

TOTAL CAPACITY OF THE RMS BANDLIMITED K -USER PAM SYNCHRONOUS CHANNEL

ROGER S. CHENG & SERGIO VERDÚ †

Department of Electrical Engineering

Princeton University, Princeton, NJ 08544

ABSTRACT

Continuous-time additive white Gaussian noise channels with strictly time-limited and root mean square (RMS) bandlimited inputs are studied. RMS bandwidth is equal to the normalized second moment of the spectrum, which has proved to be a useful and analytically tractable measure of the bandwidth of strictly time-limited waveforms.

We find the Total Capacity (TC) of the K -user channel under total power and power-weighted average RMS bandwidth constraints. A lower bound to the TC under equal-power constraint is obtained. Total Capacity Ratio (TCR) is defined as the ratio of the K -user TC to K times the single-user capacity. Power (Bandwidth) efficiency is defined as the ratio of the effective power (bandwidth) to the actual power (bandwidth). The effective power (bandwidth) is the corresponding power (bandwidth) needed for a single user channel to achieve the same capacity. We find lower bounds to the TCR and efficiencies which indicate that savings in bandwidth compared to the FDMA scheme can be achieved by the CDMA scheme at the expense of more complicated decoding hardware.

1. INTRODUCTION

In this paper, we deal with the continuous-time Pulse Amplitude Modulation (PAM) Gaussian multiple-access channel (MAC). Each user is assigned a fixed deterministic continuous-time signature waveform, $s_k(t)$, which is time-limited to $[0, T]$ and is modulated linearly by the information stream. Assuming that the transmitters are symbol-synchronous, the channel can be expressed as

$$y(t) = \sum_{i=1}^n \sum_{k=1}^K b_k(i) s_k(t - iT) + n(t) \quad (1)$$

where $n(t)$ is white Gaussian noise with spectral density, $\frac{N_0}{2}$ and $\{b_k(i)\}$ is the symbol stream transmitted by the k^{th} user.

The capacity region of this channel has been found by Verdú [1] [2]. Denoting \mathbf{W} and \mathbf{H} as the diagonal matrix with the users' powers as its diagonal entries, and the cross-correlation matrix of the normalized signature waveforms, respectively, the capacity region is expressed as

$$C_V = \left\{ (R_1, R_2, \dots, R_K) : \sum_{j \in J} R_j \leq \frac{1}{2T} \log[\det(\mathbf{I}_{|J|} + \frac{2T}{N_0} \mathbf{W}_J \mathbf{H}_J)] \quad \forall J \subset \{1, \dots, K\} \right\} \quad (2)$$

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where A_j is the $|J| \times |J|$ matrix formed by the j^{th} row and column of A for all $j \in J$. It is clear that, without other constraints, the capacity region is maximized by orthogonal signature waveforms. However, under bandwidth constraints, orthogonal signature waveforms are not necessarily optimal since orthogonality can only be achieved by lowering the symbol rate, $1/T$.

There are many different bandwidth definitions [3]. In this paper, we concentrate on the root mean square (RMS) bandwidth because it is analytically tractable and can be applied to strictly time-limited signals. The RMS bandwidth was introduced by Gabor [4] and studied subsequently in [3], [5] and [6]. It is the square root of the second moment of the energy spectral density ($|S_k(f)|^2$) of the normalized signal which is proportional to the square root of the energy of its derivative.

In the two-user case, the capacity region of the RMS bandlimited PAM channel has been found in [7] and the total capacity (the maximum rate sum over the capacity region) is larger than the single-user capacity with the power equal to the sum of the users' powers. The gain in the total capacity from the single-user to the two-user case can be explained by the increase in the dimensionality of the signal set. We can consider the transmitted signal in a symbol interval as a signal drawn from a signal set. Then, the signal set in the single-user and the two-user case are one-dimensional and two-dimensional, respectively. From this viewpoint, it is easy to see that the total capacity increases as the number of users increases while the total power remains constant.

In this paper, we find the total capacity (TC) of the K -user channel under the total power constraint

$$\text{tr}(\mathbf{W}) \leq W \quad (3)$$

and the power-weighted averaged RMS bandwidth constraint

$$\frac{1}{\text{tr}(\mathbf{W})} \sum_{k=1}^K \mathbf{W}_{kk} \int_{-\infty}^{\infty} f^2 |S_k(f)|^2 df \leq B^2 \quad (4)$$

The power constraint is placed on the total power instead of the individual power since the later requires finding all possible sets of eigenvalues of a positive definite matrix with fixed diagonal entries which is, in general, intractable. The bandwidth constraint is justified because the power-weighted average RMS bandwidth is the RMS bandwidth of the power spectral density of the transmitted signal.

Several performance measures, Total Capacity Ratio, Power efficiency and Bandwidth efficiency, are defined and analyzed. Bounds and limiting values of these measures are also obtained.

2. TOTAL CAPACITY

Theorem 2.1.

The Total Capacity of the K -user RMS bandlimited PAM Gaussian MAC with total power and power-weighted average RMS bandwidth constraints is

$$\text{TC}(B, K, \Lambda) = \max \left\{ \frac{B}{\gamma} \sum_{n=1}^N \log[1 + h_n(\lambda)] \right\} \log e \quad (5)$$

where the maximization is over $1 \leq N \leq K$, $\frac{\gamma K \Lambda}{N} \leq \lambda$, and $1 \leq \gamma \leq \sqrt{f_N}$ such that $\gamma = \sqrt{f_N}$ iff $\lambda = \frac{\gamma K \Lambda}{N}$, and

$$\sum_{n=1}^N h_n(\lambda) = \gamma K \Lambda \quad (6)$$

where

$$h_n(\lambda) \triangleq \frac{N \lambda f_N + \gamma K \Lambda [\lambda (\gamma^2 - 1) - 1] - n^2 (N \lambda - \gamma K \Lambda)}{N (f_N - 1 - \lambda) + \gamma^3 K \Lambda + n^2 (N \lambda - \gamma K \Lambda)} > 0 \quad (7)$$

$$f_N \triangleq \frac{1}{N} \sum_{n=1}^N n^2 = \frac{1}{6} (N+1)(2N+1) \quad (8)$$

and the average signal-to-noise ratio is denoted by

$$\Lambda \triangleq \frac{W}{K N_0 B} \quad (9)$$

Proof.

Since the signature waveforms are RMS bandlimited, and the set $\{\phi_i(t, T)\}_{i=1}^{\infty}$ where

$$\phi_i(t, T) \triangleq \begin{cases} \sqrt{\frac{2}{T}} \sin\left(\frac{\pi i t}{T}\right) & \text{if } t \in [0, T]; \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

forms a complete orthonormal basis for all RMS bandlimited signals, we can write

$$\begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_K(t) \end{bmatrix} = \mathbf{A}^T \Phi(t, T) \quad (11)$$

where $\Phi(t, T) \triangleq [\phi_1(t, T) \phi_2(t, T) \dots]^T$. Then, the power-weighted average RMS bandwidth constraint can be written, via the Parseval's theorem, as

$$\begin{aligned} \frac{1}{\text{tr}(\mathbf{W})} \sum_{k=1}^K \mathbf{W}_{kk} \int_0^T \left| \frac{d}{dt} s_k(t) \right|^2 dt &= \frac{1}{(2T)^2 \text{tr}(\mathbf{W})} \text{tr}(\mathbf{W} \mathbf{A}^T \mathbf{\Pi} \mathbf{A}) \\ &= \frac{1}{(2T)^2 \text{tr}(\mathbf{W})} \text{tr}(\mathbf{\Pi} \mathbf{A} \mathbf{W} \mathbf{A}^T) \\ &\leq B^2 \end{aligned} \quad (12)$$

From the capacity region in (2), it is clear that the total capacity is maximized when $\text{tr}(\mathbf{W}) = W$. We denote the time-bandwidth product by $\gamma \triangleq 2BT$, the average signal-to-noise ratio by $\Lambda \triangleq \frac{W}{K N_0 B}$, and the eigenvalues of $\frac{2T}{N_0} \mathbf{W} \mathbf{H}$ by λ_k such that $\lambda_j \leq \lambda_i, \forall i < j \leq K$. Then, the total capacity becomes

$$\text{TC}_V = \frac{B}{\gamma} \sum_{k=1}^K \log[1 + \lambda_k] \quad (13)$$

and the power constraint becomes

$$\sum_{k=1}^K \lambda_k = \gamma K \Lambda \quad (14)$$

Since the eigenvalues of $\frac{2T}{N_0} \mathbf{W} \mathbf{H}$ are also the eigenvalues of $\frac{2T}{N_0} \mathbf{A} \mathbf{W} \mathbf{A}^T$, and once $\{\lambda_k\}_{k=1}^K$ are fixed, the left hand side of (12) is minimized when $\mathbf{A} \mathbf{W} \mathbf{A}^T$ is diagonal with decreasing diagonal entries, we can rewrite (12) as

$$\sum_{k=1}^K k^2 \lambda_k \leq \gamma^3 K \Lambda \quad (15)$$

For fixed T , the total capacity is found by maximizing (13) over all $\lambda_k \geq 0$, $k = 1, \dots, K$ under the constraints (14) and (15). Using the Kuhn-Tucker Theorem, we form the Lagrangian

$$-\sum_{k=1}^K \log[1 + \lambda_k] + x \left(\sum_{k=1}^K \lambda_k - \gamma K \Lambda \right) + y \left(\sum_{k=1}^K k^2 \lambda_k - \gamma^3 K \Lambda \right) \quad (16)$$

and obtain the necessary conditions:

$$\lambda_n = \frac{1}{x + y n^2} - 1 > 0 \quad n = 1, \dots, N. \quad (17)$$

and $\lambda_n = 0$ for all $n > N$,

$$y \left(\sum_{n=1}^N n^2 \lambda_n - \gamma^3 K \Lambda \right) = 0 \quad (18)$$

and $0 \leq y$.

Rewriting (17) as $(x + y n^2)(1 + \lambda_n) = 1$, and summing over all n , we have, from (14) and (18),

$$(N + \gamma K \Lambda)x + (N f_N + \gamma^3 K \Lambda)y = N \quad (19)$$

Particularizing (17) to $n = 1$, and substituting in (19), we have

$$y = \frac{N \lambda_1 - \gamma K \Lambda}{(1 + \lambda_1)(N(f_N - 1) + \gamma K \Lambda(\gamma^2 - 1))} \quad (20)$$

and

$$x = \frac{N(f_N - 1 - \lambda_1) + \gamma^3 K \Lambda}{(1 + \lambda_1)(N(f_N - 1) + \gamma K \Lambda(\gamma^2 - 1))} \quad (21)$$

Substituting (20) and (21) into (17), and denoting λ_1 by λ and λ_n by $h_n(\lambda)$, we have (7), and the power constraint in (14) becomes (6).

When $y = 0$, $\lambda = h_n(\lambda) = \frac{\gamma K \Lambda}{N}$ for all $n = 1, \dots, N$. Upon substituting into (15), we have $\sqrt{f_N} \leq \gamma$. Since the total capacity becomes $\frac{BN}{\gamma} \log[1 + \frac{\gamma K \Lambda}{N}]$ which is monotonically decreasing in γ , the optimal γ is equal to $\sqrt{f_N}$ and (15) is satisfied with equality. If we rewrite (7) and sum up over all n , we have

$$(N \lambda_1 - \gamma K \Lambda) \left(\sum_{n=1}^N n^2 h_n(\lambda) - \gamma^3 K \Lambda \right) = 0 \quad (22)$$

When $0 < y$, $\frac{\gamma K \Lambda}{N} < \lambda$, and from (22), (15) is again satisfied with equality. Therefore, if we require $\gamma = \sqrt{f_N}$ iff $\lambda = \frac{\gamma K \Lambda}{N}$, (15) is superfluous. Finally, specifying the range of γ and λ and the condition that $\gamma = \sqrt{f_N}$ iff $\lambda = \frac{\gamma K \Lambda}{N}$, we have the desired result. ■

This theorem gives the exact calculation needed for the TC. The main reason why we cannot obtain a simpler solution is the lack of a closed form expression of $\sum_{n=1}^N \frac{1}{a + b n^2}$. However,

despite the complicated expression, the TC can be computed once the average signal-to-noise ratio, Λ , and K is given. In Figure 1, we show the TC with different values of K and Λ .

For a given \mathbf{W} , we show that any set of signature waveforms, with \mathbf{A} such that $\mathbf{A}\mathbf{W}\mathbf{A}^T$ is a diagonal matrix with the n^{th} diagonal entry equal to $h_n(\lambda)$, is optimal. However, such an \mathbf{A} does not always exist for any arbitrarily given \mathbf{W} . For fixed total power, W , finding the set of \mathbf{W} where \mathbf{A} exists is equivalent to finding the possible set of diagonal entries of a positive definite matrix with fixed eigenvalues, which seems intractable. Reversing the problem, one may want to fix the \mathbf{W} and find the total capacity. In general, this is equivalent to finding the possible set of eigenvalues of a positive definite matrix with fixed diagonal entries, which is again intractable.

In the following theorem, we give a lower bound to the TC in the equal-power case where $\mathbf{W} = \frac{W}{K}\mathbf{I}$. Clearly, this is also a lower bound to the capacity of the original channel with the total power constraint in Theorem 2.1.

Theorem 2.2.

The lower bound to the Total Capacity when the users' powers are the same is

$$\text{TC}_{\text{EP}}(B, K, \Lambda) \geq \max_{1 \leq \gamma \leq \sqrt{f_K}} \frac{B}{\gamma} \log \left\{ \left[1 + \gamma \Lambda \frac{(\gamma^2 - 1) + K(f_K - \gamma^2)}{f_K - 1} \right] \left[1 + \gamma \Lambda \frac{\gamma^2 - 1}{f_K - 1} \right]^{K-1} \right\} \quad (23)$$

Proof.

The lower bound is found by exhibiting a symmetric positive definite matrix \mathbf{H} , such that the total capacity for that particular signature waveform set is easy to find. We let \mathbf{H} be

$$\mathbf{H} = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{bmatrix} \quad (24)$$

the eigenvalues of \mathbf{H} with $0 \leq \rho \leq 1$ to be specified in the sequel are $1 + (K - 1)\rho$ and $1 - \rho$ with multiplicity $K - 1$. Then, the total capacity under the equal-power constraint becomes

$$\begin{aligned} \text{TC}_V &= \frac{B}{\gamma} \log[\det(\mathbf{I}_K + \gamma \Lambda \mathbf{H})] \\ &= \frac{B}{\gamma} \log \left\{ [1 + \gamma \Lambda(1 + (K - 1)\rho)] [1 + \gamma \Lambda(1 - \rho)]^{K-1} \right\} \end{aligned} \quad (25)$$

while the bandwidth constraint (15) becomes

$$\rho \geq \frac{f_K - \gamma^2}{f_K - 1} \quad (26)$$

Since (25) is monotonically decreasing in ρ when $0 \leq \rho \leq 1$, the TC is maximized when ρ achieves equality in (26). Substituting ρ from (26) with equality into (25), and maximizing over all γ , we have (23). ■

In Figure 1, we plot the lower bound to the TC_{EP} for different values of K and Λ . Since the TC under the total power constraint serves as an upper bound to the TC_{EP} , Figure 1 gives a tight upper and lower bound to the Total Capacity of an equal-power constrained channel, for moderate number of users.

As a performance measure, we define the Total Capacity Ratio (TCR) as the ratio of the K -user TC to K times the single-user capacity with the same RMS bandwidth and average signal-to-noise ratio constraints. Since the single-user capacity of a RMS bandlimited PAM channel is equal to $B \log[1 + \Lambda]$ (see [7]), the TCR can be written as

$$TCR(K, \Lambda) \triangleq \frac{TC(B, K, \Lambda)}{KB \log[1 + \Lambda]} \quad (27)$$

The TCR gives the ratio of the capacity available to an average user (when the channel is shared by K users) to the single-user capacity. In other words, it measures, from the user's viewpoint, the ratio of the average user capacity in a multi-user channel to the capacity in a single-user channel. Notice that the TCR depends only on K and Λ , and is independent of B . Using the lower bound in Theorem 2.2, we obtain a lower bound to the TCR under the equal-power constraint for all signal-to-noise ratios.

Corollary 2.1.

A lower bound to the TCR under the equal-power constraint for all signal-to-noise ratio is

$$TCR(K, \Lambda) \geq \frac{1}{\gamma} \quad (28)$$

where γ is the positive real root of the equation

$$\gamma(\gamma^2 - 1) = f_K - 1 \quad (29)$$

Proof.

In order to obtain (28), we simply substitute γ from (29) into (23) and (27). Since there is one and only one real positive solution in (29), there is no ambiguity in the value of γ . ■

In Figure 2, we show the TCR under the total power constraint and the lower bound to the TCR under the equal-power constraint for different number of users and different average signal-to-noise ratios.

3. EFFICIENCIES

The TCR gives the performance degradation, from the user's viewpoint, when a bandlimited channel is shared by K users instead of a single user. A natural question to be asked is "How to maintain the same rate in the presence of other users?" If we want to maintain the same information rate, we have to modify some of the parameters. In the following, we will analyze two alternatives. First, we increase the signal-to-noise ratio by increasing the power while the bandwidth remains constant. Second, we increase the bandwidth of the channel while the power of each user remains the same.

The power efficiency, denoted by $\eta_P(K, \Lambda)$, is defined as

$$\eta_P(K, \Lambda) \triangleq \frac{e^{\frac{\text{TC}(B, K, \Lambda)}{BK}} - 1}{\Lambda} \quad (30)$$

or, equivalently, implicitly as

$$\text{TC}(B, K, \Lambda) = BK \log[1 + \eta_P(K, \Lambda)\Lambda] \quad (31)$$

The bandwidth efficiency, denoted by $\eta_B(K, \Lambda)$, is defined implicitly as

$$\text{TC}(B, K, \Lambda) = \eta_B(K, \Lambda)BK \log\left[1 + \frac{\Lambda}{\eta_B(K, \Lambda)}\right] \quad (32)$$

The power efficiency, $\eta_P(K, \Lambda)$ (bandwidth efficiency, $\eta_B(K, \Lambda)$) gives the ratio of the effective power (bandwidth) to the actual power (bandwidth) when the actual signal-to-noise ratio is Λ . The actual power (bandwidth) is the power (bandwidth) used in transmission while the effective power (bandwidth) is the corresponding power (bandwidth) needed for a single user channel to achieve the same capacity. In other words, $-10 \log[\eta_P(K, \Lambda)]$ gives the power in db that we have to add to each user in order to maintain the single-user capacity. Similarly, $1/\eta_B(K, \Lambda)$ gives the ratio that we have to increase the bandwidth in order to maintain the same information rate.

Theorem 3.1.

The power efficiency satisfies,

$$\lim_{\Lambda \rightarrow \infty} \eta_P(K, \Lambda) = 0 \quad (33)$$

A lower bound to the bandwidth efficiency, $\eta_B(K, \Lambda)$, for all signal-to-noise ratio under the equal-power constraint is

$$\eta_B(K, \Lambda) \geq \frac{1}{\sqrt{f_K}} \quad (34)$$

where f_K is defined in (8).

Proof.

From the definition of $\eta_P(K, \Lambda)$, we have,

$$\prod_{n=1}^N (1 + h_n(\lambda)) = [1 + \eta_P(K, \Lambda)\Lambda]^{K\gamma} \quad (35)$$

where N , $h_n(\lambda)$ and γ are all optimally selected for that Λ . Subtracting (14) from (15), it is easy to get

$$h_n(\lambda) \leq \gamma K \Lambda (\gamma^2 - 1) \quad n = 1, \dots, N. \quad (36)$$

Substituting (36) into (35), and dividing both side by $\Lambda^{K\gamma}$, we have, in the limit as $\Lambda \rightarrow \infty$,

$$\lim_{\Lambda \rightarrow \infty} \Lambda^{N-K\gamma} \prod_{n=1}^N \left\{ \frac{1}{\Lambda} + \gamma K (\gamma^2 - 1) \right\} \geq \eta_P(K, \Lambda)^{K\gamma} \quad (37)$$

If $\gamma \rightarrow 1$ as $\Lambda \rightarrow \infty$, the second factor on the left hand side of (37) tends to 0, while the first factor tends either to 0 ($N < K$) or to 1 ($N = K$). On the other hand, if $\gamma \rightarrow \alpha > 1$ as $\Lambda \rightarrow \infty$, the first factor on the left hand side of (37) tends to 0 for any $N \leq K$, while the second factor is bounded. Therefore, in both cases, the left hand side tends to 0 and since $1 \leq \gamma \leq \sqrt{f_K}$, we have $\eta_P(K, \Lambda) \rightarrow 0$ as $\Lambda \rightarrow \infty$.

Substituting $\gamma = \sqrt{f_K}$ in Theorem 2.2, we have

$$\text{TC}_{\text{EP}}(B, K, \Lambda) \geq \frac{1}{\sqrt{f_K}} BK \log[1 + \sqrt{f_K} \Lambda] \quad (38)$$

Since the right hand side of (32) is monotonically increasing in $\eta_B(K, \Lambda)$, we have (34) when compared to (38). ■

The TC is obtained by optimizing the balance between the "symbol rate" factor, B/γ , and the "information sent per symbol" factor, $\log[\cdot]$. As the average signal-to-noise ratio tends to infinity, the "symbol rate" factor dominates and the optimal users' signature waveforms are asymptotically identical. Then, the product term of the signal-to-noise ratios inside the log function in the TC becomes relatively small, and the asymptotic power efficiency is equal to zero. The bandwidth efficiency indicates the increase in bandwidth needed to maintain the same user rate when a single-user channel is shared by K users.

In Figure 3 and 4, we plot the Power and Bandwidth efficiency for different values of K and Λ . Also, in the same graphs, we show the lower bound to the efficiencies for the equal-power constrained channel. It shows that regardless of the signal-to-noise ratios, increasing the bandwidth by a factor of 10, we can accommodate about 50 users on the multi-user channel. This indicates a 80 percent reduction in the bandwidth required by Frequency Division Multiple Access (FDMA). Clearly, the tradeoff is a more complicated demodulating and decoding process in the Synchronous Code Division Multiple Access (CDMA) channel, which is a special case of the current model.

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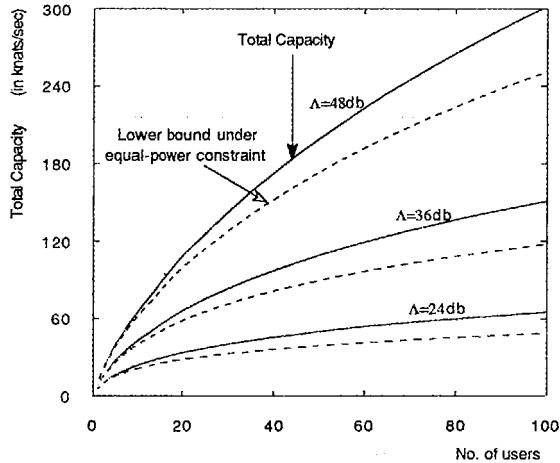


Figure 1. Total Capacity for different number of users and average signal-to-noise ratios when $B=1\text{kHz}$

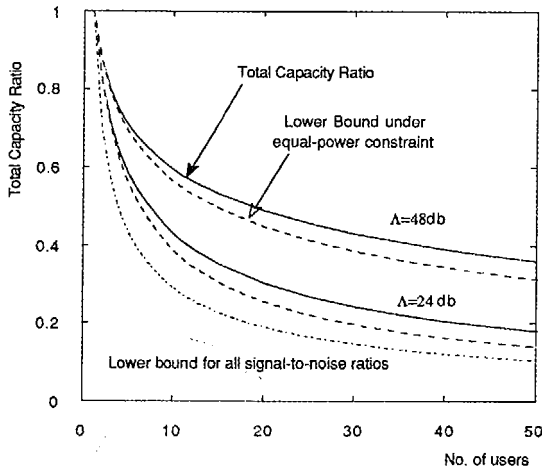


Figure 2. Total Capacity Ratio for different number of users and average signal-to-noise ratios

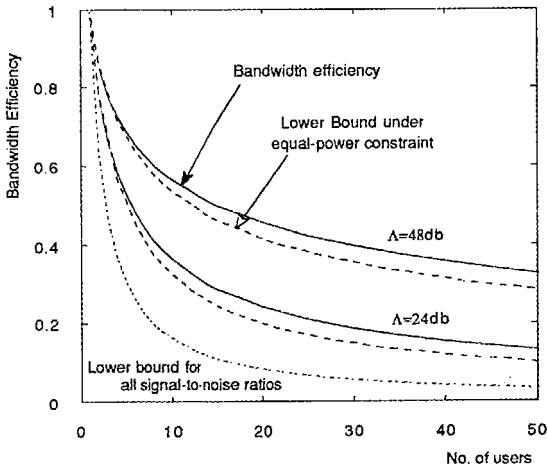


Figure 3. Bandwidth efficiency for different number of users and average signal-to-noise ratios

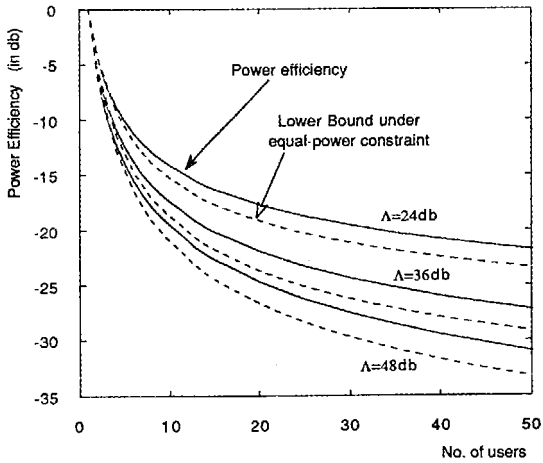


Figure 4. Power efficiency for different number of users and average signal-to-noise ratios