

SPECTRAL EFFICIENCY OF THE NONCOHERENT RICIAN FADING CHANNEL IN THE LOW-POWER REGIME

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ABSTRACT

Transmission of information over a discrete-time memoryless Rician fading channel is considered where neither the receiver nor the transmitter knows the fading coefficients and the input amplitude is subject to second and fourth moment constraints. The main focus is on the power-limited (i.e., wideband, low-power) regime. First the structure of the capacity-achieving input signals is investigated. It is shown that uniform phase is optimal if there is a line of sight component and that the capacity-achieving input amplitude distribution is discrete in the low-power regime. The spectral-efficiency/bit-energy tradeoff in the power-limited regime is examined and it is shown that the minimum bit energy is not always achieved at zero spectral efficiency. In this case, it is argued that one should avoid operating in the region where the spectral efficiency is lower than the spectral efficiency of the minimum bit energy point.

INTRODUCTION

Recently, the information theoretic analysis of fading channels has received much attention [9]. This interest is motivated by the rapid advances in wireless technology and the need to use scarce resources such as bandwidth and power as efficiently as possible under severe fading conditions. Providing the ultimate performance, information theoretic measures such as capacity, spectral efficiency and error exponents can be used as benchmarks to which we can compare the performance of practical communication systems.

A significant amount of effort has been expended to study fading channel models where side information about the fading is available at either the receiver or the transmitter or both (see [9] and references therein). However, under fast fading conditions noncoherent communications, where neither party knows the fading, often becomes the only available alternative. Richters [3] considered the problem of communicating over an average power limited discrete-time memoryless noncoherent Rayleigh fading channel. He conjectured that the capacity-

achieving amplitude distribution is discrete with a finite number of mass points. Recently, Abou-Faycal *et al.* [5] gave a rigorous proof of Richters' conjecture. This result shows that when the fading is known by neither the transmitter nor the receiver, the optimal amplitude distribution has a notably different character than that of unfaded Gaussian channels.

Another characterization for the capacity achieving input signals can be given in the low-power regime. For a general class of fading channels, Verdú [1] has recently shown that if there are no constraints other than average power, flash signaling, a class of unbounded peak-to-average ratio inputs, is necessary to achieve the capacity as $\text{SNR} \rightarrow 0$ when the channel realization is unknown at the receiver. Flash signaling can be practically employed in systems where sudden discharge of energy (e.g., using capacitors) is allowed, thus sidestepping the use of RF amplifiers. However, these peaky signals are not feasible in communication systems with strict peak-to-average ratio requirements. Furthermore, in some systems CDMA-type white signals which spread their energy over the available bandwidth are used because of their anti-jamming and low probability of intercept capabilities. Hence, it is of interest to investigate the effect upon the capacity of imposing a finite kurtosis constraint, especially in the low-power regime.

Médard and Gallager [6] considered noncoherent broadband fading channels with no specular component and limited the peakedness of the input signals by imposing a fourth moment input constraint. Then they show that such a constraint forces the mutual information to zero inversely with increasing bandwidth. Other results on this theme are obtained by Telatar and Tse [8] and Subramanian and Hajek [7].

In this paper, we study the capacity and the spectral efficiency in the low-power regime of the discrete-time memoryless Rician fading channel subject to second and fourth moment input constraints.

CHANNEL MODEL

We consider the following discrete-time memoryless Rician fading channel model

$$y_i = mx_i + a_i x_i + n_i \quad (1)$$

where a_i and n_i are independent and identically distributed (i.i.d.) circular zero mean complex Gaussian random variables,

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independent of each other and of the input, with variances $E\{|a_i|^2\} = \gamma^2$ and $E\{|n_i|^2\} = N_0$, m is a deterministic complex constant, x_i is the complex channel input and y_i is the complex channel output. $\{a_i\}$ and $\{n_i\}$ represent sequences of fading coefficients and background noise samples respectively.

The Rician fading channel model is particularly appropriate when there is a direct propagating line of sight (LOS) component in addition to the faded component arising from multipath propagation. Moreover, the Rician model includes both the unfaded Gaussian channel and the Rayleigh fading channel as two special cases. Hence, results obtained for this model provide a unifying perspective.

In the channel model (1), fading is assumed to be flat and hence has a multiplicative effect on the channel input. This is a valid assumption if the delay spread of the channel is much smaller than the symbol duration. Moreover, frequency selective fading channels can be decomposed into parallel non-interacting flat fading subchannels using orthogonal multicarrier techniques. Note also that the fading coefficients assume independent realizations at every symbol period. Under such fast fading conditions, reliable estimation of the fading coefficients may be quite difficult because of the short duration between independent fades. Therefore, we consider the noncoherent scenario where neither the receiver nor the transmitter knows the fading coefficients $\{a_i\}$.

CHANNEL CAPACITY AND OPTIMAL INPUT DISTRIBUTION

In this section, we elaborate on the structure of the capacity achieving input distribution for the Rician fading channel when the input has limited peakedness. To that end, we impose second and fourth moment constraints on the input amplitude:

$$E\{|x_i|^2\} \leq P_{av} \quad \forall i \quad (2)$$

$$E\{|x_i|^4\} \leq \kappa P_{av}^2 \quad \forall i \quad (3)$$

where P_{av} is the average power constraint and $1 < \kappa < \infty$. Since the average power constraint is always active for capacity-achieving sources, the fourth moment constraint is identical to $\frac{E\{|x_i|^4\}}{(E\{|x_i|^2\})^2} \leq \kappa$. Hence imposing a fourth moment constraint (3) is equivalent to putting a limitation on the kurtosis (fourth moment divided by squared second moment), which is a measure of the peakedness of the input signal. For the Rician channel model (1), the capacity is the supremum of the input-output mutual information over the set of all input distributions satisfying the constraints (2) and (3)

$$C = \sup_{\substack{F_x(\cdot) \\ E\{|x|^2\} \leq P_{av} \\ E\{|x|^4\} \leq \kappa P_{av}^2}} \int_{\mathbb{C}} \int_{\mathbb{C}} f_{y|x}(y|x) \log \frac{f_{y|x}(y|x)}{f_y(y)} dy dF_x(x) \quad (4)$$

where the conditional density of the output given the input,

$$f_{y|x}(y|x) = \frac{1}{\pi(\gamma^2|x|^2 + N_0)} \exp\left(-\frac{|y - mx|^2}{\gamma^2|x|^2 + N_0}\right),$$

is circular complex Gaussian. Moreover note that $f_y(y) = \int_{\mathbb{C}} f_{y|x}(y|x) dF_x(x)$ is the marginal output density, F_x is the distribution function of the input and \mathbb{C} denotes the complex domain.

First we can prove the following preliminary result on the optimal phase distribution.

Proposition 1: For the noncoherent Rician fading channel (1) with $|m| > 0$ and input constraints (2) and (3), uniformly distributed phase that is independent of the input amplitude is capacity-achieving¹.

In contrast to the noncoherent Rayleigh fading channel ($|m| = 0$) where the input phase information is completely destroyed, the above proposition shows that if there is a line of sight component, i.e., $|m| > 0$, input phase can indeed carry information and the uniform distribution maximizes the transmission rate.

With this characterization, we have reduced the optimization problem (4) to optimal selection of the distribution function of the input amplitude. For the sake of simplification in the notation, we introduce new random variables $R = \frac{1}{N_0}v^2$ and $r = \frac{\gamma}{\sqrt{N_0}}u$. After a straightforward transformation, the mutual information becomes

$$I(F_r) \stackrel{def}{=} I(x; y) = - \int_0^\infty f_R(R; F_r) \log f_R(R; F_r) dR - \int_0^\infty \ln(1+r^2) dF_r(r) - 1 \quad (5)$$

where $f_R(R; F_r) = \int_0^\infty g(R, r) dF_r(r)$ is the density function of R with the kernel given by $g(R, r) = \frac{1}{1+r^2} \exp\left(-\frac{R+Kr^2}{1+r^2}\right) I_0\left(\frac{2\sqrt{Kr}\sqrt{R}}{1+r^2}\right)$. Furthermore, F_r is the distribution function of r and $K = \frac{|m|^2}{\gamma^2}$ is the Rician factor. Now, the capacity formula can be recast as follows

$$C(\alpha, \kappa, K) = \sup_{\substack{F_r \\ E\{r^2\} \leq \alpha \\ E\{r^4\} \leq \kappa\alpha^2}} I(F_r) \quad (6)$$

depending on three parameters, namely, $\alpha = \gamma^2 \frac{P_{av}}{N_0}$, the normalized SNR; κ , the kurtosis constraint; and K , the Rician factor.

Using the techniques employed in [5], we first obtain the following sufficient and necessary condition.

Proposition 2: (Kuhn-Tucker Condition) For the noncoherent Rician channel (1) with input constraints (2) and (3), F_0 is a capacity achieving amplitude distribution if and only if there exist $\lambda_2 \geq 0$ and λ_1 such that the following is satisfied

$$\int_0^\infty g(R, r) \log f_R(R, F_0) dR + \log(1+r^2) + \lambda_1(r^2 - \alpha) + \lambda_2(r^4 - \kappa\alpha^2) + C + 1 \geq 0 \quad \forall r \geq 0 \quad (7)$$

with equality if $r \in E_0$ where E_0 is the set of points of increase of F_0 .

¹The result holds in wider generality: the feasible set defined by (2) and (3) can be replaced by any set of constraints that are imposed only on the input magnitude.

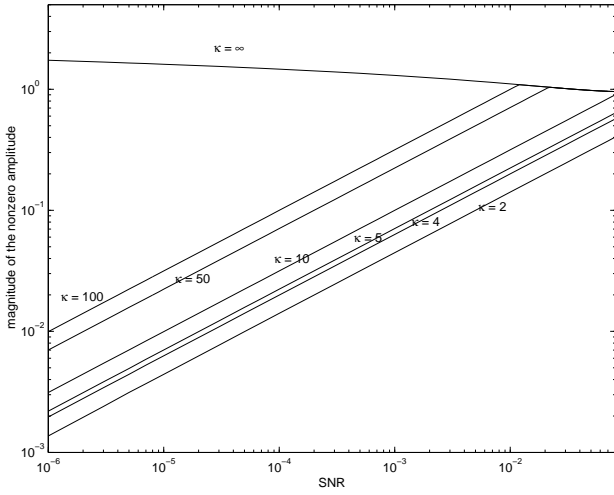


Fig. 1. Magnitude of the nonzero amplitude vs. SNR in the Rician channel $K = 1$.

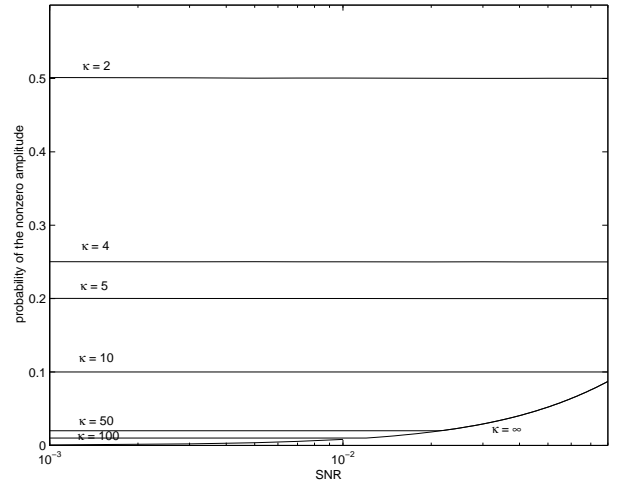


Fig. 2. Probability of the nonzero amplitude vs. SNR in the Rician channel $K = 1$.

Note that in the above formulation, λ_1 and λ_2 are the Lagrange multipliers for the second and fourth moment constraints respectively. Using the Kuhn-Tucker condition (7), we have the following result on the capacity achieving amplitude distribution.

Theorem 1: For the noncoherent Rician fading channel (1) with input amplitude constraints (2) and (3), if the fourth moment constraint (3) is active then the optimal input amplitude distribution is discrete with a finite number of mass points².

The significance of the above result comes from the fact that for the Rician fading channel, any finite kurtosis constraint will eventually be active for sufficiently small SNR because, as observed in [1], if there is no such constraint, the kurtosis of the optimal input grows without bound as $\text{SNR} \rightarrow 0$. Therefore, Theorem 1 establishes the discrete nature of the optimal input in the low-power regime which is the focus of this paper.

For the noncoherent Rician fading channel (1) subject to second and fourth moment input constraints, numerical results obtained using the Kuhn-Tucker condition (7) indicate that for sufficiently small SNR values, a particular two-mass-point discrete distribution with the following location and probabilities

$$\begin{cases} |x_1| = 0 & p_1 = 1 - \frac{1}{\kappa} \\ |x_2| = \sqrt{\kappa N_0 \text{SNR}} & p_2 = \frac{1}{\kappa} \end{cases} \quad (8)$$

is optimal. Note that this distribution does not depend on the Rician factor K .

Figures 1 and 2 plot the magnitude and the probability of the nonzero amplitude respectively as a function of SNR ($N_0 = 1$) for various values of κ . We immediately notice the significant impact of imposing a finite kurtosis constraint. When there is no kurtosis constraint, the nonzero amplitude migrates away from the origin as $\text{SNR} \rightarrow 0$ while its probability decreases sufficiently fast to satisfy the average power constraint. This

²The result holds in a more general setting where the fourth moment constraint (3) is replaced by a constraint in the following form: $E|x|^{2+\delta} \leq M$ for some $\delta > 0$ and $M < \infty$.

type of input is called *flash signaling* in [1]. However, as we see from the figures, if there is any finite kurtosis constraint, then the behavior is quite different. The nonzero amplitude approaches the origin as $\text{SNR} \rightarrow 0$ while its probability is kept constant.

It is also of interest to investigate the structure of the optimal input for the noncoherent Rician fading channel (1) with only an average power limitation (2). By Proposition 1, we know that the uniform phase is optimal for this model. Moreover, we have the following partial result on the optimal amplitude distribution which proves the suboptimality of input amplitude distributions with unbounded support such as the Rayleigh distribution.

Theorem 2: For the noncoherent Rician fading channel (1) with only average power limitation $E\{|x|^2\} \leq P_{av}$, the optimal input amplitude distribution has bounded support.

SPECTRAL EFFICIENCY VS. BIT ENERGY

In the power-limited regime, spectral-efficiency/bit-energy tradeoff is the key concept reflecting the fundamental tradeoff between bandwidth and power. We will denote the spectral efficiency as a function of bit energy by $C\left(\frac{E_b}{N_0}\right)$. If we assume without loss of generality that one complex symbol occupies $1s \times 1\text{Hz}$ time-frequency slot, then the maximum achievable spectral efficiency can be obtained from the Shannon capacity (bits/symbol)

$$C\left(\frac{E_b}{N_0}\right) = C(\text{SNR}) \quad \text{bits/s/Hz} \quad (9)$$

where

$$\frac{E_b}{N_0} = \frac{\text{SNR}}{C(\text{SNR})} \quad (10)$$

is the bit energy normalized to the noise power. For average power limited channels, the bit energy required for reliable communications decreases monotonically with decreasing spectral efficiency, and the minimum bit energy is achieved

at zero spectral efficiency, $\frac{E_b}{N_0} \Big|_{\min} = \frac{E_b}{N_0} \Big|_{C=0} = \frac{\log_e 2}{C(0)}$. Hence for fixed rate transmission, reduction in the required power comes only at the expense of increased bandwidth. Recently, Verdú [1] has analyzed the spectral-efficiency/bit-energy function in the power limited regime for a general class of average power limited fading channels and has shown that the minimum bit energy is $\log_e 2 = -1.59$ dB as long as the additive background noise is Gaussian. This minimum bit energy is achieved only in the asymptotic regime of infinite bandwidth. If one is willing to spend more power, then reliable communication over a finite bandwidth is possible. Hence achieving the minimum bit energy is not a sufficient criterion for finite bandwidth analysis. Verdú [1] has recently given the following formula for the wideband slope defined as the slope of spectral efficiency curve $C\left(\frac{E_b}{N_0}\right)$ in bits/s/Hz/3dB at zero spectral efficiency:

$$S_0 \stackrel{\text{def}}{=} \lim_{\frac{E_b}{N_0} \downarrow \frac{E_b}{N_0} \Big|_{C=0}} \frac{C\left(\frac{E_b}{N_0}\right)}{10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_0} \Big|_{C=0}} 10 \log_{10} 2 = \frac{2\left(\dot{C}(0)\right)^2}{-\ddot{C}(0)}, \quad (11)$$

where $\dot{C}(0)$ and $\ddot{C}(0)$ denote the first and second derivatives of capacity in nats. The wideband slope closely approximates the growth of the spectral efficiency curve in the power-limited regime and hence is proposed as a new tool providing insightful results when bandwidth is a resource to be conserved.

Under quite general conditions on the input and the channel, Prelov and Verdú [2] obtained the exact asymptotic second-order behavior of the input-output mutual information for vanishing SNR. Using this expression we have the following result.

Proposition 3: For the noncoherent Rician channel model (1) subject to second and fourth moment input constraints (2) and (3), the first and second derivatives of capacity at SNR = 0 are

$$\dot{C}(0) = |m|^2, \quad (12)$$

and

$$\ddot{C}(0) = \kappa\gamma^4 - (|m|^2 + \gamma^2)^2 \quad (13)$$

respectively.

As a corollary to the above Proposition, we can easily get expressions for the bit energy at zero spectral efficiency and the wideband slope.

Corollary 1: For the noncoherent Rician fading channel (1) subject to second and fourth moment input constraints (2) and (3), the normalized received bit energy $\frac{E_b^r}{N_0}$ required at zero spectral efficiency and the wideband slope are

$$\frac{E_b^r}{N_0} \Big|_{C=0} = \left(1 + \frac{\gamma^2}{|m|^2}\right) \log_e 2 \quad (14)$$

and

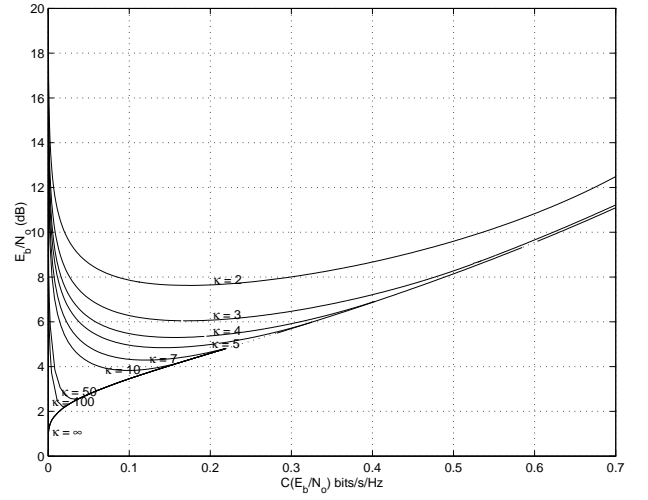


Fig. 3. $\frac{E_b^r}{N_0}$ (dB) vs. $C\left(\frac{E_b^r}{N_0}\right)$ bits/s/Hz for the Rayleigh Channel

$$S_0 = \frac{2\frac{|m|^4}{\gamma^4}}{\left(1 + \frac{|m|^2}{\gamma^2}\right)^2 - \kappa} \quad (15)$$

From (14), we immediately see that for the Rayleigh fading channel where $|m| = 0$, the bit energy required at zero spectral efficiency is infinite. Therefore reliable communications is not possible at this point. This is in sharp contrast with the behavior observed in average power limited channels where the bit energy required at zero spectral efficiency is indeed the minimum one. For Rician fading channels where $|m| > 0$, (14) is finite and interestingly does not depend on the kurtosis constraint κ . Moreover (14) monotonically decreases to -1.59 dB with increasing Rician factor $K = \frac{|m|^2}{\gamma^2}$. This is intuitively appealing because the channel is becoming more Gaussian where the bit energy at zero spectral efficiency is -1.59 dB.

For average power limited channels, the wideband slope is always nonnegative. In the noncoherent Rician fading channel subject to second and fourth moment input limitations, we again observe a markedly different behavior. From (15), we see that if $\kappa > \left(1 + \frac{|m|^2}{\gamma^2}\right)^2$, then the wideband slope is negative, leading to the conclusion that the minimum bit energy is achieved at a nonzero spectral efficiency. If $\kappa \leq \left(1 + \frac{|m|^2}{\gamma^2}\right)^2$, the wideband slope is positive and, based on numerical evidence, we conjecture that the minimum bit energy is achieved at zero spectral efficiency and is equal to (14).

Figures 3 and 4 plot the $\frac{E_b^r}{N_0}$ (dB) vs. $C\left(\frac{E_b^r}{N_0}\right)$ bits/s/Hz curves for the Rayleigh and Rician ($K = 1$) channels, respectively, for various values of κ . In the Rayleigh fading channel, for any finite kurtosis constraint κ , the bit energy curve is bowl-shaped, achieving its minimum at a nonzero spectral efficiency C^* . In this case, as observed in [1, p.1341], one should avoid operating in the region where the spectral efficiency is lower than C^* because decreasing the spectral efficiency further (i.e., increasing the bandwidth for fixed rate transmission) only increases the required power. In the Rician fading channel ($K = 1$), we observe the same behavior when $\kappa > 4$.

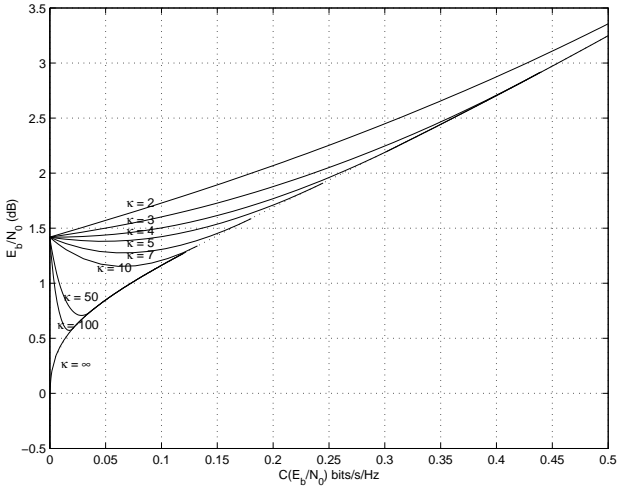


Fig. 4. $\frac{E_b^r}{N_0}$ (dB) vs. $C(\frac{E_b^r}{N_0})$ bits/s/Hz for the Rician Channel with $K = |m|^2/\gamma^2 = 1$

The minimum bit energy is achieved at a nonzero spectral efficiency. However note that now the bit energy required at zero spectral efficiency is finite and is the same for all finite κ . If $\kappa \leq 4$, the bit energy decreases monotonically with decreasing spectral efficiency and the minimum bit energy is achieved at zero spectral efficiency. Therefore in this case, the bandwidth-power tradeoff is the usual one that we encounter in average power limited channels; i.e., for fixed rate transmission, increasing the bandwidth decreases the power required for reliable communications.

A more stringent approach to limit the peakedness of the input will be to impose a peak power constraint in addition to the second moment constraint, i.e.,

$$E\{|x|^2\} \leq P_{av} \quad (16)$$

$$|x|^2 \leq \kappa P_{av} \quad \text{a.s.} \quad (17)$$

where $1 \leq \kappa < \infty$. For this case we have the following result.

Proposition 4: For the noncoherent Rician fading channel (1) with average and peak power limitations (16) and (17), the first and the second derivatives of capacity at zero SNR are

$$\dot{C}(0) = |m|^2 \quad \text{and} \quad \ddot{C}(0) = \kappa\gamma^4 - (|m|^2 + \gamma^2)^2. \quad (18)$$

Note that these derivative expressions are the same as the first and second derivative expressions (12) and (13) obtained under input constraints $E\{|x|^2\} \leq P_{av}$ and $E\{|x|^4\} \leq \kappa P_{av}^2$. Thus, imposing fourth moment or peak constraints on the input amplitude has the same effect in the low-power regime.

CONCLUSION

We have considered transmission over a discrete-time memoryless noncoherent Rician fading channel and have analyzed the effects upon the capacity and the spectral efficiency in the power limited regime of using input signals having limited peakedness, which is achieved by imposing a fourth moment input constraint.

First, we focused on the structure of the optimal input distribution. We showed the optimality of uniform phase if there is a line of sight component. Then, using a sufficient and necessary condition, we proved that the optimal input amplitude is discrete with a finite number of levels in the low-power regime.

We then analyzed the spectral-efficiency/bit-energy tradeoff in the low-power regime. In contrast to the behavior observed in average power limited channels, the minimum bit energy is not always achieved at zero spectral efficiency. In this case, we identified a forbidden region where one should not operate. We also established that replacing the fourth moment constraint with a peak constraint does not affect the second order asymptotics of the capacity and hence the achievable low-power performance.

DISCLAIMER

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