SOME MINIMAX DETECTION AND ESTIMATION PROBLEMS IN A HILBERT SPACE SETTING

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Abstract

The minimax approach to the design of systems which are robust with respect to modeling uncertainties is studied employing a game theoretic formulation in which the performance functional and the sets of modeling uncertainties and admissible design policies are arbitrary. It is shown that if the performance functional and the uncertainty set are convex then a certain type of regularity condition on the functional is sufficient in order to ensure that the optimal strategy for a least favorable element of the uncertainty set is minimax robust. The efficacy of this approach is tested in a Hilbert space formulation of the problems of matched filtering, Wiener filtering, quadratic detection and output energy filtering, in which uncertainties in their respective signal and noise models are assumed to exist.

I. Introduction

One of the major techniques for designing systems which are robust with respect to modeling uncertainties is the minimax approach, in which the goal is the optimization of worst-case performance. In most of the decision theoretic works that follow this approach, a common structure can be identified in the form of a game in which a certain performance function depends on the elements selected by the minimizing and maximizing players from a pair of prespecified sets containing the uncertain quantities of the model and the admissible design strategies. Motivated by the applications considered here, the elements of the uncertainty set and of the set of design strategies will henceforth be referred to as operating points and filters respectively. The cases in which there exists an amenable analytical solution for finding robust filters are those for which saddle points exist. A filter H and an operating point P are said to form a saddle point if, fixing P, any other filter different from H has worse performance, i.e., H is the optimal filter for P, and if, fixing H, any other operating point different from P gives better performance, i.e. H has its worst performance when P is present. If there exists such a filter H, then it is the sought-after minimax robust filter, because its worst-case performance is attained at P and any other filter has worse behavior at P. Further, suppose that we use optimal filters for every operating point in the uncertainty class, then P is the element whose filter achieves the worst optimal performance, and hence is referred to as the least favorable operating point. Note that the saddle point property is not necessary for the robust filter to exist; however, if it holds, the robust filter has the convenient feature of being the optimal filter for one of the operating points (the least favorable) and of performing better at any other point.

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Due to the fact that the computation of least favorable points is often a conceptually straightforward optimization problem, it is of interest to obtain conditions on the performance functional and on the maximizing and minimizing sets under which the optimal filter of every least favorable operating point is minimax robust. It turns out that the concept of regular pair of filter and operating point introduced here provides a useful sufficient condition on convex functionals over convex uncertainty sets for the above implication to hold. This result generalizes significantly the approach taken by Huber (and followed in a number of subsequent works) in his precursory solution of the robust statistical estimation of location problem [7], in which he extended to a non-Bayesian functional the well established (see e.g. [2], [9]) method of obtaining minimax procedures by using Bayes solutions with respect to least favorable a priori distributions.

In this paper we study the application of this approach to
i) matched filtering,
ii) linear minimum mean-square-error filtering,
iii) quadratic detection, and
iv) output energy filtering

when there are uncertainties in their respective signal and noise models. The emphasis in the study of each one of these problems is in the investigation of the conditions that ensure the validity of the computation of minimax filters via least favorable operating points.

II. Saddle Point Solutions and Regular Pairs

Denote by $\mathcal{H}$ a space of filters, and by $\mathcal{Q}$ a space of operating points. The payoff function M is a real functional

$$M(\cdot,\cdot) : \mathcal{H} \times \mathcal{Q} \rightarrow \mathbb{R} \quad . \tag{2.1}$$

Suppose that $H \subset \mathcal{H}$ is the set of allowable filters and $Q \subset \mathcal{Q}$ is the set of possible operating points (i.e., the uncertainty class). According to the standard terminology, the triple (H,Q,M) will be referred to as a game, in which the function M is maximized over H and minimized over Q. The following definitions will be used:

(i) $h^*(q)$ is an optimal filter for $q \in Q$ if

$$M(h^*(q),q) = M^*(q) \triangleq \sup_{h \in H} M(h,q). \tag{2.2}$$

(ii) $q^*(h)$ is the worst operating point for $h \in H$ if

$$q^*(h) = \arg\min_{q \in Q} M(h,q) \tag{2.3}$$

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(iii) h_R is a minimax robust filter for the game (H,Q,M) if

$$h_R \in \arg\max_{h \in H} \inf_{q \in Q} M(h,q). \tag{2.4}$$

(iv) q_L is a least favorable operating point for the game (H,Q,M) if

$$q_L \in \arg\min_{q \in Q} M^*(q) \quad , \tag{2.5}$$

(v) $(h_L,q_L) \in H \times Q$ is a saddle point solution to the game (H,Q,M) if for every $(h,q) \in H \times Q$,

$$M(h,q_L) \leq M(h_L,q_L) \leq M(h_L,q) \tag{2.6}$$

(vi) $(h_L,q_L) \in H \times Q$ is a regular pair for (H,Q,M) if, for every $q \in Q$ such that $q_\alpha = (1-\alpha)q_L + \alpha q \in Q$ for $\alpha \in [0,1]$, we have

$$M^*(q_\alpha) - M(h_L,q_\alpha) = o(\alpha), \tag{2.7}$$

If (h_L,q_L) is a saddle point solution to the game (H,Q,M), then h_L is the sought-after minimax robust filter for (H,Q,M). The reverse implication does not hold, i.e., a minimax robust filter and a least favorable operating point do not need to form a saddle point solution, however here we will focus attention on the existence and characterization of minimax robust filters that form saddle points.

Deriving sufficient conditions for a game to have saddle point solutions is the main goal of minimax theory. In general, these conditions require some topological properties on the sets of the game and continuity and quasi concave-convexity of the payoff function on the maximizing and minimizing sets respectively (cf. for example [3] and [14]). Frequently, when the convexity requirements are not fulfilled for a particular payoff function, the problem is reformulated by allowing randomized strategies, i.e., the original sets are replaced by sets of probability distributions defined on them and the payoff function is replaced by its expected value. Due to the linearity of the expected value (on the probability distribution) this approach "convexifies" the original problem ([2], [8]). Although the payoff function is convex in the uncertainty sets for the majority of filtering problems of interest, more often than not it is not concave in the set of filters. In such cases, use of the standard minimax theorems can still be made by allowing a randomized robust filtering solution (not always attractive from an implementation point of view). Alternatively, realizing that an explicit expression for the optimal performance, $M^*(\cdot)$, is usually available, an approach to solving the original nonrandomized problem can be devised as follows.

If sufficient conditions can be found such that every least favorable operating point forms a saddle point with its optimal filter, then the existence of a least favorable (the solution to a minimization problem) will guarantee the existence of a saddle point solution to the game. To this end, we have the following result ([16]).

Theorem 1.

Suppose that the game (H,Q,M) is such that
(i) Q is a convex set
(ii) $M(h,\cdot)$ is convex on Q for every $h \in H$.
Then, if (h_L,q_L) is a regular pair for (H,Q,M), the following are equivalent:
(a) q_L is a least favorable operating point for (H,Q,M)
(b) (h_L,q_L) is a saddle point solution for (H,Q,M).

Proof

The left inequality in the definition of a saddle point (2.6) is satisfied for every regular pair, because particularizing the regularity condition (2.7) for $q = q_L$, it follows that

$$M^*(q_L) = M(h_L,q_L), \tag{2.9}$$

i.e., h_L is the optimal filter for q_L. Then it remains to be shown that under the regularity of (h_L,q_L), q_L is the worst operating point for h_L if and only if it is the least favorable operating point for the game. $M^*(\cdot)$ is convex in Q, since with $q_0 = (1-\alpha)q_1 + \alpha q_2$, for all $0 \leq \alpha \leq 1$, and $q_1, q_2 \in Q$, we have that

$$\sup_{h \in H} M(h,q_0) \leq \sup_{h \in H} \{(1-\alpha)M(h,q_1) + \alpha M(h,q_2)\}$$

$$\leq (1-\alpha) \sup_{h \in H} M(h,q_1) + \alpha \sup_{h \in H} M(h,q_2) \tag{2.10}$$

where the first inequality follows from the second assumption in the theorem. Therefore, using Eq. (2.5) and the assumed convexity of Q, q_L is a least favorable operating point for (H,Q,M) if and only if, for every $q \in Q$,

$$L(q,q_L) \triangleq \lim_{\alpha \downarrow 0} \frac{1}{\alpha} [M^*(q_L+\alpha(q-q_L)) - M^*(q_L)] \geq 0. \tag{2.11}$$

Analogously, q_L is the worst operating point for h_L if and only if

$$W(q,h_L,q_L) \triangleq \lim_{\alpha \downarrow 0} \frac{1}{\alpha} [M(h_L,q_L+\alpha(q-q_L)) - M(h_L,q_L)] \geq 0. \tag{2.12}$$

Setting $q_\alpha = (1-\alpha)q_L + \alpha q$, considering that

$$M^*(q_\alpha) - M^*(q_L) = M^*(q_\alpha) - M(h_L,q_\alpha) + M(h_L,q_\alpha) - M(h_L,q_L), \tag{2.13}$$

and taking $\lim_{\alpha \downarrow 0} \frac{1}{\alpha}[\;]$ of both sides of this equation, we obtain that

$$L(q,q_L) = W(q,h_L,q_L) \tag{2.14}$$

for every $q \in Q$, if and only if (h_L,q_L) is a regular pair for (H,Q,M). But from (2.11) and (2.12) we have that (2.14) is sufficient in order for q_L to be the worst operating point for h_L if and only if it is the least favorable operating point for (H,Q,M).

This result reduces (under its convexity assumptions and the regularity condition) the problem of existence of a saddle point to the problem of existence of a minimizing argument of the (convex) function $M^*(\cdot)$ over the (convex) uncertainty set. On one hand, this allows the solution of problems in which the payoff function is not concave on the set of filters (not required to be topologized). On the other hand, only the existence of a least favorable operating point (not the compactness of the uncertainty set) is required to ensure the existence of a saddle point solution.

Of principal interest is to test the restrictiveness of the regularity condition for particular payoff functions. In [17] it is shown that Theorem 1 provides an elegant framework for studying the problems of minimax state estimation and linear quadratic control of linear systems with uncertain second order statistics. In the following sections various sufficient conditions are shown to guarantee the regularity of pairs of filters and operating points in several problems in signal detection and estimation with modeling uncertainties, imbedded in a common Hilbert space setting that allows the accommodation of the usual formulations in continuous or discrete time and in the time or frequency domains.

III. Matched Filtering

A signal-to-noise ratio is defined at some instant of time at the output of a linear system driven by a deterministic signal embedded in additive noise. The system that maximizes the signal-to-noise ratio is known as the matched filter for that particular pair of signal and noise statistics. The following formulation of the problem (see [12]) is used here. Let $s \in \mathcal{H}$ and $h \in \mathcal{H}$ be the signal and filter functions, and $\Sigma \in \mathcal{P}$ be the noise operator, where $\mathcal{H}$ is a Hilber space with inner product $\langle\cdot,\cdot\rangle$ and $\mathcal{P}$ is a set of bounded, linear, real, (self-adjoint) positive operators mapping $\mathcal{H}$ into itself. Then the signal-to-noise ratio of the filter at some selected time instant is the real valued functional defined by

$$\mathrm{SNR}(h;s,\Sigma) = |\langle h,s\rangle|^2/\langle h,\Sigma h\rangle \quad . \tag{3.1}$$

Suppose now that the signal and noise operator pair is known to belong to some fixed uncertainty set Q, and we are interested in finding a minimax robust filter for $(\mathcal{H},Q,\mathrm{SNR})$. Following the definitions given in Section II, a matched filter for (s,Σ) will be denoted by $h^*(s,\Sigma)$. It is shown via the Schwarz inequality that $h^*(s,\Sigma)$ is a matched filter for (s,Σ) if and only if there exists a nonzero scalar β such that

$$\Sigma h^*(s,\Sigma) = \beta s \quad . \tag{3.2}$$

Note that since Σ is not necessarily invertible, the existence of a matched filter for a particular pair of signal and noise is not assured. However, the positivity of Σ implies that when equation (3.2) has a solution it is uniquely defined. The optimum SNR attainable for a particular pair is denoted by

$$\mathrm{SNR}^*(s,\Sigma) = \sup_{h \in \mathcal{H}} \mathrm{SNR}(h;s,\Sigma) = \langle s,h^*(s,\Sigma)\rangle \quad , \tag{3.3}$$

and a pair of signal and noise $(s_0,\Sigma_0) \in Q$ is said to be least favorable for $(\mathcal{H},Q,\mathrm{SNR})$ if

$$(s_0,\Sigma_0) = \arg \min_{(s,\Sigma)\in Q} \mathrm{SNR}^*(s,\Sigma) \quad . \tag{3.4}$$

It can be shown (see [12]) that the signal-to-noise ratio defined by (3.1) is convex in any convex uncertainty set for every $h \in \mathcal{H}$ (however it is not concave in the set of filters). Therefore it is of interest to find under what conditions a given pair of filter and signal/noise is regular. The following result provides an answer to this question.

Theorem 2

Denote $h_L = h^*(s_L,\Sigma_L)$ and define the functional $f:\mathcal{H} \times \mathcal{H} \times \mathcal{P} \times \mathcal{P} \times [0,1] \to \mathbb{C}$ where $\mathbb{C}$ is the complex scalar field of $\mathcal{H}$ by

$$f(a,b,A,B,\alpha) \triangleq \langle b-Bh^*(a,A),h^*(a+\alpha(a-b),A+\alpha(A-B))\rangle . \tag{3.5}$$

If for every $(s,\Sigma) \in Q$ such that $(s_\alpha,\Sigma_\alpha) = (1-\alpha)(s_L,\Sigma_1) + \alpha(s,\Sigma) \in Q$ for all $\alpha \in [0,1]$, we have

(CC) $f(s_L,s,\Sigma_L,\Sigma,\cdot)$ is right continuous at the origin

then $(h_L,(s_L,\Sigma_L))$ is a regular pair for $(\mathcal{H},Q,\mathrm{SNR})$.

Proof

In order to prove the regularity of $(h_L,(s_L,\Sigma_L))$ we need to show that

$$\mathrm{SNR}^*(s_\alpha,\Sigma_\alpha) - \mathrm{SNR}(h_L;s_\alpha,\Sigma_\alpha)$$
$$= [\langle s_\alpha,h^*(s_\alpha,\Sigma_\alpha)\rangle\langle h_L,\Sigma_\alpha h_L\rangle - |\langle s_\alpha,h_L\rangle|^2]/\langle h_L,\Sigma_\alpha h_L\rangle = o(\alpha) \tag{3.6}$$

for every $(s,\Sigma) \in Q$ such that $(s_\alpha,\Sigma_\alpha) \in Q$ for $\alpha \in [0,1]$. Then, manipulating the numerator of (3.6) we obtain the following equalities,

$$\begin{aligned}
&\langle s_\alpha,h^*(s_\alpha,\Sigma_\alpha)\rangle\langle h_L,\Sigma_\alpha h_L\rangle - \langle s_\alpha,h_L\rangle\langle h_L,s_\alpha\rangle \\
&= \langle s_\alpha - \Sigma_\alpha h_L,h^*(s_\alpha,\Sigma_\alpha)\rangle\langle h_L,\Sigma_\alpha h_L\rangle \\
&+ \langle \Sigma_\alpha h_L,h^*(s_\alpha,\Sigma_\alpha)\rangle\langle h_L,\Sigma_\alpha h_L\rangle \\
&- \langle s_\alpha,h_L\rangle\langle h_L,s_\alpha\rangle \\
&= \langle s_\alpha-\Sigma_\alpha h_L,h^*(s_\alpha,\Sigma_\alpha)\rangle\langle h_L,\Sigma_\alpha h_L\rangle \\
&- \langle s_\alpha - \Sigma_\alpha h_L,h_L\rangle\langle h_L,s_\alpha\rangle \\
&= \langle s_\alpha - \Sigma_\alpha h_L,h^*(s_\alpha,\Sigma_\alpha) - h_L\rangle\langle h_L,\Sigma_\alpha h_L\rangle \\
&- |\langle s_\alpha - \Sigma_\alpha h_L,h_L\rangle|^2
\end{aligned} \tag{3.7}$$

where the first and third equations follow subtracting and adding a term to the previous equality and the second equation follows from the definition of $h^*(\cdot,\cdot)$ and the fact that Σ_α is self-adjoint. Now, notice that

$$s_\alpha-\Sigma_\alpha h_L = s_L+\alpha(s-s_L)-(\Sigma_L+\alpha(\Sigma-\Sigma_L))h_L = \alpha(s-\Sigma h_L), \tag{3.8}$$

and therefore the last expression of (3.7) is equal to

$$\alpha\langle s-\Sigma h_L,h^*(s_\alpha,\Sigma_\alpha)-h_L\rangle\langle h_L,\Sigma_\alpha h_L\rangle-\alpha^2|\langle s-\Sigma h_L,h_L\rangle|^2$$
$$= \alpha\langle s-\Sigma h_L,h^*(s_\alpha,\Sigma_\alpha)-h_L\rangle\langle h_L,s_L\rangle + o(\alpha) \quad . \tag{3.9}$$

Thus, taking $\lim_{\alpha \downarrow 0} \frac{1}{\alpha}[\quad]$, and using the continuity condition of the theorem, the desired result is obtained.

We have seen in the last theorem that under a mild continuity condition on the behavior of the matched filter around a given operating point (s_L,Σ_L), this point and its optimal filter form a regular pair. Furthermore, it can be proved (see Appendix) that the invertibility of Σ_L is sufficient for that continuity condition to hold. Therefore, the conclusion is that when the uncertainty class Q is convex (and when the above continuity condition holds) the problem of finding a minimax robust matched filter is reduced to that of finding a least favorable pair of signal and noise.

IV. Linear Minimum Mean-Square Error Filtering

Suppose that the output of a given linear filter when driven by a stochastic process $\{Z_\tau,\tau \in T\}$ is denoted by $\{\hat{X}_\tau,\tau \in T\}$. Given a fixed time $\underline{t}$ and the joint second order statistics of $\{Z_\tau,\tau \in T\}$ and another process $\{X_\tau,\tau \in T\}$, the classical (Wiener) filtering problem is to find the filter for which the mean-square difference $E[|X_t-\hat{X}_t|^2]$ is minimized. In a Hilbert space setting, suppose that $h \in \mathcal{H}$ is the filter function, $\psi \in \mathcal{H}$ represents the cross statistics (e.g. crosscorrelation or cross power spectrum) between $\{Z_\tau,\tau \in T\}$ and X_t, $\Sigma \in \mathcal{P}$ is an operator representing the second order statistics (e.g. autocorrelation or power spectrum) of $\{Z_\tau,\tau \in T\}$, and $\mathcal{S}_x(t) = E[|X_t|^2]$, with $\mathcal{H}$ a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\mathcal{P}$ a set of bounded, linear, real, (self-adjoint) positive operators mapping $\mathcal{H}$ into itself. Then, the real functional

$$MSE(h;\psi,\Sigma) = \langle h,\Sigma h\rangle - \langle h,\psi\rangle - \langle \psi,h\rangle + \mathcal{S}_x(t) \tag{3.18}$$

represents the aforementioned mean-square error $E[|X_t-\hat{X}_t|^2]$ for a suitably defined Hilbert Space . If $\psi \in \mathcal{H}$ is in the range of $\Sigma \in \mathcal{P}$, there exists a unique solution, $h^*(\psi,\Sigma)$, to the equation

$$\Sigma h^*(\psi,\Sigma) = \psi \ , \tag{3.19}$$

and it is easy to see that the penalty function in (3.18) can be put in the form:

$$MSE(h;\psi,\Sigma) = \langle h-h^*(\psi,\Sigma),\Sigma(h-h^*(\psi,\Sigma))\rangle-\langle h^*(\psi,\Sigma),\psi\rangle+\mathcal{S}_x(t). \tag{3.20}$$

Therefore, it follows from the positivity of the operator Σ that $h^*(\psi,\Sigma)$ is indeed the optimal filter for (ψ,Σ).

If there exists a convex uncertainty set $Q \subset \mathcal{H} \times \mathcal{P}$, such that $(\psi,\Sigma) \in Q$, and the minimax filtering game is $(\mathcal{H},Q,-MSE)$, the payoff function is convex in Q for every filter, and is concave in $\mathcal{H}$ for every operating point. Therefore, the application of minimax theorems and Theorem 1 can be investigated for this filtering game. Concerning the application of Theorem 1, the following result is relevant.

Theorem 3

Denote $h_L = h^*(\psi_L,\Sigma_L)$. If for every $(\psi,\Sigma) \in Q$ such that $(\psi_\alpha,\Sigma_\alpha) = (1-\alpha)(\psi_L,\Sigma_L) + \alpha(\psi,\Sigma) \in Q$ for all $\alpha \in [0,1]$, we have that
(CC) $f(\psi_L,\psi,\Sigma_L,\Sigma,\cdot)$ is right continuous at the origin, then $(h_L,(\psi_L,\Sigma_L))$ is a regular pair for $(\mathcal{H},Q,-MSE)$.

Proof

Using expression (3.20) (if the continuity condition holds $f(\psi_L,\psi,\Sigma_L,\Sigma,\cdot)$ must be defined in a neighborhood of the origin, and consequently $h^*(\psi_\alpha,\Sigma_\alpha)$ exists for α sufficiently small) it is easy to check that

$$\begin{aligned} MSE(h_L;\psi_\alpha,\Sigma_\alpha) &- MSE^*(\psi_\alpha,\Sigma_\alpha) \\ &= \langle h^*(\psi_\alpha,\Sigma_\alpha) - h_L,\Sigma_\alpha(h^*(\psi_\alpha,\Sigma_\alpha) - h_L)\rangle \\ &=\langle\psi_\alpha - \Sigma_\alpha h_L,(h^*(\psi_\alpha,\Sigma_\alpha) - h_L)\rangle \end{aligned} \tag{3.21}$$

where the last equation follows from Equation (3.19) and the fact that Σ_α is self-adjoint. Analogously to Equation (3.8), we have

$$\psi_\alpha - \Sigma_\alpha h_L = \alpha(\psi - \Sigma h_L) \ . \tag{3.22}$$

Thus, (3.21) and the continuity condition of the theorem result in

$$MSE(h_L;\psi_\alpha,\Sigma_\alpha) - MSE^*(\psi_\alpha,\Sigma_\alpha) = o(\alpha) \tag{3.23}$$

We have just seen how a Hilbert space formulation of the problems of matched filtering and Wiener filtering results in sufficient conditions for regularity of pairs of filters and operating points that possess the same structure. It is perhaps worth mentioning that not every particular case of both problems fits into a Hilbert space setting; however, the results presented here give an indication of what type of conditions for regularity should be expected in such cases.

V. Quadratic Receiver

In a quadratic receiver the detection of a stochastic signal embedded in additive independent zero-mean noise is based on a quadratic form of the input with operator $H \in \mathcal{R}$ (a set of bounded, linear, real, self-adjoint operators mapping a separable Hilbert space $\mathcal{H}$ into itself). A possible criterion for choosing H given the second order statistics of signal and noise is to maximize the signal-to-noise ratio of the quadratic test statistic; in particular here we consider the (deflection) ratio between the mean of the output due to the signal squared, and the variance of the output due to the noise. If $\Lambda \in \mathcal{J}$ and $\Sigma \in \mathcal{P}$ (where $\mathcal{J}$ and $\mathcal{P}$ are sets of bounded, linear, real, self-adjoint, nonnegative operators mapping $\mathcal{H}$ into itself) represent the power spectrum or autocorrelation of signal and noise respectively, and the noise is a Gaussian process, it can be shown (cf. [1]) that the deflection can be expressed in the form

$$DEF(H;\Lambda,\Sigma) = \tfrac{1}{2}\, tr^2\{H\Lambda\}/tr\{H\Sigma H\Sigma\} \ . \tag{3.26}$$

By application of the Schwarz inequality it can be seen that the filter that maximizes (3.26) for a given pair of signal and noise operators satisfies,

$$\Sigma H^*(\Lambda,\Sigma)\Sigma = \Lambda \tag{3.27}$$

achieving a maximum deflection of

$$DEF^*(\Lambda,\Sigma) = \tfrac{1}{2}\, tr\{\Lambda H^*(\Lambda,\Sigma)\} \ . \tag{3.28}$$

It is well known that the output of this quadratic receiver, optimal in terms of deflection, approximates a sufficient statistic for Gaussian signal detection in Gaussian noise for small signal-to-noise ratio situations (see [20]).

In order to apply the results of existence of minimax robust solutions for convex uncertainty classes, the first step is, as before, to check the concavity/convexity of the payoff function in (3.26). Unfortunately, it can be shown that it is neither concave in the filter set for arbitrary Λ and Σ, nor convex in $\mathcal{P}$ for arbitrary H and Σ. A fortiori it is not convex in the uncertainty set for a given filter. However from the linearity of trace and the convexity of the parabola it is easy to see that the payoff function is convex in $\mathcal{J}$ for arbitrary H and Σ. Therefore, the application of Theorem 1 is restricted to the cases in which the noise operator is known and is founded on the following result.

Theorem 4

If $H_L = H^*(\Lambda_L,\Sigma_L)$, and there exists an optimal filter $H^*(\Lambda,\Sigma_L)$ for every $\Lambda \in \mathcal{J}$, then $(H_L,(\Lambda_L,\Sigma_L))$ is a regular pair for $(\mathcal{H},\mathcal{J} \times \{\Sigma_L\},DEF)$.

Proof

For every $\Lambda \in \mathcal{J}$ such that $\Lambda_\alpha = (1-\alpha)\Lambda_L+\alpha\Lambda \in Q$ for $\alpha \in [0,1]$, we want to show that

$$\mathrm{DEF}^*(\Lambda_\alpha,\Sigma_L) - \mathrm{DEF}(H_L;\Lambda_\alpha,\Sigma_L)$$
$$= \frac{[\mathrm{tr}\{H^*(\Lambda_\alpha,\Sigma_L)\Lambda_\alpha\}\mathrm{tr}\{H_L\Lambda_L\} - \mathrm{tr}^2\{H_L\Lambda_\alpha\}]}{2\mathrm{tr}\{H_L\Sigma_L\}} = o(\alpha). \quad (3.29)$$

By definition of $H^*(\Lambda_\alpha,\Sigma_L)$ and rotation of the self-adjoint operators under the trace functional, we have that

$$\mathrm{tr}\{H_L\Lambda_\alpha\}\mathrm{tr}\{H_L\Lambda_L\} - \mathrm{tr}\{H^*(\Lambda_\alpha,\Sigma_L)\Lambda_L\}\mathrm{tr}\{H_L\Lambda_L\} = 0 \; . \quad (3.30)$$

Adding the left side of the last equation to the numerator of (3.29) we obtain

$$\mathrm{tr}\{H^*(\Lambda_\alpha,\Sigma_L)\Lambda_\alpha\}\mathrm{tr}\{H_L\Lambda_L\} - \mathrm{tr}^2\{H_L\Lambda_\alpha\}$$
$$= \mathrm{tr}\{H^*(\Lambda_\alpha,\Sigma_L)[\Lambda_\alpha-\Lambda_L]\}\mathrm{tr}\{H_L\Lambda_L\}-\mathrm{tr}\{H_L[\Lambda_\alpha-\Lambda_L]\}\mathrm{tr}\{H_L\Lambda_\alpha\}$$
$$= \alpha\ \mathrm{tr}\{[H^*(\Lambda_\alpha,\Sigma_L) - H_L]\Lambda\}\mathrm{tr}\{H_L\Lambda_L\} + o(\alpha)$$
$$= \alpha\ \mathrm{tr}\{H^*(\Lambda,\Sigma_L)[\Lambda_\alpha-\Lambda_L]\}\mathrm{tr}\{H_L\Lambda_L\} + o(\alpha)$$
$$= o(\alpha) \quad (3.31)$$

where the fact that for every $\Lambda \in \mathcal{A}$ there exists an optimal filter has been used. □

We have seen that because of the lack of convexity in the noise operator of the deflection (3.26), the presently available results concerning existence of robust filters are not applicable to the case in which both the signal and noise second order statistics are inaccurately known. Note that other generalized signal-to-noise ratios that have been proposed in the literature for quadratic receiver design (see [5],[18]) fail even to provide convexity in the signal operator.

VI. Output Energy Filtering

When a stochastic signal embedded in zero-mean additive noise is processed by a linear filter, a performance criterion used in certain applications such as seismic data processing and sonar (see [13]) is the ratio between the energy of the output due to the signal and the energy of the output due to the noise at a certain time instant. Analogously to the above filtering situations, $h \in \mathcal{H}$ represents the filter transfer function or impulse response $\Lambda \in \mathcal{A}$ and $\Sigma \in \mathcal{P}$ represent the signal and noise operators (autocorrelation or power spectrum), where $\mathcal{A}$ and $\mathcal{P}$ are sets of bounded, linear, real, self-adjoint nonnegative and positive operators, respectively, mapping the Hilbert space $\mathcal{H}$ into itself. Then, the energy signal-to-noise ratio can be readily seen to be

$$E(h;\Lambda,\Sigma) = \frac{\langle h,\Lambda h\rangle}{\langle h,\Sigma h\rangle} \quad . \quad (3.34)$$

This payoff function displays several interesting relationships with those cases treated above. For instance, in the special case in which the signal is deterministic and square integrable ($s \in \mathcal{H}$), (3.34) reduces to (3.1) by defining the signal operator by

$$\Lambda h = \langle h,s\rangle s \quad . \quad (3.35)$$

On the other hand, consider the special type of quadratic receiver consisting of a linear filter followed by a square law device. If the signal and noise are jointly Gaussian processes, then the variance of the output of the receiver is proportional to the square of the variance of the output of the linear filter, and therefore the deflection of the output of the receiver is proportional to the square of the energy ratio (3.34).

Following the above usage, we denote by $E^*(\Lambda,\Sigma)$ the optimum performance attainable in (Λ,Σ). Then, $E^*(\Lambda,\Sigma)$ can be shown to be the maximum generalized eigenvalue of the regular pencil $\langle h,(\Lambda - \lambda\Sigma)h\rangle$, (cf. [4]), and the optimum filter, if it exists, is a nonzero element satisfying

$$[E^*(\Lambda,\Sigma)\Sigma - \Lambda]h^*(\Lambda,\Sigma) = 0. \quad (3.36)$$

Now we suppose that the operating point belongs to some uncertainty set $Q \subset \mathcal{A} \times \mathcal{P}$, then the corresponding regularity result is the following.

Theorem 5

Suppose that h_L satisfies

$$[E^*(\Lambda_L,\Sigma_L)\Sigma_L - \Lambda_L]h_L = 0 \quad . \quad (3.37)$$

If for every $(\Lambda,\Sigma) \in Q$ such that $(\Lambda_\alpha,\Sigma_\alpha) = (1-\alpha)(\Lambda_L,\Sigma_L) + \alpha(\Lambda,\Sigma) \in Q$ for all $\alpha \in [0,1]$, we have that $\langle h,h^*(\Lambda_\alpha,\Sigma_\alpha)\rangle$ is right continuous at $\alpha = 0$ for all $h \in \mathcal{H}$, then $(h_L,(\Lambda_L,\Sigma_L))$ is a regular pair for $(\mathcal{H},Q,E)$.

Proof

Since $h^*(\Lambda_\alpha,\Sigma_\alpha)$ satisfies (3.36) and $[E^*(\Lambda_\alpha\ \Sigma_\alpha)\Sigma_\alpha - \Lambda_\alpha]$ is self-adjoint we have that

$$[E^*(\Lambda_\alpha,\Sigma_\alpha) - E(h_L;\Lambda_\alpha,\Sigma_\alpha)]\langle h_L,\Sigma_\alpha h_L\rangle$$
$$= \langle h_L-h^*(\Lambda_\alpha,\Sigma_\alpha),[E^*(\Lambda_\alpha,\Sigma_\alpha)\Sigma_\alpha-\Lambda_\alpha]h_L\rangle \quad . \quad (3.38)$$

On the other hand, denoting $E_L = E^*(\Lambda_L,\Sigma_L)$, the definition of $E^*(\cdot,\cdot)$ results in

$$|E^*(\Lambda_\alpha,\Sigma_\alpha) - E_L| \leq \sup_{h\in\mathcal{H}}\left|\frac{\langle h,\Lambda_\alpha h\rangle}{\langle h,\Sigma_\alpha h\rangle} - \frac{\langle h,\Lambda_L h\rangle}{\langle h,\Sigma_L h\rangle}\right|$$
$$= \sup_{h\in\mathcal{H}}\left|\frac{\langle h,(\Lambda_\alpha-\Lambda_L)h\rangle}{\langle h,\Sigma_\alpha h\rangle} + \langle h,\Lambda_L h\rangle\left[\frac{1}{\langle h,\Sigma_\alpha h\rangle} - \frac{1}{\langle h,\Sigma_L h\rangle}\right]\right|$$
$$= O(\alpha) \quad . \quad (3.39)$$

Therefore, we have that

$$[E^*(\Lambda_\alpha,\Sigma_\alpha)\Sigma_\alpha - \Lambda_\alpha]h_L$$
$$= [E^*(\Lambda_\alpha,\Sigma_\alpha) - E_L]\Sigma_\alpha h_L + [E_L\Sigma_\alpha - \Lambda_\alpha]h_L$$
$$= O(\alpha)[\Sigma_L + E_L\Sigma - \Lambda]h_L \quad . \quad (3.40)$$

By way of (3.40) and the continuity condition of the theorem, (3.38) leads to the desired relation

$$E^*(\Lambda_\alpha,\Sigma_\alpha) - E(h_L;\Lambda_\alpha,\Sigma_\alpha) = o(\alpha) \; . \quad (3.41)$$

In order to employ this theorem in the problem of existence of saddle points (Theorem 1) when Q is a convex set, we need to investigate the convexity of the payoff function in Q. $E(h;\cdot,\Sigma)$ is linear and therefore convex in the signal operator. Moreover it can be shown using the method in [7, Lemma 6] that $E(h;\Lambda,\cdot)$ is convex in the noise operator. Unfortunately for arbitrary $h \in \mathcal{H}$, the payoff function is not convex in Q (recall that the function x/y is not convex). Thus, in this case Theorem 5 can only be used when there is uncertainty in either the signal or noise operator, but not in both.

A formally similar continuity condition was sufficient for regularity in the cases of matched and Wiener filtering. We showed (see Appendix) that in those instances the invertibility of Σ_L is sufficient for such continuity condition to be fulfilled. It is interesting to notice that this fact does not hold for the minimax robust output energy filtering problem. To see this, consider the following simple counter-example. Let $\mathcal{H} = \mathbb{R}^2$ and suppose that the noise operator is known to be the identity, and that the signal operator uncertainty set is described by a set of diagonal matrices $S = \{\mathrm{diag}\{1+\varepsilon, 1+\delta\}; \varepsilon, \delta \in [0,1]\}$. Then, no vector h_L forms a regular pair with $(\Lambda_L, \Sigma_L) = (I, I)$, because of the multiplicity of the dimensionality of the maximum eigenvalue eigenspace of $\Sigma_L^{-1}\Lambda_L$. This is easily checked by noting that if $h_L = [a,b]^T$, $\Lambda = \mathrm{diag}\{1+c, 1+d\}$, and $\Lambda_\alpha = \Lambda_L + \alpha(\Lambda - \Lambda_L)$, then

$$E^*(\Lambda_\alpha, \Sigma_L)\|h_L\|^2 - h_L^T \Lambda_\alpha h_L = \alpha \max\{a^2(d-c), b^2(c-d)\} \qquad (3.42)$$

which implies that for any given nonzero h_L, there exist elements in the convex uncertainty class S for which the right side of (3.42) is not $o(\alpha)$.

Appendix

Lemma

If A is invertible and $(a_\alpha, A_\alpha) = (1-\alpha)(a,A) + \alpha(b,B)$, then

$$\|h^*(a_\alpha, A_\alpha) - h^*(a,A)\| = O(\alpha) \qquad (A.1)$$

for all $(b,B) \in \mathcal{H} \times P$, where $h^*(\cdot,\cdot)$ is defined by $A_\alpha h^*(a_\alpha, A_\alpha) = a_\alpha$.

Proof

Using the triangle inequality and the boundedness of P we have that for every x ,

$$\|A_\alpha x\| \geq (1-\alpha)\|Ax\| - \alpha\|B\| \ \|x\| \qquad (A.2)$$

If A is invertible (one to one and onto) then it follows (see Theorem 21.3, [6]) that there exists $\varepsilon > 0$ such that

$$\|A_\alpha x\| \geq \varepsilon \|x\| \ . \qquad (A.3)$$

Now, fixing t such that $0 < t < \varepsilon/(\varepsilon + \|B\|)$, the last two previous inequalities result in

$$\|A_\alpha x\| \geq \delta \|x\| \qquad (A.4)$$

for all $\alpha \in [0,t]$ and $\delta = (1-t)\varepsilon - t\|B\| > 0$. Since A_α is self-adjoint positive, its range is dense. This fact and (A.4) imply that A_α is invertible for $\alpha \in [0,t]$ (see Theorem 21.3,[6]), and it is easy to see that the (operator) norm of its inverse is uniformly bounded, i.e.

$$\|A_\alpha^{-1}\| \leq \delta^{-1} \qquad (A.5)$$

Considering that, by the definition of $h^*(a_\alpha, A_\alpha)$, we have that

$$A_\alpha[h^*(a_\alpha, A_\alpha) - h^*(a,A)] = \alpha(a - Bh^*(a,A)) \qquad (A.6)$$

and applying A_α^{-1} to both sides of (A.6) and using the bound in (A.5), we obtain

$$\|h^*(a_\alpha, A_\alpha) - h^*(a,A)\| = \alpha\|A^{-1}[a - Bh^*(a,A)]\|$$

$$\leq \alpha\delta^{-1}\|a - Bh^*(a,A)\| \ , \qquad (A.7)$$

which, in turn, implies (A.1).

Note that, by the Banach inverse theorem (e.g., [10]), since A_α is linear and continuous its inverse (if it exists) is bounded. However, this would not be enough for the previous proof, since it requires that A_α^{-1} is uniformly bounded in a neighborhood of $\alpha = 0$.

Applying the Schwarz inequality and the result of the above lemma, the invertibility of the operator Σ_L is sufficient in order for the continuity condition (CC) of Theorem 2 (matched filtering) and Theorem 3 (Linear mean square error filtering).

References

[1] C. R. Baker, "Optimum quadratic detection of a random vector in Gaussian noise," IEEE Trans. Comm., vol COM-14, no. 6, pp. 802-805, Dec. 1966.

[2] D. Blackwell and M. A. Girshick, Theory of Games and Statistical Decisions, New York: Wiley, 1954.

[3] K. Fan, "Minimax theorems," Proc. Nat. Acad. Sci. U.S.A., vol. 39, pp. 42-47, 1953.

[4] F. R. Gantmacher, The Theory of Matrices, vol. 1, New York: Chelsea, 1960.

[5] W. A. Gardner, "Anomalous behavior of receiver output SNR as a predictor of signal detection performance exemplified for quadratic receivers and incoherent fading Gaussian channels," IEEE Trans. Inform. Theory, vol. IT-25, no. 6, pp. 743-745, Nov. 1979.

[6] P. R. Halmos, Introduction to Hilbert Space, 2nd Ed., New York: Chelsea, 1957.

[7] P. J. Huber, "Robust estimation of a location parameter," Annals of Math. Statistics, vol. 35, pp. 73-101, 1964.

[8] H. W. Kuhn and A. W. Tucker, Eds., Contributions to the Theory of Games, Princeton, NJ: Princeton University Press, 1950.

[9] E. L. Lehmann, Testing Statistical Hypotheses, New York: Wiley, 1959.

[10] D. G. Leunberger, Optimization by Vector Space Methods, New York: Wiley, 1969.

[11] H. V. Poor, "On robust Wiener filtering," IEEE Trans. Automatic Control, vol. AC-25, no. 3, pp. 531-536, June 1980.

[12] H. V. Poor, "Robust matched filters," IEEE Trans. Inform. Theory, vol. IT-29, no. 5, Sept. 1983.

[13] E. A. Robinson, Statistical Communication and Detection, New York: Hafner P.C., 1967.

[14] M. Sion, "On general minimax theorems," Pacific J. Mathematics, vol. 8, pp. 171-176, 1958.

[15] K. S. Vastola and H. V. Poor, "Robust Wiener-Kolmogorov theory," IEEE Trans. Inform. Theory, to appear.

[16] S. Verdú, "A general approach to minimax robust filtering," Tech. Rep. No. R-933, UILU-ENG 81-2264, Coordinated Sci. Lab., Univ. of Illinois, Dec. 1981.

[17] S. Verdú and H. V. Poor, "Minimax linear observers and regulators for stochastic systems with uncertain second order statistics," IEEE Trans. Automatic Control, to appear.

[18] S. Verdú, "Comments on 'Anomalous behavior of receiver output SNR as a predictor of signal detection performance exemplified for quadratic receivers and incoherent fading Gaussian channels,'" IEEE Trans. Information Theory, vol. IT-28, no. 6, Nov. 1982.

[19] S. Verdú and H. V. Poor, "Minimax robust discrete-time matched filters," IEEE Trans. Comm., vol. COM-31, no. 2, Feb. 1983.

[20] C. L. Weber, Elements of Detection and Signal Design, New York: McGraw-Hill, 1968.