}|\mathsf{A}|^2}{1+\mathsf{SNR}\beta}\right)}-1\right)\left(\frac{1}{\mathsf{SNR}}+\beta\right)\right] \end{aligned} \tag{239}$$

$$\begin{aligned} &P\left[R^{\mathsf{deco}}(\beta,\,\mathsf{SNR}) \le rER^{\mathsf{deco}}(\beta,\,\mathsf{SNR})\right] \\ &\quad = P\left[|\mathsf{A}|^2 \le \frac{2^{rE\log_2(1+\mathsf{SNR}(1-\beta)|\mathsf{A}|^2)}-1}{\mathsf{SNR}(1-\beta)}\right], \qquad \beta \le 1 \end{aligned} \tag{240}$$

$$\begin{aligned} &P\left[R^{\mathsf{mmse}}(\beta,\,\mathsf{SNR}) \le rER^{\mathsf{mmse}}(\beta,\,\mathsf{SNR})\right] \\ &\quad = P\left[|\mathsf{A}|^2 \le \frac{2^{rE\log_2(1+\mathsf{SNR}|\mathsf{A}|^2\eta(\mathsf{SNR}))}-1}{\mathsf{SNR}\eta(\mathsf{SNR})}\right] \end{aligned} \tag{241}$$

where $\eta(\mathsf{SNR})$ is specified in (89).

Numerical evaluations of (241) are given in [38], [29].

Appendix G

Proof of Theorem IV.2: Define the monotone continuous function

$$f(g) \stackrel{\text{def}}{=} E\left[\left(\frac{1}{g} - \frac{1}{|\mathsf{A}|^2}\right)^+\right]. \tag{242}$$

First, we show that for any $\mathsf{SNR} > 0$, the claimed solution $\overline{g}$ to

$$f(g) = \frac{\mathsf{SNR}}{1+\beta g\mathsf{SNR}} \tag{243}$$

does exist. To see this, note first that for large g, the right side of (243) behaves as $1/(\beta g)$, and, thus, lies above the left side which satisfies $f(g) = o(g^{-1})$. On the other hand, as $g \to 0$, $f(g) \to \infty$, whereas the right side converges to SNR.

Let $\overline{\eta}$ be the positive solution to the equation

$$\eta + \beta E\left[\frac{p(|\mathsf{A}|^2)|\mathsf{A}|^2\eta}{p(|\mathsf{A}|^2)|\mathsf{A}|^2\eta + 1}\right] = 1 \tag{244}$$

with $p(|\mathsf{A}|^2)$ given by (164). Then, the solution to (244) admits a particularly simple expression

$$\overline{\eta} = \frac{1}{1+\beta\overline{g}\mathsf{SNR}}. \tag{245}$$

To verify this, substitute (164), (245) in the left side of (244)

$$\begin{aligned} &\eta + \beta E\left[\frac{p(|\mathsf{A}|^2)|\mathsf{A}|^2\eta}{p(|\mathsf{A}|^2)|\mathsf{A}|^2\eta+1}\right] \\ &= \frac{1}{1+\beta\overline{g}\mathsf{SNR}} + \beta E\left[\frac{\left(\frac{|\mathsf{A}|^2}{\overline{g}}-1\right)^+}{\left(\frac{|\mathsf{A}|^2}{\overline{g}}-1\right)^+ + 1}\right] && (246) \\ &= \frac{1}{1+\beta\overline{g}\mathsf{SNR}} + \beta E\left[\left(1 - \frac{\overline{g}}{|\mathsf{A}|^2}\right)^+\right] && (247) \\ &= \frac{1}{1+\beta\overline{g}\mathsf{SNR}} + \frac{\beta\overline{g}\mathsf{SNR}}{1+\beta\overline{g}\mathsf{SNR}} && (248) \\ &= 1 && (249) \end{aligned}$$

where (248) follows from (165). Thus, (245) is confirmed.

Using (245), the power control law in (164) achieves the following capacity

$$\begin{aligned} \mathcal{C}^{\mathsf{opc}}(\beta,\,\mathsf{SNR}) &= \beta E[\log_2(1+p(|\mathsf{A}|^2)|\mathsf{A}|^2\overline{\eta})] \\ &\quad + \log_2\frac{1}{\overline{\eta}} + (\overline{\eta}-1)\log_2 e && (250) \\ &= \beta E\left[\left(\log_2\frac{|\mathsf{A}|^2}{\overline{g}}\right)^+\right] + \log_2(1+\beta\overline{g}\mathsf{SNR}) \\ &\quad - \frac{\beta\overline{g}\mathsf{SNR}}{1+\beta\overline{g}\mathsf{SNR}}\log_2 e. && (251) \end{aligned}$$

To conclude the proof we show that (251), the capacity achieved by (164), is an upper bound to the capacity achievable with any power control law. To that end, we use the unexpected property of (126) stated in (167). To check (167), one can just verify that the derivative with respect to x of the function in the right side is

$$\beta E\left[\frac{\mathsf{SNR}|\mathsf{A}|^2}{\mathsf{SNR}|\mathsf{A}|^2x+1}\right] - \frac{1}{x} + 1 \tag{252}$$

divided by $\log_2 e$. Upon comparison with (89) we see that (252) vanishes at $x = \eta$. Furthermore, letting

$$Y = \frac{\mathsf{SNR}|\mathsf{A}|^2x}{\mathsf{SNR}|\mathsf{A}|^2x+1} \tag{253}$$

the derivative of (252) with respect to x, evaluated at the stationary point $x = \eta$, is

$$\begin{aligned} \frac{1}{x^2}\left(1-\beta E\left[Y^2\right]\right) &= \frac{1}{\eta^2}\left(1-(1-\eta)\frac{E\left[Y^2\right]}{E[Y]}\right) && (254) \\ &\ge 0 && (255) \end{aligned}$$

where the inequality follows from the fact that $\eta \le 1$ and $Y \le 1$.

Using (167), we see that finding the optimum power control is a maximin problem

$$\max_{\substack{p:\\E[p(|\mathsf{A}|^2)]=\mathsf{SNR}}} \min_{0\le x\le 1} \beta E\left[\log_2(1+p(|\mathsf{A}|^2)|\mathsf{A}|^2 x)\right] + \log_2 \frac{1}{x} + (x-1)\log_2 e$$
$$\le \min_{0\le x\le 1} \max_{\substack{p:\\E[p(|\mathsf{A}|^2)]=\mathsf{SNR}}} \beta E\left[\log_2(1+p(|\mathsf{A}|^2)|\mathsf{A}|^2 x)\right] + \log_2 \frac{1}{x} + (x-1)\log_2 e. \tag{256}$$

Affecting only the first term therein, the maximization in the right side of (256) is immediate to accomplish for any x

$$\max_{\substack{p:\\E[p(|\mathsf{A}|^2)]=\mathsf{SNR}}} \beta E\left[\log_2(1+p(|\mathsf{A}|^2)|\mathsf{A}|^2 x)\right] = \beta E\left[\left(\log_2 \frac{|\mathsf{A}|^2}{g(x)}\right)^+\right] \tag{257}$$

achieved by

$$p(|\mathsf{A}|^2) = \frac{1}{x}\left[\frac{1}{g(x)} - \frac{1}{|\mathsf{A}|^2}\right]^+ \tag{258}$$

with the Lagrange multiplier $g(x)$ chosen so as to satisfy

$$f(g(x)) = x\mathsf{SNR}. \tag{259}$$

It is possible to further upper-bound (256) by fixing a given value of x. Let us choose (cf. (245))

$$x = \overline{\eta} = \frac{1}{1+\beta\overline{g}\mathsf{SNR}}. \tag{260}$$

Then, according to (243), $g(x) = \overline{g}$ and the upper bound to the right side of (256) becomes

$$\beta E\left[\log_2(1+p(|\mathsf{A}|^2)|\mathsf{A}|^2 x)\right] + \log_2 \frac{1}{x} + (x-1)\log_2 e$$
$$= \beta E\left[\left(\log_2 \frac{|\mathsf{A}|^2}{\overline{g}}\right)^+\right] + \log_2(1+\beta\overline{g}\mathsf{SNR}) - \frac{\beta\overline{g}\mathsf{SNR}}{1+\beta\overline{g}\mathsf{SNR}}\log_2 e \tag{261}$$

which is precisely the capacity (251) achievable by (164).

Appendix H

Proof of Theorem IV.3: First note the following lower bound to the capacity (251) achieved by the optimum power control policy:

$$\beta E\left[\left(\log_2 \frac{|\mathsf{A}|^2}{\overline{g}}\right)^+\right] + \log_2(1+\beta\overline{g}\mathsf{SNR}) - \frac{\beta\overline{g}\mathsf{SNR}}{1+\beta\overline{g}\mathsf{SNR}}\log_2 e$$
$$= \beta E\left[\left(\log_2 \frac{|\mathsf{A}|^2}{\overline{g}}\right)^+ - \left(1-\frac{\overline{g}}{|\mathsf{A}|^2}\right)^+ \log_2 e\right] + \log_2(1+\beta\overline{g}\mathsf{SNR}) \tag{262}$$
$$\ge \log_2(1+\beta\overline{g}\mathsf{SNR}) \tag{263}$$

where we used (165) to write (262), and (263) follows from $\log_e x \ge 1 - 1/x$.

Fix an arbitrarily small $\delta > 0$ and choose

$$\mathsf{SNR} = \frac{\delta}{\beta}. \tag{264}$$

Since the random variable $|\mathsf{A}|$ is unbounded, it can be checked in (165) that with this choice of SNR, $\overline{g} \to \infty$ as $\beta \to \infty$.

Thus, as $\beta \to \infty$, the lower bound in (263) also goes to infinity. Moreover, this is achieved with vanishing $\frac{E_b}{N_0}$

$$\frac{E_b^{\mathsf{opc}}}{N_0} = \frac{\beta\mathsf{SNR}}{\mathcal{C}} = \frac{\delta}{\mathcal{C}} \le \frac{\delta}{\log_2(1+\delta\overline{g})}, \tag{265}$$

and the proof is complete.

Appendix I

Proof of (55): Use (48) and (50) to write

$$\frac{E_b}{N_0}_{\min} = \lim_{\mathsf{SNR}\downarrow 0} \frac{\mathsf{SNR}}{\mathcal{C}(\mathsf{SNR})} \tag{266}$$
$$= -\lim_{x\downarrow 0} \frac{E\left[\frac{1}{|\mathsf{A}|^2_{\max}-x} - \frac{1}{\theta}\right]}{E\left[\log_2(|\mathsf{A}|^2_{\max}-x) - \log_2(\theta)\right]} \tag{267}$$
$$= -\lim_{x\downarrow 0} \frac{x}{|\mathsf{A}|^2_{\max}(|\mathsf{A}|^2_{\max}-x)\log_2\left(1-\frac{x}{|\mathsf{A}|^2_{\max}}\right)} \tag{268}$$
$$= \frac{\log_e 2}{|\mathsf{A}|^2_{\max}} \tag{269}$$

where the expectation in (267) is with respect to the random variable θ which is $|\mathsf{A}|^2$ restricted to the interval $(|\mathsf{A}|^2_{\max} - x, |\mathsf{A}|^2_{\max}]$.

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