Spectral Efficiency of Direct-Sequence Spread-Spectrum Multiaccess with Random Spreading

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Abstract — Information theoretic aspects of Code Division Multiple Access (CDMA) random direct-sequence spread-spectrum (DSSS) are investigated. The CDMA-DSSS channel with randomly and independently chosen spreading sequences accurately models the situation where pseudo-noise sequences span many symbol periods. We analyze the spectral efficiency (total capacity per chip) as a function of the number of users, spreading gain and signal-to-noise ratio, and we quantify the loss in efficiency relative to an optimally chosen set of signature sequences and to an optimal multiaccess system without spreading. Particular attention is given to the limiting spectral efficiency as the number of users grows without bound. White Gaussian background noise and equal-power synchronous users are assumed. The analysis comprises the following receivers: a) optimal joint processing, b) single-user matched filtering; c) decorrelation and d) minimum mean square error linear processing.

I. Channel Model

Coded direct-Sequence Spread-Spectrum (DSSS) Code Division Multiple Access (CDMA) has well-known advantages over time/frequency division multiple access: dynamic channel sharing, robustness to channel impairments, graceful degradation, ease of cellular planning, etc. These advantages result from the assignment of "signature waveforms" with large time-bandwidth products to every potential user of the system. The coded DSSS K-user model is as in [1], [2]

$$y(t) = \sum_{k=1}^{K} \sum_{l} b_k(l)\, s_k(t - lT) + n(t).$$

The signature waveform of the k-th user is designated by $s_k(t)$, the l-th coded information symbol of the k-th user is $b_k(l)$. The additive white Gaussian noise with a two sided power spectral density of $N_0/2$ is designated by $n(t)$. In the sequel we confine our attention to piecewise constant bipolar valued signatures with N chips per symbol. The ratio of the number of users to the spreading gain is denoted by $\beta = K/N$. The central question addressed in this paper is the capacity loss incurred by the imposition of the direct-sequence spread-spectrum structure on the transmitted signals, and by the imposition of several suboptimal but practically appealing receiver structures for which single-user decoding are sufficient. Our analysis considers a white Gaussian channel with users constrained to have identical average powers. We shall focus on randomly selected signatures modeling, among other scenarios, the long signature sequences used in some practical systems such as the IS-95 standard.

II. Spectral Efficiency

The fundamental figure of merit is the *spectral efficiency*, C, defined as the total number of information bits per chip that can be transmitted arbitrarily reliably. Since the bandwidth of the CDMA system is (roughly) equal to the reciprocal of the chip duration, the spectral efficiency can be viewed as the bits/s/Hz supported by the system. In our asymptotic (in K) analysis we do not just average spectral efficiency with respect to the spreading sequences, but we show convergence of the spectral efficiencies to deterministic quantities. Optimal spectral efficiency in non-orthogonal CDMA requires joint processing and decoding of the users. As advocated in a number of recent works, see [5, and references therein], it is sensible in terms of complexity-performance tradeoff to adopt as a front-end a (soft-output) *multiuser detector* [6] followed by autonomous single-user error-control decoders. In our analysis of spectral efficiency we consider, in addition to optimal decoding, some popular linear multiuser detector front-ends: Single-user matched filter, Decorrelator and Linear Minimum Mean-Square-Error (MMSE).

III. Optimum Spectral Efficiency–No Spreading

The maximum spectral efficiency in the absence of spreading is the solution to

$$\mathsf{C}^*\left(\frac{E_b}{N_0}\right) = \frac{1}{2}\log\left(1 + 2\mathsf{C}^*\left(\frac{E_b}{N_0}\right)\frac{E_b}{N_0}\right) \qquad (1)$$

Since (1) does not depend on K, when the transmitted signals are not constrained to the spread-spectrum format, the spectral efficiency is the same as in a single-user system.

IV. Optimum Spectral Efficiency–CDMA Spreading

The total capacity (sum-rate) of the power-constrained synchronous CDMA channel is equal to [1]

$$\frac{1}{2}\log\left(\det\left[\mathbf{I} + \sigma^{-2}\mathbf{ARA}\right]\right), \qquad (2)$$

where $\mathbf{A} = \mathrm{diag}\{A_1, \ldots, A_K\}$, and $\mathbf{R}$ is the matrix of normalized crosscorrelations of the bipolar signature sequences.

If the users have equal power, then $A_k = A$, and the optimum spectral efficiency is equal to

$$\mathsf{C}^{\mathsf{opt}}(\mathsf{SNR}, \mathbf{R}, K, N) = \frac{1}{2N}\log\left(\det\left[\mathbf{I} + \mathsf{SNR}\mathbf{R}\right]\right). \qquad (3)$$

Using (3), it can be shown that in the case of orthogonal sequences the spectral efficiency is equal to

$$\mathsf{C}^{\mathsf{orth}}\left(\frac{K}{N}, \frac{E_b}{N_0}\right) = \frac{K}{N}\mathsf{C}^*\left(\frac{E_b}{N_0}\right). \qquad (4)$$

The equality of C^{orth} and C^* for $K = N$ is a consequence of the well-known fact [3] that for equal-rate equal-power users orthogonal multiple access incurs no loss in capacity relative to unconstrained multiple access. It is also known [4] that even if $K > N$, there exist spreading codes (*Welch-Bound-Equality*) that incur no loss in capacity. With random spreading sequences, the optimum spectral efficiency for $0 < \beta$ converges almost surely as $K \to \infty$ to

$$\lim_{K\to\infty} C^{opt} = \frac{1}{4\pi}\int_{a(\beta)}^{b(\beta)} \log(1+\mathsf{SNR}x)\sqrt{(b(\beta)-x)(x-a(\beta))}\frac{dx}{x},$$

where $a(\beta) = (\sqrt{\beta}-1)^2$, and $b(\beta) = (\sqrt{\beta}+1)^2$. Note that it is not necessary that the signature waveforms change from bit to bit in order for the spectral efficiency to become deterministic in the limit, the averaging effect comes from the consideration of many users with randomly assigned waveforms. When $K = N = 2$, binary random sequences achieve 75% of the spectral efficiency of orthogonal sequences [5]. When K is large, the loss in spectral efficiency as a function of E_b/N_0 due to a random choice of sequences (as opposed to optimal) vanishes as $E_b/N_0 \to \infty$ or as $\beta \to \infty$. The maximum loss is 50% and occurs at $K = N$, $E_b/N_0 \downarrow \log_e 2$.

V. Matched Filter Spectral Efficiency

Using a recent form of the divergence-gaged central limit theorem, we have shown that the spectral efficiency of the single-user matched filter converges almost surely as $K \to \infty$ to

$$\lim_{K\to\infty} C^{sumf} = \frac{\beta}{2}\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{SNR}\beta}\right).$$

The maximum (over K/N) spectral efficiency of the single-user matched filter receiver is

$$\lim_{\beta\to\infty} C^{sumf}(\beta, \frac{E_b}{N_0}) = \frac{\log_2 e}{2} - \frac{1}{2}\frac{N_0}{E_b} \qquad \frac{E_b}{N_0} > \log_e 2.$$

The use of random signatures as opposed to optimally chosen sequences brings about substantial losses in spectral efficiency for the single-user matched filter, unless $\frac{E_b}{N_0}$ is relatively low and K/N is high.

VI. Decorrelator Spectral Efficiency.

If $\beta \le 1$, the spectral efficiency of the decorrelator converges in mean-square as $K \to \infty$ to

$$\lim_{K\to\infty} C^{deco} = \frac{\beta}{2}\log(1+\mathsf{SNR}(1-\beta)),$$

which yields

$$C^{deco}\left(\beta, \frac{E_b}{N_0}\right) = \beta C^*\left((1-\beta)\frac{E_b}{N_0}\right). \qquad (5)$$

VII. MMSE Spectral Efficiency.

If $\beta > 0$, the spectral efficiency of the linear MMSE transformation converges in mean-square sense as $K \to \infty$ to

$$\lim_{K\to\infty} C^{mmse} = \frac{\beta}{2}\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right)$$

where

$$\mathcal{F}(x,z) \stackrel{\text{def}}{=} \left(\sqrt{x(1+\sqrt{z})^2+1}-\sqrt{x(1-\sqrt{z})^2+1}\right)^2.$$

VIII. Conclusions

A misconception that has arisen in the last few years [7] claims that in CDMA systems with large number of users, error-control-coding, perfect power control and long codes, little can be gained by exploiting the structure of the multiaccess interference at the receiver. Our results have shown that exactly the opposite conclusion is true. Because of the deleterious effects of imperfect power control on the single-user matched filter, we would expect that the spectral inefficiency of that receiver to be even greater in that situation. Another misconception [7] predicts that multiuser detectors suffer from high sensitivity to the actual signature waveforms. On the contrary, our convergence results have shown that, as the number of users grows, the variability in achievable signal-to-noise ratio and spectral efficiency due to the choice of signature waveforms vanishes. The optimum coding-spreading tradeoff favors negligible spreading for either optimum or single-user matched filter processing. In contrast, non-negligible spreading is optimum for linear multiuser detectors such as the decorrelator and the MMSE receiver. With an optimal choice of spreading factor, the spectral efficiencies of the decorrelator and MMSE receivers grow without bound as $\frac{E_b}{N_0}$ increases, in contrast to the single-user matched filter for which large signal-to-noise ratios offer little incentive. For large K/N, even if the signal-to-noise ratio is very low, the spectral efficiency of the single-user matched filter is a fraction of the optimum one. For low K/N systems (such as state-of-the-art CDMA), either the decorrelator or the MMSE are excellent choices and little inefficiency results from random rather than orthogonal signatures.

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