Some Minimax Detection and Estimation Problems in a Resetless Space Setting

Sieghe Verdé and H. Vincent Poor
Department of Electrical Engineering and
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Urbana, IL 61801

Abstract

The minimax approach to the design of systems which are robust with respect to modeling uncertainties is studied employing a game theoretic formulation in which the performance functional and the sets of modeling uncertainties and admissible design policies are arbitrary. It is shown that if the performance functional and the uncertainty set are convex then a certain type of regularity condition on the functional is sufficient in order to ensure that the optimal strategy for a least favorable element of the uncertainty set is minimax robust. The efficacy of this approach is tested in a MILOS space formulation of the problem of matched filtering, Weiner filtering, quadratic detection and output energy filtering, in which uncertainties in their respective signal and noise models are assumed to exist.

I. Introduction

One of the major techniques for designing systems which are robust with respect to modeling uncertainties in the minimax approach, in which the goal is the optimization of worst-case performance. In most of the decision theoretic works that follow this approach, a common structure can be identified in the form of a game in which a certain performance functional depends on the elements selected by the minimizing and maximizing players from a pair of prespecified sets containing the uncertain quantities of the model and the admissible design strategies. Motivated by the applications considered here, the elements of the uncertainty set and of the set of design strategies will henceforth be referred to as operating points and filters respectively. The cases in which there exists an admissible analytical solution for finding robust filters are those for which saddle points exist. A filter F and an operating point P are said to form a saddle point if, for fitting F, any other filter different from H has worse performance, i.e., H is the optimal filter for P, and if, fitting H, any other operating point different from P gives better performance, i.e., H has its worst performance when P is present. If there exists such a filter H, then it is in the sought-after minimax robust filter, because its worst-case performance is attained at P and any other filter has worse behavior at P. Further, suppose that we have a finite set of filters F and a discrete set of points in the uncertainty space, then P is the element whose filter achieves the worst optimal performance, and hence is referred to as the least favorable operating point. Note that the saddle point property is not necessary for the robust filter to exist; however, if it holds, the robust filter has the convenient feature of being the optimal filter for one of the operating

II. Saddle Point Solutions and Regular Pairs

Denote by $\mathbf{X}$ a space of filters, and by $\mathbf{Z}$ a space of operating points. The payoff function $F$ is a real function $F: \mathbf{X} \times \mathbf{Z} \rightarrow \mathbb{R}$.

Suppose that $F: \mathbf{X} \times \mathbf{Z} \rightarrow \mathbb{R}$ is the set of allowable filters and $C: \mathbf{Z} \rightarrow \mathbb{R}$ is the set of possible operating points (i.e., the uncertainty set). According to the standard terminology, the triplet $(F, C, \mathbf{Z})$ will be referred to as a game, the function $F$ is maximized over $\mathbf{X}$ and minimized over $\mathbf{Z}$. The following definitions will be used:

1. $\hat{F}(\mathbf{X})$ is an optimal filter for $\mathbf{X}$ if $\hat{F}(\mathbf{X}) = \max_{\mathbf{X}} F(\mathbf{X}, \mathbf{Z})$.

2. $\hat{F}(\mathbf{X})$ is the worst operating point for $\mathbf{Z}$ if $\hat{F}(\mathbf{X}) = \min_{\mathbf{Z}} F(\mathbf{X}, \mathbf{Z})$.

This research was supported in part by the U. S. Army research Office under Contract DAAG29-81-E-0062, in part by the U. S. Office of Naval Research under Contract N00014-82-E-0014, and in part by the U. S. National Science Foundation under Grant ECS-82-12080.

921-2216/83/0000-292 $1.00 © 1983 IEEE
(iii) \( h_r \) is a minimax robust filter for the game
\[ h_r \in \arg \max_{h} \inf_{f \in Q} M(h, f). \]

(iv) \( q_h \) is a least favorable operating point for the game
\[ (h, q_h) \] if
\[ q_h \in \arg \min_{q} M(h, q). \]

(v) \( (h, q_h) \) is a saddle point solution to the game
\[ (h, q_r) \] if for every \( (h^*, q^*) \in Q \times Q \),
\[ M(h^*, q^*) \leq M(h, q^*) \leq M(h, q^*) \]

(vi) \( (h, q_h) \) is a saddle point solution to the game
\[ (h, q_r) \] if for every \( q \neq q \) such that \( q = (1 - \alpha) q_r + \alpha q \) for \( \alpha \in [0, 1] \), we have
\[ M(h, q) - M(h, q_r) = o(\alpha). \]

If \( (h, q_r) \) is a saddle point solution to the game
\[ (h, q_r) \], then \( h \) is the worst-case minimax robust filter for \( (h, q_r) \). The reverse implication also holds, i.e., a minimax robust filter and a least favorable operating point do not need to form a saddle point solution, however we will focus attention on the existence and characterization of minimax robust filters that form saddle points.

Noting sufficient conditions for a game to have saddle point solutions is the main goal of minimax theory. In general, these conditions require some topological properties of the sets of the game and continuity and quasi concave-convexity of the payoff function on the maximizing and minimizing sets respectively (cf. for example [1] and [5]). Frequently, when the convexity requirements are not fulfilled for a particular payoff function, the problem is reformulated by allowing randomized strategies, i.e., the original sets are replaced by sets of probability distributions defined on them and the payoff function is replaced by its expected value. Due to the linearity of the expected value (on the probability distribution) this approach "convexifies" the original problem (11), [11]. Although the payoff function is convex in the uncertainty sets for the majority of filtering problems of interest, more often than not it is not concave in the set of filters. In such cases, one of the standard minimax theorems can still be made by allowing a randomized robust filtering solution (not always attractive from an implementation point of view). Alternatively, realizing that an explicit expression for the optimal performance, \( M^*(h) \), is usually available, an approach to solving the original nonrandomized problem can be devised as follows.

If sufficient conditions can be found such that every least favorable operating point forms a saddle point with its optimal filter, then the existence of a least favorable solution to a minimization problem will guarantee the existence of a saddle point solution to the game. To this end, we have the following result (11b).

Theorem 1:
Suppose that the game
\[ (h, q_r) \] is such that
(i) \( Q \) is a convex set.
(ii) \( M(\cdot, q) \) is convex on \( h \) for every \( h \in H \).
Then, if \( (h_r, q_r) \) is a regular pair for \( (h, q_r) \), the following are equivalent:
(a) \( q_r \) is a least favorable operating point for \( (h, q_r) \)
(b) \( (h_r, q_r) \) is a saddle point solution for \( (h, q_r) \).

Proof:
The left inequality in the definition of a saddle point (2.6) is satisfied for every regular pair, because particularizing the regularity condition (2.7) for \( \alpha = 0 \), it follows that
\[ M(h, q) \leq M(h, q_r) \]

i.e., \( h \) is the optimal filter for \( q \). Then it remains to be shown that under the regularity of \( (h_r, q_r) \), \( q_r \) is the worst operating point for \( h \) if and only if it is the least favorable operating point for the game.

\[ M(h, q) \]

is convex in \( h \), since with \( q = (1 - \alpha) q_r + \alpha q \), for all \( 0 \leq \alpha \leq 1 \), and \( q_r, q \in Q \), we have that
\[ M(h, q) \leq (1 - \alpha) M(h, q_r) + \alpha M(h, q). \]

If \( (h, q_r) \) is the worst operating point for \( h \), and only if
\[ M(h, q) \leq M(h, q_r) \]

\[ (1 - \alpha) M(h, q_r) + \alpha M(h, q). \]

Moreover, \( q_r \) is the worst operating point for \( h \) if and only if
\[ M(h, q) \leq M(h, q_r) \]

and taking \( (1 - \alpha) q_r + \alpha q \) as in both sides of this equation.

We obtain that
\[ M(h, q) \leq M(h, q_r) \]

for every \( q \in Q \), if and only if \( (h, q) \) is a regular pair for \( (h, q) \). But then (2.11) and (2.12) we have that (2.14) is sufficient in order for \( q_r \) to be the worst operating point for \( h \) if and only if it is the least favorable operating point for \( (h, q) \).

This result reduces (under its convexity assumptions and the regularity condition) the problem of existence of a saddle point to the problem of existence of a minimizing argument of the (concave) function \( M(\cdot, q) \) over the (convex) uncertainty set. On the other hand, this allows the solution of problems in which the payoff function is not concave on the set of filters (not required to be topological). On the other hand, only the existence of a least favorable operating point (not the compactness of the uncertainty set) is required to ensure the existence of a saddle point solution.
of principal interest is to test the restrictiveness of the regularity condition for particular payoff functions. In [12] it is shown that theorem 1 provides an elegant framework for studying the problems of minimum state estimation and linear quatic control of linear systems with uncertain second order statistics. In the following sections various sufficient conditions are shown to guarantee the regularity of pairs of filters and operating points in several problems in signal detection and estimation with modeling uncertainties, tabulated in a uniform Hilbert space setting that allows the accommodation of the usual formulations in continuous or discrete time and in the time or frequency domains.

III. Matched Filtering

A signal-to-noise ratio is defined at some instant of time at the output of a linear system driven by a deterministic signal embedded in additive noise. The system that maximizes the signal-to-noise ratio is known as the matched filter for that particular pair of signal and noise statistics. The following formulation of the problem (see [12]) is used here. Let $X$ and $h \in X$ be the signal and filter functions, and $Z \notin \theta$ be the noise operator, where $\theta$ is a filter space with inner product $(\cdot, \cdot)$ and $h$ is a bounded, linear, real, (self-adjoint) positive operator mapping $X$ into itself. Then the signal-to-noise ratio of the filter at some selected time is the real-valued function defined by

$$SNR(h, \alpha) = \frac{(h, \alpha)^2}{(h, h)}.$$  

(3.1)

Suppose now that the signal and noise operator pair is known to belong to some fixed uncertainty set $\Theta$, and we are interested in finding a minimum robust filter $h$ in $\Theta$. Following the definitions given in section II, a matched filter for $(\alpha, h)$ will be denoted by $M(h, \alpha)$. It is shown via the Schwarz inequality that for all $\alpha \in \Theta$, if $h$ is a matched filter for $(\alpha, h)$, then for any $\beta \in \Theta$, we have

$$M(h, \alpha) = \sup_{\beta \in \Theta} SNR(h, \beta) = (h, \beta^*) (h, h)^{-1}.$$  

(3.3)

and a pair of signal and noise $(\alpha, h) \in \Theta$ is said to be least favorable for $\Theta$ if $M(h, \alpha)$ is the smallest relative to $\Theta$.

(3.4)

It can be shown (see [12]) that the signal-to-noise ratio defined by (3.1) is convex in any convex uncertainty set for which the Hilbert space is contained in the set of filters. Therefore it is of interest to find under what conditions a given pair of filter and signal/noise is regular. The following result provides an answer to this question.

Theorem 2

Denote $h = M(h, \alpha)$ and define the functional

$$\phi(h, \alpha, \theta, X) \equiv \sup_{\beta \in \Theta} SNR(h, \beta) = \sup_{\beta \in \Theta} (h, \beta^*) (h, h)^{-1}$$

where $h$ is the complex scalar field of $h$ by

$$\phi(h, \alpha, \theta, X) \equiv (h, \beta^*) (h, h)^{-1}.$$  

(3.5)

If for every $(\alpha, h) \in \Theta$ such that $(\alpha, h) = (1-n)(\alpha, h) + n(\alpha, h) \in \Theta$ for all $n \in [0, 1]$, we have

$$\phi(h, \alpha, \theta, X) = \sup_{\beta \in \Theta} SNR(h, \beta) = \sup_{\beta \in \Theta} (h, \beta^*) (h, h)^{-1} = c(\alpha),$$  

(3.6)

for every $(\alpha, h) \in \Theta$ such that $(\alpha, h) \in \Theta$ for all $n \in [0, 1]$. Then, manipulating the numerator of (3.6) we obtain the following equalities,

$$\phi(h, \alpha, \theta, X) = \sup_{\beta \in \Theta} SNR(h, \beta) = (h, \beta^*) (h, h)^{-1}$$

(3.7)

where the first and third equations follow subtracting and adding a term to the previous equality and the second equation follows from the definition of $\phi(h, \alpha, \theta, X)$ and the fact that $\phi(h, \alpha, \theta, X)$ is a matched filter for $(\alpha, h)$. Thus, taking $\lim_{n \to 1}$ of (3.7) and using the continuity condition of the theorem, the desired result is obtained.

We have seen in the last theorem that under a mild continuity condition on the behavior of the matched filter around a given operating point $(\alpha, h)$, this point and its optimal filter form a regular pair. Furthermore, it can be proved (see appendix) that if a necessary condition of theorem 2 is satisfied, then the continuity condition is sufficient.

IV. Linear Minimum Mean-Quadratic Error Filtering

Suppose that the output of a given linear filter when driven by a stochastic process $(z(t) \in T)$ is denoted by $(X(t) \in T)$. Given a fixed time $t$ and the joint second order statistics of $(z(t), X(t), X(t))$ of the classical (linear) filtering
problem is to find the filter for which the mean-
square difference \( E[I(x, y)|z|] \) is minimized. In a
Hilbert space setting, suppose that \( b \in \mathbb{R}^2 \) is the
filter function, \( y \in \mathbb{R}^2 \) represents the cross statis-
tics (e.g., cross-correlation in the power spectrum) between
\( z \) and \( x \), and \( y(x) \in \mathbb{R} \) is an operator representing
the second order statistics (e.g., autocorrelation or power
spectrum) of \( z \), where \( \mathbb{R} \) is the right hand side of
a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). Then, the
filter is minimally 
\[
\text{MSE}(b) = E[I(x, y)|z|] = \langle a^T, \theta \rangle \quad (3.18)
\]
represents the determined mean-square error
\( E[I(x, y)|z|] \) for a suitably defined Hilbert space. If
\( z \in \mathbb{R} \) is in the range of \( c \in \mathbb{R} \), there exists a unique
solution, \( b(x, y) \), to the equation
\[
\text{MSE}(b) = \min.
\]
and it is easy to see that the penalty function in
(3.19) can be put in the form
\[
\text{MSE}(b) = \langle c^T, H(x, y) \rangle \quad (3.20)
\]
Therefore, it follows from the positivity of the operator
\( c \) that \( \text{MSE}(b) \) is the optimal filter for
(3.21).

If there exists a convex uncertainty set \( \mathbb{Q} \subseteq \mathbb{R} \), such that
\( b(x, y) \in \mathbb{Q} \) and the Hilbert (filtering) norm is \( \|h(x, y)\|_{\mathbb{Q}} \), the penalty function is convex in \( c \) for
\( c \) in \( \mathbb{R} \) for every operating point. Therefore, the application of the minimax theorem and Theorem 1 can be investigated for this filtering game concerning the application of Theorem 1, the following result is relevant.

**Theorem 2**

Denote \( b = (\psi_1, \psi_2) \). If for every \( \psi_1 \in \mathbb{Q} \) such that \( \psi_1(\psi_2) = (\psi_2(\psi_1(\psi_2)) + \psi_1(\psi_2)) = \langle a^T, \theta \rangle \) for all \( \psi_1 \in \mathbb{Q} \), we have that
\[
\text{MSE}(b) \text{ is regular for all } b(x, y),
\]
then \( b(x, y) \) is a regular pair for \( \mathbb{Q}, \text{MSE} \).

**Proof**

Using expression (3.20) for the continuity condition
\( \text{MSE}(b) \text{ is regular for all } b(x, y) \), it must be defined in a neighborhood of the origin and consequently \( \text{MSE}(b) \text{ is regular for small } \theta \).

**Theorem 3**

Denote \( b = (\psi_1, \psi_2) \). If for every \( \psi_1 \in \mathbb{Q} \) such that \( \psi_1(\psi_2) = (\psi_2(\psi_1(\psi_2)) + \psi_1(\psi_2)) = \langle a^T, \theta \rangle \) for all \( \psi_1 \in \mathbb{Q} \), we have that
\[
\text{MSE}(b) \text{ is regular for all } b(x, y),
\]
then \( b(x, y) \) is a regular pair for \( \mathbb{Q}, \text{MSE} \).

**Proof**

Using expression (3.20) for the continuity condition
\( \text{MSE}(b) \text{ is regular for all } b(x, y) \), it must be defined in a neighborhood of the origin and consequently \( \text{MSE}(b) \text{ is regular for small } \theta \).

**Theorem 4**

Denote \( b = (\psi_1, \psi_2) \). If for every \( \psi_1 \in \mathbb{Q} \) such that \( \psi_1(\psi_2) = (\psi_2(\psi_1(\psi_2)) + \psi_1(\psi_2)) = \langle a^T, \theta \rangle \) for all \( \psi_1 \in \mathbb{Q} \), we have that
\[
\text{MSE}(b) \text{ is regular for all } b(x, y),
\]
then \( b(x, y) \) is a regular pair for \( \mathbb{Q}, \text{MSE} \).

**Proof**

Using expression (3.20) for the continuity condition
\( \text{MSE}(b) \text{ is regular for all } b(x, y) \), it must be defined in a neighborhood of the origin and consequently \( \text{MSE}(b) \text{ is regular for small } \theta \).

**Theorem 5**

Denote \( b = (\psi_1, \psi_2) \). If for every \( \psi_1 \in \mathbb{Q} \) such that \( \psi_1(\psi_2) = (\psi_2(\psi_1(\psi_2)) + \psi_1(\psi_2)) = \langle a^T, \theta \rangle \) for all \( \psi_1 \in \mathbb{Q} \), we have that
\[
\text{MSE}(b) \text{ is regular for all } b(x, y),
\]
then \( b(x, y) \) is a regular pair for \( \mathbb{Q}, \text{MSE} \).

**Proof**

Using expression (3.20) for the continuity condition
\( \text{MSE}(b) \text{ is regular for all } b(x, y) \), it must be defined in a neighborhood of the origin and consequently \( \text{MSE}(b) \text{ is regular for small } \theta \).
Theorem 3 Suppose that \( h \) satisfies

\[
| E[h(t_1, \omega)]^2 | \leq L(h) > 0.
\]

If for every \( (x, t) \in \mathbb{R}^2 \) such that \( (x, \omega) \neq (0, 0) \), we have that \( h(t_1, \omega) \) is right continuous at \( t = 0 \) for all \( t \in \mathbb{R} \), then \( h(t_1, \omega) \) is a regular pair for \( (0, a, \beta) \).

Proof

Since \( h(t_1, \omega) \) satisfies (3.36) and

\[
E[h(t_1, \omega) | \mathcal{F} - (0, a, \beta)] = 0,
\]

it is adjoint so we have that

\[
E[h(t_1, \omega)] = E(h(t_1, \omega), h(t_1, \omega))^*.
\]

On the other hand, denoting \( E = R_{h_1, h_1} \),

\[
E[h(t_1, \omega)] = \sup_{h \in \mathbb{R}} \left| \left( \frac{h(t_1, \omega)}{h(t_1, \omega)} \right) \right|.
\]

By way of (3.40) and the continuity condition of the theorem, (3.38) leads to the desired relation

\[
E[h(t_1, \omega)] = E(h(t_1, \omega), h(t_1, \omega))^* = o(a). \tag{3.41}
\]

In order to employ this theorem in the problem of existence of saddle points (Theorem 1) when \( o(a) \) is a convex set, we need to investigate the convexity of the payoff function in \( o(t_1, \omega) \). Since \( o(t_1, \omega) \) is linear and therefore convex in the signal operator. Moreover, it can be shown using the method in [7, Lemma 6] that \( E(h(t_1, \omega)) \) is convex in the noise operator. Unfortunately for arbitrary \( h \in \mathbb{R}^2 \), the payoff function is not convex in \( \mathbb{R} \) (recall that the function \( f(x, y) = x \) is convex). Thus, in this case Theorem 2 can only be used when there is uncertainty in both the signal or noise operator, but not both.

A formally similar continuity condition was sufficient for regularity in the case of matched and Wiener filtering. We showed (see appendix) that in these instances the invertibility of \( h \) is sufficient.
for such continuity condition to be fulfilled. It is interesting to notice that this fact does not hold for the problem of robust output energy filtering problem.

To see this, consider the following simple counterexample: let $Y = \mathbb{R}$ and suppose that the noise operator is known to be the identity, and that the usual operator uncertainty set as described by a set of diagonal matrices $\Gamma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_N\}$. Then, no vector $\theta \in \mathbb{R}^N$ exists for which $\gamma = (1, 1)$, because of the multiplicity of the dimensionality of the minimum eigenvalue eigenspace of $\gamma = (1, 1)$. This is easily checked by noting that if $h_0 = [a_0, b_0]^T = \text{diag}[x_0, y_0]$, and $h_0 = [-a_0, -b_0]^T$, then

$$\max_{\theta \in \Gamma} \mathbb{E} \left[ \left( x_0^T \theta \right)^2 \right] = \max_{\theta \in \Gamma} \mathbb{E} \left[ \left( y_0^T \theta \right)^2 \right] = \infty$$

which implies that for any given nonzero $h_0$, there exist elements in the convex uncertainty class $\Gamma$ for which the right side of (3.42) is not $0/0$.

Appendix

Lemma

If $A$ is invertible and $(x, x_0) = (1, 0)\alpha(x, 0, \beta, \delta)$, then

$$|A| = \alpha(0, \beta, \delta)$$

(4.1)

for all $(0, \beta, \delta) \in \mathbb{R}^2$, where $B(I, \alpha, \beta)$ is defined by $B(I, \alpha, \beta)^T = \alpha(0, \beta, \delta)$.

Proof

Using the triangle inequality and the boundedness of $\mathbb{P}$ if we have that for every $x$,

$$|A| \leq \mathbb{E} [a_0^T x]$$

(4.2)

If $A$ is invertible (one to one onto) then it is unique (see Theorem 21.1, (6)) that there exists $c > 0$ such that $|A| \leq \mathbb{E} [x]$, (4.3)

Now, fixing $x$ such that $c > 0 \leq c/(c + [a_0]^T)$, the last two inequalities result in

$$|A| \leq \mathbb{E} [x]$$

(4.4)

for all $x \in [0, c]$ and $c/(c + [a_0]^T) > 0$. Since $A$ is self-adjoint positive, its range is dense. This fact and (4.4) imply that $A$ is invertible for all $x \in [0, c]$ (see Theorem 21.3, (6)), and it is easy to see that the (operator) norm of its inverse is uniformly bounded, i.e.

$$|A| \leq c$$

(4.5)

Considering that, by the definition of $(x, x_0)$, we have that

$$\lim_{n \to \infty} \left[ A \left( x_0^T \theta \right)^n \right] = (-a_0, b_0)^T \theta$$

(4.6)

and applying $\lim_{n \to \infty}$ to both sides of (4.6) and using the bound in (4.5), we obtain

$$|A| \max \left( \left[ x_0^T \theta \right] \right)^n \leq \left( \frac{1}{|A|} \max \left( \left[ x_0^T \theta \right] \right) \right)^n$$

(4.7)

which, in turn, implies (4.1).

Note that, by the Banach inverse theorem (8, 20), since $A$ is linear and continuous in its inverse (if it exists), bounded. However, this would not be enough for the proof of the lemma, since it requires that $|A|$ is uniformly bounded in a neighborhood of $a = 0$.

Applying the Schwarz inequality and the result of the above lemma, the invertibility of the operator $A$ is sufficient for $c_0$ to be the continuity condition (CC) of Theorem 2 (matched filtering) and Theorem 3 (linear mean square error filtering).

References


