

# On the Capacity of Cognitive Radios in Multiple Access Networks

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**Abstract**—This paper analyzes the fundamental limits of performance of cognitive radios in a multiple access setting. In the traditional information theoretic model for cognitive radio channel, there is a primary and cognitive transmitter-receiver pair, and the cognitive transmitter knows the message transmitted by the primary transmitter. In the multiple access setting, the primary network is an uplink system with multiple transmitters communicating with the primary receiver with the cognitive transmitter having access to the messages of all the transmitters. This paper analyzes a system where two primary transmitters communicate with a primary receiver in the presence of a cognitive transmitter-receiver pair. The capacity region of this system is derived when the channel gain from the cognitive transmitter to the primary receiver is weak.

## I. INTRODUCTION

Interference channels are prevalent in most communication systems today. However, determining the capacity region of the interference channel has been a long standing open problem for more than three decades except for a few special cases [1]–[4]. Over the last few years significant advances have been made in understanding the performance limits of interference networks [5]–[8]. The cognitive radio channel has been studied as a special form of interference channel where one of the transmitters (the “cognitive” transmitter) gains some knowledge about the transmissions of the other transmitter. Networks with cognitive users are gaining prominence with the development of cognitive radio technology, which is aimed at improving the spectral efficiency and the system performance by designing nodes which can adapt their strategy based on the network setup. The information theoretic model for the cognitive radio channel [9] models the channel as a two user interference channel in which one transmitter (the cognitive transmitter) knows apriori the message transmitted by the other transmitter. Prior work on this channel model includes [9]–

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[13]. More recently, the interference channel with a cognitive relay has been studied in [14]–[16].

In this paper, we study the performance limits of a cognitive radio channel in a multiple access setting. In particular, we consider a system where two primary transmitters communicate their messages to a primary receiver in a multiple access setting, and one cognitive transmitter transmits its message to a cognitive receiver. We assume that the cognitive transmitter knows apriori the messages of both the primary transmitters. In this paper, we derive an outer bound on the capacity region of the cognitive radio channel in a multiple access setting (MACRC) when the channel gain from the cognitive transmitter to the primary receiver is “weak” ( $\leq 1$ ) and show that Gaussian distributions maximize the single letter outer bound. We also derive an achievable region for the MACRC which combines superposition and dirty paper coding techniques [17]. We show that while in the general case the bounds do not meet, in the Gaussian case, the achievable region meets the outer bound when the cross channel gain from the cognitive transmitter to the primary receiver is weak ( $\leq 1$ ).

The rest of the paper is organized as follows: In Section II, we describe the system model for the MACRC. We derive an outer bound on the capacity region for the MACRC in Section III. In Section IV, we derive an achievable region for the MACRC using a combination of superposition and dirty paper coding. In Section V, we show that the achievable region meets the outer bound when the cross channel gain from the cognitive transmitter to the primary receiver is weak ( $\leq 1$ ). Finally, we conclude in Section VI.

Throughout the paper, we denote random variables by capital letters, their realizations by lower case and their alphabets by calligraphic letters (eg.  $X, x$  and  $\mathcal{X}$  respectively). We denote vectors of length  $n$  with boldface letters (e.g.  $\mathbf{x}^n$ ), and the  $i^{th}$  element of a vector  $\mathbf{x}^n$  by  $x_i$ . For any set  $S$ ,  $\bar{S}$  denotes the closure of the convex hull of  $S$ .

## II. SYSTEM MODEL

In this section, we describe the system model for the cognitive radio channel in a multiple access setting (MACRC). In this system, we have two primary transmitters communicating their messages to a primary receiver in a multiple access manner, and one cognitive transmitter communicating its message to a cognitive receiver. We assume that the cognitive transmitter

knows apriori the messages of both the primary transmitters. The system model is described in Figure 1. The channel is described by  $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_c, \mathcal{Y}_1, \mathcal{Y}_c, p(y_1, y_c|x_1, x_2, x_c))$  where  $\mathcal{X}_1, \mathcal{X}_2$  denote the input alphabets of the primary transmitters,  $\mathcal{X}_c$  denotes the input alphabet of the cognitive transmitters, and  $\mathcal{Y}_1$  and  $\mathcal{Y}_c$  denote the output alphabets of the primary and the cognitive receiver.

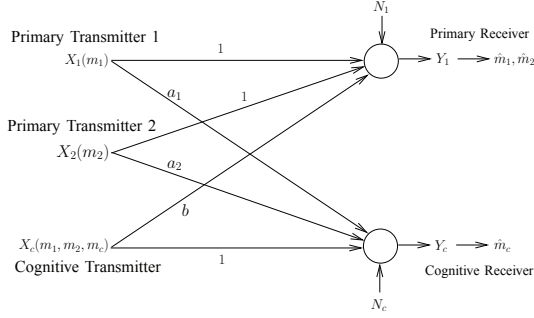


Fig. 1. System Model for Gaussian Cognitive Radio Channel in a Multiple Access Setting

Transmitter  $i$ ,  $i \in \{1, 2\}$  has message  $m_i \in \{1, 2, \dots, 2^{nR_i}\}$  that it wishes to communicate to the primary receiver in a multiple access manner. The cognitive transmitter has message  $m_c \in \{1, 2, \dots, 2^{nR_c}\}$  that it wishes to communicate to the cognitive receiver. The cognitive transmitter has non-causal access to messages of both the primary transmitters. Let  $X_{1i}, X_{2i}, X_{ci}$  and  $Y_{1i}, Y_{ci}$  denote the variables representing the respective channel inputs and outputs at time  $i$ . Note that the channel input from the cognitive transmitter ( $X_{ci}$ ) is a function of all the three messages. For the Gaussian channel, the input-output relationship at time  $i$  can be expressed by the system equations given below:

$$\begin{aligned} Y_{1i} &= X_{1i} + X_{2i} + bX_{ci} + N_{1i} \\ Y_{ci} &= a_1X_{1i} + a_2X_{2i} + X_{ci} + N_{ci}. \end{aligned} \quad (1)$$

where  $a_1, a_2$ , and  $b$  represent the channel gains as shown in Figure 1. Throughout the paper, we assume that the channel gains are positive, and the results can be readily extended when the channel gains are negative.  $N_{1i}$  and  $N_{ci}$  denote the additive noise at the two receivers which are i.i.d. Gaussian random variables distributed as  $\mathcal{N}(0, 1)$ . The channel inputs must satisfy the following power constraints:

$$\frac{1}{n} \sum_{i=1}^n E[X_{ji}^2] \leq P_j, \quad j \in \{1, 2, c\}. \quad (2)$$

A  $(2^{nR_1}, 2^{nR_2}, 2^{nR_c}, n, Pe)$  code consists of message sets  $M_1 = \{1, \dots, 2^{nR_1}\}$ ,  $M_2 = \{1, \dots, 2^{nR_2}\}$  and  $M_c \in \{1, \dots, 2^{nR_c}\}$ , three encoding functions

$$\begin{aligned} f_1: M_1 &\rightarrow \mathcal{X}_1^n, & f_2: M_2 &\rightarrow \mathcal{X}_2^n, \\ f_c: M_1 \times M_2 \times M_c &\rightarrow \mathcal{X}_c^n, \end{aligned} \quad (3)$$

and two decoding functions

$$g_1: \mathcal{Y}_1^n \rightarrow M_1 \times M_2, \quad g_2: \mathcal{Y}_c^n \rightarrow M_c, \quad (4)$$

such that the transmitted codewords  $\mathbf{X}_1^n, \mathbf{X}_2^n$  and  $\mathbf{X}_c^n$  satisfy the power constraints given by (2) and the overall decoding error probability at both the receivers is  $\leq Pe$ .

A rate triple  $(R_1, R_2, R_c)$  is achievable if there exists a sequence of  $(2^{nR_1}, 2^{nR_2}, 2^{nR_c}, Pe^{(n)})$  codes such that  $Pe^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$ . The capacity region of the MACRC is then the set of all rate triples  $(R_1, R_2, R_c)$  that are achievable, and is denoted by  $\mathcal{C}_{MACRC}$ .

### III. OUTER BOUND ON THE CAPACITY REGION OF MACRC

In this section, we derive an outer bound on the capacity region of the MACRC when the cross channel gain from the cognitive transmitter to the primary receiver,  $b \leq 1$ . Let  $\mathcal{P}_o$  denote the set of all probability distributions  $P_o(\cdot)$  given by

$$P_o(q, x_1, x_2, u, v, x_c) = p(q)p(x_1|q)p(x_2|q)p(u, v|x_1, x_2, q)p(x_c|u, v, x_1, x_2, q). \quad (5)$$

Let  $\mathcal{R}_{out}(P_o)$  denote the set of rate triples  $(R_1, R_2, R_c)$  given by

$$\begin{aligned} R_1 &\leq I(X_1, U; Y_1|V, X_2, Q) \\ R_2 &\leq I(X_2, V; Y_1|U, X_1, Q) \\ R_1 + R_2 &\leq I(X_1, U, X_2, V; Y_1|Q) \\ R_c &\leq I(X_c; Y_c|X_1, U, X_2, V, Q) \\ R_1, R_2, R_c &\geq 0 \end{aligned} \quad (6)$$

Let  $\mathcal{R}_{out}$  denote the set of rate triples given by

$$\mathcal{R}_{out} = \overline{\bigcup_{P_o(\cdot) \in \mathcal{P}_o} \mathcal{R}_{out}(P_o)} \quad (7)$$

Then, the following theorem describes an outer bound on the capacity region of the discrete memoryless MACRC.

*Theorem 1:* For the Gaussian cognitive radio channel in a MAC setting, when the cross channel gain satisfies  $b \leq 1$ , the capacity region  $\mathcal{C}_{MACRC}$  satisfies

$$\mathcal{C}_{MACRC} \subseteq \mathcal{R}_{out}. \quad (8)$$

*Proof:* We fix a probability distribution  $P_o(\cdot) \in \mathcal{P}_o$ . Then, we have

$$\begin{aligned} nR_1 &\stackrel{(a)}{=} H(M_1|M_2) \\ &\stackrel{(b)}{\leq} I(M_1; Y_1^n|M_2) + n\epsilon_n^1 \\ &\stackrel{(c)}{=} \sum_{i=1}^n H(Y_{1i}|M_2, Y_1^{i-1}, X_{2i}) + n\epsilon_n^1 \\ &\quad - \sum_{i=1}^n H(Y_{1i}|M_2, Y_1^{i-1}, M_1, X_{1i}, X_{2i}) \\ &= \sum_{i=1}^n I(U_i, X_{1i}; Y_{1i}|V_i, X_{2i}) + n\epsilon_n^1 \end{aligned} \quad (9)$$

where  $V_i = M_2, Y_1^{i-1}$  and  $U_i = M_1, Y_1^{i-1}$ . Here, (a) follows from the independence of  $M_1$  and  $M_2$ , (b) follows from Fano's inequality and (c) follows from the fact that  $X_{2i}$  is a function of  $M_2$ .

A similar set of inequalities can be derived to show that

$$nR_2 \leq (V_i, X_{2i}; Y_{1i}|U_i, X_{1i}) + n\epsilon_n^2 \quad (10)$$

Subsequently, we can show that

$$\begin{aligned} n(R_1 + R_2) &= H(M_1, M_2) \\ &\leq I(M_1, M_2; Y_1^n) + n\epsilon_n^{1,2} \\ &\leq \sum_{i=1}^n H(Y_{1i}) + n\epsilon_n^{1,2} \\ &\quad - \sum_{i=1}^n H(Y_{1i}|M_1, M_2, Y_1^{i-1}, X_{1i}, X_{2i}) \\ &\leq \sum_{i=1}^n I(U_i, X_{1i}, V_i, X_{2i}; Y_{1i}) + n\epsilon_n^{1,2} \end{aligned} \quad (11)$$

and

$$\begin{aligned} nR_c &\stackrel{(d)}{=} H(M_c|M_1, M_2, X_1^n, X_2^n) \\ &\stackrel{(e)}{\leq} I(M_c; Y_c^n|M_1, M_2, X_1^n, X_2^n) + n\epsilon_n^c \\ &\stackrel{(f)}{\leq} \sum_{i=1}^n H(Y_{ci}|Y_c^{i-1}, M_1, M_2, X_1^n, X_2^n) \\ &\quad - \sum_{i=1}^n H(Y_{ci}|X_{ci}, X_{1i}, X_{2i}) + n\epsilon_n^c \\ &\stackrel{(g)}{=} \sum_{i=1}^n H(Y_{ci}|Y_c^{i-1}, Y_1^{i-1}, M_1, M_2, X_1^n, X_2^n) \\ &\quad - \sum_{i=1}^n H(Y_{ci}|X_{ci}, X_{1i}, X_{2i}) + n\epsilon_n^c \\ &\leq \sum_{i=1}^n H(Y_{ci}|Y_1^{i-1}, M_1, M_2, X_1^n, X_2^n) \\ &\quad - \sum_{i=1}^n H(Y_{ci}|X_{ci}, X_{1i}, X_{2i}) + n\epsilon_n^c \\ &\leq \sum_{i=1}^n I(X_{ci}; Y_{ci}|X_{1i}, U_i, X_{2i}, V_i) + n\epsilon_n^c \end{aligned} \quad (12)$$

where (d) follows from independence between  $M_c$ ,  $M_1$  and  $M_2$ , (e) follows from Fano's inequality, (f) follows from the memoryless nature of the channel and (g) follows from the degraded nature of the channel (with the assumption  $b < 1$ ).

Defining  $Q$  to be the time-sharing random variable that is uniformly distributed over  $\{1, \dots, n\}$  and defining  $(Q, X_1, X_2, U, V, X_c, Y_1, Y_c) = (Q, X_{1,Q}, X_{2,Q}, U_Q, V_Q, X_{c,Q}, Y_{1,Q}, Y_{c,Q})$  yields the desired outer bound. ■

#### IV. ACHIEVABLE REGION

In this section, we describe an achievable region for the MACRC. The coding strategy combines superposition and

dirty paper coding techniques. Let  $\mathcal{P}_{in}$  denote the set of probability distributions  $P_{in}(\cdot)$  given by

$$P_{in}(q, x_1, u, x_2, v, x_c, t) = p(q)p(u, x_1|q)p(v, x_2|q) p(t, x_c|u, v, x_1, x_2). \quad (13)$$

Let  $\mathcal{R}_{in}(P_{in})$  denote the set of rate triples  $(R_1, R_2, R_c)$  given by

$$\begin{aligned} R_1 &\leq I(X_1, U; Y_1|V, X_2, Q) \\ R_2 &\leq I(X_2, V; Y_1|U, X_1, Q) \\ R_1 + R_2 &\leq I(X_1, U, X_2, V; Y_1|Q) \\ R_c &\leq I(T; Y_c|Q) - I(T; X_1, U, X_2, V|Q) \\ R_1, R_2, R_c &\geq 0. \end{aligned} \quad (14)$$

Let  $\mathcal{R}_{in}$  denote the set of rate triples  $(R_1, R_2, R_c)$  given by

$$\mathcal{R}_{in} = \overline{\bigcup_{P_{in}(\cdot) \in \mathcal{P}_{in}} \mathcal{R}_{in}(P_{in})}. \quad (15)$$

Then, the following theorem describes an achievable region for the MACRC.

*Theorem 2:* The capacity region of the MACRC satisfies

$$\mathcal{R}_{in} \subseteq \mathcal{C}_{MACRC}. \quad (16)$$

*Proof:* For simplicity, we shall present the coding-scheme for the case where the time-sharing random variable  $Q$  is deterministic. It should be kept in mind that the introduction of time-sharing may increase the region by convexification. We fix a  $P_{in}(\cdot) \in \mathcal{P}_{in}$  and show that the region  $\mathcal{R}_{in}(P_{in})$  is achievable. We now describe codebook generation at the transmitters.

**Codebook Generation:** Transmitter 1 generates  $2^{nR_1}$  vector pairs  $X_1^n, U^n \sim \prod_{i=1}^n p(x_{1i}, u_i)$  and indexes them using  $j \in \{1, \dots, 2^{nR_1}\}$ . Similarly, transmitter 2 generates  $2^{nR_2}$  vector pairs  $X_2^n, V^n \sim \prod_{i=1}^n p(x_{2i}, v_i)$  and indexes them using  $k \in \{1, \dots, 2^{nR_2}\}$ . The cognitive transmitter generates  $2^{nR_c}$   $T^n \sim \prod_{i=1}^n p(t_i)$  and places them uniformly in  $2^{nR_c}$  bins. We next describe the transmission strategy at the three transmitters.

**Transmission strategy:** Given message  $m_1 \in \{1, \dots, 2^{nR_1}\}$ , transmitter 1 determines  $X_1^n(m_1)$  and transmits it. Similarly, for message  $m_2 \in \{1, \dots, 2^{nR_2}\}$ , transmitter 2 transmits  $X_2^n(m_2)$ . As the cognitive transmitter has access to messages  $m_1$  and  $m_2$ , it can determine  $X_1^n(m_1), U^n(m_1), X_2^n(m_2), V^n(m_2)$ . For message  $m_c \in \{1, \dots, 2^{nR_c}\}$ , the cognitive transmitter looks for a sequence  $T^n$  in bin  $m_c$  such that  $(T^n(M_c), X_1^n(m_1), U^n(m_1), X_2^n(m_2), V^n(m_2))$  is jointly typical. If such a  $T^n$  is located, then an  $X_c^n$  is generated according to the conditional  $\prod_{i=1}^n p(x_{ci}|x_{1i}, u_i, x_{2i}, v_i)$  and transmitted. We next describe the decoding strategy at the two receivers.

**Reception:** The primary receiver determines indices  $(\hat{m}_1, \hat{m}_2)$  such that  $(X_1^n(\hat{m}_1), U^n(\hat{m}_1), X_2^n(\hat{m}_2), V^n(\hat{m}_2), Y_1^n)$  is

jointly typical. The cognitive receiver looks for a  $T^n$  such that  $(T^n, Y_c^n)$  is jointly typical. The cognitive receiver then determines the bin index of  $T^n$  and declares that as the decoded message. We next describe the probability of error of encoding and decoding process.

**Decoding Error at Primary Receiver:** Let  $E_{j,k}$  denote the event that  $(X_1^n(j), U^n(j), X_2^n(k), V^n(k), Y_1^n)$  is jointly typical. We assume that the transmitters transmitted messages  $m_1$  and  $m_2$ . Then the probability of decoding error is given by

$$Pe = \Pr \left( E_{m_1, m_2}^c \cup \bigcup_{(j,k) \neq (m_1, m_2)} E_{j,k} \right).$$

The probability of decoding error can be upper bounded by

$$Pe \leq \Pr(E_{m_1, m_2}^c) + \sum_{j \neq m_1} \Pr(E_{j, m_2}) + \sum_{k \neq m_2} \Pr(E_{m_1, k}) + \sum_{j \neq m_1, k \neq m_2} \Pr(E_{j, k}).$$

For any  $\epsilon > 0$ , there exists  $n$  large enough such that the first term  $\Pr(E_{m_1, m_2}^c) \leq \epsilon$ . The other three terms can be made smaller than  $\epsilon$  if

$$\begin{aligned} R_1 &\leq I(X_1, U; Y_1 | X_2, V) - 3\epsilon \\ R_2 &\leq I(X_2, V; Y_1 | X_1, U) - 3\epsilon \\ R_1 + R_2 &\leq I(X_1, U, X_2, V; Y_1) - 4\epsilon. \end{aligned} \quad (17)$$

**Encoding Error at Cognitive Transmitter:** An encoding error occurs at the cognitive transmitter if no  $T^n$  in bin index  $m_c$  can be found such that  $(T^n, X_1^n(m_1), U^n(m_1), X_2^n(m_2), V^n(m_2))$  is jointly typical. The probability of this happening can be upper bounded by

$$Pe \leq (1 - 2^{-nI(T; X_1, U, X_2, V)})^{2^{n(\tilde{R}_c - R_c)}}.$$

The probability of encoding error can be made arbitrarily small if

$$\tilde{R}_c \geq R_c + I(T; X_1, U, X_2, V). \quad (18)$$

**Decoding Error at Cognitive Receiver:** The cognitive receiver determines a bin index  $\hat{m}_c$  and a sequence  $T^n$  from that bin such that  $(T^n, Y_c^n)$  is jointly typical. To analyze the probability of error, we assume that the transmitter wished to communicate message  $m_c$  and no error occurred at the cognitive encoder. Then, a decoding error occurs if no  $T^n$  in bin  $m_c$  is jointly typical with  $Y_c^n$ , or if a  $T^n$  from a different bin is jointly typical with  $Y_c^n$ . The probability that no  $T^n$  in bin  $m_c$  is jointly typical with  $Y_c^n$  can be made arbitrarily small for suitably large  $n$ . The probability that a  $T^n$  from a different bin is jointly typical with  $Y_c^n$  can be made small if

$$\tilde{R}_c \leq I(T; Y_c) - 3\epsilon. \quad (19)$$

Choosing  $\tilde{R}_c = R_c + I(T; X_1, U, X_2, V) + \epsilon$ , we get

$$R_c \leq I(T; Y_c) - I(T; X_1, U, X_2, V) - 4\epsilon. \quad (20)$$

Hence the region described by  $\mathcal{R}_{in}$  is achievable. ■

## V. OPTIMALITY OF THE ACHIEVABLE REGION

In this section, we show that for the Gaussian MACRC, when the cross channel gain from the cognitive transmitter to the primary receiver is small enough (i.e.  $b \leq 1$ ), the achievable region described by Theorem 2 meets the outer bound described in Theorem 1. Let  $\rho_1, \rho_2 \in [0, 1]$  such that  $\rho_1^2 + \rho_2^2 \leq 1$ . Define  $\Delta = 1 - \rho_1^2 - \rho_2^2$ . Define the function  $L: \mathbb{R}_+ \rightarrow \mathbb{R}$  by  $L(x) = \frac{1}{2} \log(1+x)$ . Let  $\mathcal{R}(\rho_1, \rho_2)$  denote the set of rate triples  $(R_1, R_2, R_c) \in \mathbb{R}_+^3$  given by

$$\begin{aligned} R_1 &\leq L \left( \frac{(\sqrt{P_1} + b\sqrt{P_c}\rho_1)^2}{1 + b^2 P_c \Delta} \right) \\ R_2 &\leq L \left( \frac{(\sqrt{P_2} + b\sqrt{P_c}\rho_2)^2}{1 + b^2 P_c \Delta} \right) \\ R_1 + R_2 &\leq L \left( \frac{(\sqrt{P_1} + b\sqrt{P_c}\rho_1)^2 + (\sqrt{P_2} + b\sqrt{P_c}\rho_2)^2}{1 + b^2 P_c \Delta} \right) \\ R_c &\leq L(P_c \Delta). \end{aligned} \quad (21)$$

Let  $\mathcal{R}$  denote the set of rate triples  $(R_1, R_2, R_c)$  described by

$$\mathcal{R} = \overline{\bigcup_{\rho_1, \rho_2 \in [0, 1]: \rho_1^2 + \rho_2^2 \leq 1} \mathcal{R}(\rho_1, \rho_2)}. \quad (22)$$

Then, the following theorem describes the capacity region of the MACRC when the cross channel gain satisfies  $b \leq 1$ .

**Theorem 3:** For the Gaussian MACRC, when the cross channel gain satisfies  $b \leq 1$  in a MACRC, the capacity region of the channel is given by

$$\mathcal{C}_{MACRC} = \mathcal{R}. \quad (23)$$

### A. Proof of Inner Bound

Consider the achievable region given by (15). Take in (14),  $(X_1, X_2, X_c)$  jointly Gaussian, with zero means and variances  $(P_1, P_2, P_c)$  respectively and where  $E(X_1 X_2) = 0$  and  $E(X_c X_i) = \rho_i \sqrt{P_i P_c}$  for  $i = 1, 2$ . Choose  $U$  and  $V$  to be deterministic random variables.

The random variable  $T$  is defined as follows

$$T = X_c + \alpha_1 X_1 + \alpha_2 X_2,$$

where  $\alpha_1$  and  $\alpha_2$  are constants to be specified. It is evident that for this choice of random variables we have,

$$\begin{aligned} R_c &= I(T; Y_c) - I(T; X_1, U, X_2, V) \\ &= I(T; Y_c) - I(T; X_1, X_2) \\ &= I(T; Y_c | X_1, X_2) - I(T; X_1, X_2 | Y_c) \\ &= I(X_c; Y_c | X_1, X_2) - I(T; X_1, X_2 | Y_c). \end{aligned} \quad (24)$$

From [18, Lemma 1], there exists  $\alpha_1^*, \alpha_2^*$  such that  $I(T; X_1, X_2 | Y_c) = 0$ . We choose  $\alpha_1 = \alpha_1^*$  and  $\alpha_2 = \alpha_2^*$ .

Therefore, we get

$$\begin{aligned}
R_c &= I(T; Y_c) - I(T; X_1, U, X_2, V) \\
&= I(T; Y_c) - I(T; X_1, X_2) \\
&= I(X_c; Y_c | X_1, X_2, U) \\
&= L(P_c(1 - \rho_1^2 - \rho_2^2)).
\end{aligned} \tag{25}$$

With these choice of random variables, we observe that

$$\begin{aligned}
h(Y_1 | X_2) &= \frac{1}{2} \log(2\pi e(1 + P_1 + 2b\sigma_1 + b^2 P_c(1 - \rho_2^2))) \\
h(Y_1 | X_1) &= \frac{1}{2} \log(2\pi e(1 + P_2 + 2b\sigma_2 + b^2 P_c(1 - \rho_1^2))) \\
h(Y_1) &= \frac{1}{2} \log(2\pi e(1 + P_1 + P_2 + 2b(\sigma_1 + \sigma_2) + b^2 P_c)) \\
h(Y_1 | X_1, X_2) &= \frac{1}{2} \log(2\pi e(1 + b^2 P_c(1 - \sigma_1^2 - \sigma_2^2))).
\end{aligned}$$

Substituting the above expressions and (25) into the achievable region in (15), it is easy to see that the achievable region matches the rate region given by  $\mathcal{R}$ .

### B. Outer Bound

In this section, we show that Gaussian distributions maximize the outer bound derived in Section III. From Section III, we have the outer bound as the union over all the rate triples that satisfy

$$\begin{aligned}
R_1 &\leq h(Y_1 | V, X_2, Q) - h(Y_1 | X_1, U, X_2, V, Q) \\
R_2 &\leq h(Y_1 | U, X_1, Q) - h(Y_1 | X_1, U, X_2, V, Q) \\
R_1 + R_2 &\leq h(Y_1 | Q) - h(Y_1 | X_1, U, X_2, V, Q) \\
R_c &\leq h(Y_c | X_1, U, X_2, V, Q) - h(N_c).
\end{aligned}$$

for some  $P_{Q, X_1, U, X_2, V}$  where  $Y_1 = X_1 + X_2 + bX_c + N_1$ ,  $Y_c = X_c + a_1 X_1 + a_2 X_2 + N_c$  and  $X_1$  and  $X_2$  are independent given  $Q$ . In this section, we derive the outer bound for a degenerate  $Q$  (that is, we assume that  $X_1$  and  $X_2$  are independent). The overall outer bound is in fact the convex hull over the entire obtained region.

Since  $0 \leq I(X_c; Y_c | X_1, U, X_2, V) \leq \frac{1}{2} \log(1 + P_c)$ , there exists some  $\gamma \in [0, 1]$  such that

$$I(X_c; Y_c | X_1, U, X_2, V) = \frac{1}{2} \log(1 + \gamma P_c),$$

and consequently

$$h(Y_c | X_1, U, X_2, V) = \frac{1}{2} \log(2\pi e(1 + \gamma P_c)). \tag{26}$$

Let  $J$  be a Gaussian noise with variance  $1 - b^2$ . Using the Entropy Power Inequality, we obtain

$$\begin{aligned}
2^{2h(Y_1 | X_1, U, X_2, V)} &= 2^{2h(bX_c + N_1 | X_1, U, X_2, V)} \\
&= 2^{2h(bY_c + J | X_1, U, X_2, V)} \\
&\geq 2^{2h(bY_c | X_1, U, X_2, V)} + 2^{2h(J)} \\
&= 2\pi e(b^2(1 + \gamma P_c) + 1 - b^2) \\
&= 2\pi e(1 + \gamma b^2 P_c).
\end{aligned} \tag{27}$$

Next, we recall that for a given covariance matrix of  $(X_1, X_2, X_c, U, V)$ , the conditional entropies  $h(Y_1 | V, X_2)$ ,

$h(Y_1 | U, X_1)$  and  $h(Y_1)$  are maximized if  $(X_1, X_2, X_c, U, V)$  is a Gaussian vector. Also, we have that

$$h(Y_1 | X_1, U) \leq h(Y_1 | X_1) \text{ and } h(Y_1 | X_2, V) \leq h(Y_1 | X_2).$$

Finally, for Gaussian  $X_1, X_2, X_c$  such that  $X_1$  and  $X_2$  are independent and  $E[X_i X_c] = \rho_i \sqrt{P_i P_c}$ , we observe that

$$\begin{aligned}
\frac{1}{2} \log(2\pi e(1 + \gamma P_c)) &= h(Y_c | X_1, U, X_2, V) \\
&= h(X_c + N | X_1, U, X_2, V) \\
&\leq h(X_c + N | X_1, X_2) \\
&= \frac{1}{2} \log(2\pi e(1 + \Delta P_c)).
\end{aligned} \tag{28}$$

Hence, we have  $\gamma \leq \Delta = 1 - \rho_1^2 - \rho_2^2$ .

Hence, the outer bound reduces to

$$\begin{aligned}
R_1 &\leq \frac{1}{2} \log\left(\frac{1 + P_1 + 2b\sigma_1 + b^2 P_c(1 - \rho_2^2)}{1 + b^2 P_c \gamma}\right) \\
R_2 &\leq \frac{1}{2} \log\left(\frac{1 + P_2 + 2b\sigma_2 + b^2 P_c(1 - \rho_1^2)}{1 + b^2 P_c \gamma}\right) \\
R_1 + R_2 &\leq \frac{1}{2} \log\left(\frac{1 + P_1 + P_2 + 2b(\sigma_1 + \sigma_2) + b^2 P_c}{1 + b^2 P_c \gamma}\right) \\
R_c &\leq \frac{1}{2} \log(1 + \gamma P_c).
\end{aligned} \tag{29}$$

where the outer bound is optimized over all  $\rho_1, \rho_2 \in [0, 1]$  such that  $\rho_1^2 + \rho_2^2 \leq 1$  and  $\gamma \leq \Delta$ .

We note that if one substitutes  $\gamma = \Delta$  into (29), we get the desired region (22). The following lemma concludes the proof of the outer bound of Theorem 3, by showing that it is sufficient to consider  $\gamma = \Delta$ .

*Lemma 1:* The region of all rate triples  $(R_1, R_2, R_c)$  given by

$$\begin{aligned}
R_1 &\leq \frac{1}{2} \log\left(\frac{1 + P_1 + 2b\sigma_1 + b^2 P_c(1 - \rho_2^2)}{1 + b^2 P_c \gamma}\right) \\
R_2 &\leq \frac{1}{2} \log\left(\frac{1 + P_2 + 2b\sigma_2 + b^2 P_c(1 - \rho_1^2)}{1 + b^2 P_c \gamma}\right) \\
R_1 + R_2 &\leq \frac{1}{2} \log\left(\frac{1 + P_1 + P_2 + 2b(\sigma_1 + \sigma_2) + b^2 P_c}{1 + b^2 P_c \gamma}\right) \\
R_c &\leq \frac{1}{2} \log(1 + \gamma P_c)
\end{aligned}$$

for some  $(\sigma_1, \sigma_2) = (\sqrt{P_1 P_c} \rho_1, \sqrt{P_2 P_c} \rho_2)$  such that  $0 \leq \rho_1^2 + \rho_2^2 \leq 1$  and some  $\gamma \in [0, \Delta]$ ,  $\Delta = (1 - \rho_1^2 - \rho_2^2)$  remains the same if one takes  $\gamma = \Delta$  (and therefore equal to the region (22)).

*Proof:* Fix  $R_c = \frac{1}{2} \log(1 + d P_c)$ . To obtain this rate,  $\Delta$  cannot be smaller than  $d$ . Consider therefore  $\Delta \in [d, 1]$ . Denote

$$\begin{aligned}
c(\Delta) &= L(b^2 \Delta P_c) \\
f_1(\rho_1, \rho_2) &= L(P_1 + 2b\sigma_1 + b^2 P_c(1 - \rho_2^2)) \\
f_2(\rho_1, \rho_2) &= L(P_2 + 2b\sigma_2 + b^2 P_c(1 - \rho_1^2)) \\
f_3(\rho_1, \rho_2) &= L(P_1 + P_2 + 2b(\sigma_1 + \sigma_2) + b^2 P_c).
\end{aligned} \tag{30}$$

For  $\gamma = \Delta$  and the rate  $R_c$  we fixed, the region becomes

$$\begin{aligned} R_1 &\leq f_1(\rho_1, \rho_2) - c(\Delta) \\ R_2 &\leq f_2(\rho_1, \rho_2) - c(\Delta) \\ R_1 + R_2 &\leq f_3(\rho_1, \rho_2) - c(\Delta) \\ R_c &= \frac{1}{2} \log(1 + dP_c) \end{aligned} \quad (31)$$

where  $\rho_1^2 + \rho_2^2 = 1 - \Delta$  and  $\Delta \in [d, 1]$ .

If we allow  $\gamma \leq \Delta$ , it is obvious that the optimal  $\gamma$  is  $d$  and the region becomes

$$\begin{aligned} R_1 &\leq f_1(\rho_1, \rho_2) - c(d) \\ R_2 &\leq f_2(\rho_1, \rho_2) - c(d) \\ R_1 + R_2 &\leq f_3(\rho_1, \rho_2) - c(d) \\ R_c &= \frac{1}{2} \log(1 + dP_c) \end{aligned} \quad (32)$$

where  $\rho_1^2 + \rho_2^2 = 1 - \Delta$  and  $\Delta \in [d, 1]$ .

The regions (31) and (32) would coincide iff the optimal  $\Delta$  in (31) as well as in (32) is  $d$ . We would show that this is indeed the case and will establish that the optimal  $\gamma$  is equal to  $\Delta$ .

**The optimal  $\Delta$  in (31) is  $d$ :** First, we observe that the sum of the bounds on the individual rates  $R_1, R_2$  in (31) is never smaller than the sum-rate bound. That is,

$$f_1(\rho_1, \rho_2) - c(\Delta) + f_2(\rho_1, \rho_2) - c(\Delta) > f_3(\rho_1, \rho_2) - c(\Delta).$$

This implies that region (31) is basically determined by the vertex points of pentagons. Hence, a vertex point of interest in (31) is determined either by the bounds on  $R_1 + R_2$  and  $R_1$ , or by the bounds on  $R_1 + R_2$  and  $R_2$  (but not simultaneously by the two bounds on the individual rates  $R_1$  and  $R_2$ ). First, assume that the determining bounds are those of  $R_1 + R_2$  and  $R_2$ . Let  $\tilde{\rho}_1 \in [0, \sqrt{1-d}]$  be the correlation coefficient that achieve this vertex point and let  $\tilde{\sigma}_1$  be the corresponding correlation. It is easy to realize that for fixed  $\rho_1$  the functions  $f_2, f_3$  are decreasing with  $\Delta$ , and therefore the minimal possible  $\Delta$  for this vertex point is the optimal, i.e.,  $\Delta = d$ .

Similarly, if the determining bounds are those of  $R_1 + R_2$  and  $R_1$  we notice that for fixed  $\rho_2$ , the functions  $f_1, f_3$  are decreasing with  $\Delta$ , and therefore the optimal  $\Delta$  for this vertex point is the minimal, i.e.,  $\Delta = d$ .

**The optimal  $\Delta$  in (32) is  $d$ :** We observe that the sum of the bounds on the individual rates  $R_1, R_2$  is never smaller than the sum-rate bound in (32) too. That is,

$$f_1(\rho_1, \rho_2) - c(d) + f_2(\rho_1, \rho_2) - c(d) > f_3(\rho_1, \rho_2) - c(d).$$

Hence, similarly to (31), a vertex point of interest in (32) is determined either by the bounds on  $R_1 + R_2$  and  $R_1$ , or by the bounds on  $R_1 + R_2$  and  $R_2$ . And, similarly, the arguments

- for fixed  $\rho_1$  the functions  $f_2, f_3$  are decreasing with  $\Delta$
- for fixed  $\rho_2$  the functions  $f_1, f_3$  are decreasing with  $\Delta$ ,

are sufficient to prove that the optimal  $\Delta$  is  $d$ . This concludes the proof of Lemma 1 and Theorem 3. ■

## VI. CONCLUSIONS

In this paper, we analyzed the capacity region of the cognitive radio channel in a MAC setting. We derived an outer bound on the capacity region of the MACRC when the cross channel gain satisfies  $b \leq 1$  and showed that Gaussian distributions maximize the outer bound. We derived an achievable region using superposition and dirty paper coding at the cognitive transmitter. Finally, we proved that when the cross channel gain  $b \leq 1$ , the achievable region achieves the entire capacity region of the Gaussian MACRC.

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