Asymptotic Analysis of Improved Linear Receivers for BPSK-CDMA Subject to Fading

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***Abstract*—In this paper, we design and analyze a new class of linear multiuser detectors, which can be applied when the users employ BPSK modulation and the fading coefficients of the active users are known at the receiver (such as base-station demodulation). The tools of asymptotic distribution of the spectrum of large random matrices are used to show that relative to the classical minimum mean-square-error (MMSE) receiver, the output signal-to-noise ratio (SNR) improves by halving the number of effective interferers and adding 3 dB to the input SNR. We also propose sensible approximations to the proposed linear receivers so as to facilitate their use in CDMA systems that employ long codes.**

***Index Terms*—Decorrelator, fading channels, MMSE, multiuser detection, random spreading sequence, spectral efficiency.**

I. Introduction

THE CLASSICAL linear multiuser detectors (decorrelating and minimum mean-square-error (MMSE) receivers) exploit the geometry of the space spanned by the interference to mitigate the effect of multiaccess interference (MAI). However, unlike nonlinear multiuser detectors (maximum-likelihood, successive cancellation, etc.), those detectors do not exploit the structure of the modulation format of the desired user and the MAI. There is much to be gained by exploiting the structure of the modulation. For example, when an interferer is very strong, a nonlinear receiver can essentially nullify its effect, whereas a decorrelator (or an MMSE receiver) would pay a penalty in performance due to the presence of that interferer. In this paper, we take a hybrid approach that considers linear receivers designed taking into account certain information available when the interferers use antipodal one-dimensional modulation [binary phase-shift keying (BPSK)] and their channel fades are known at the receiver.

Unlike additive white Gaussian noise (AWGN), when the users employ BPSK modulation, the baseband equivalent of the MAI process has a nonzero pseudo-autocovariance function when conditioned on the received fading coefficients. To see this, consider the contribution to the observable vector of an interferer with signature $\mathbf{s}_k$, symbol b_k and fading coefficient α_k. The pseudocovariance matrix[1] of that random vector is given by:

$$E\left[\alpha_k^2\right] E\left[b_k^2\right] \mathbf{s}_k\mathbf{s}_k^T$$

which is equal to 0 because the fading coefficient α_k is circularly symmetric. However, if α_k is known at the receiver, we are actually interested in the (generally complex valued) conditional pseudocovariance matrix

$$\alpha_k^2 E\left[b_k^2\right] \mathbf{s}_k\mathbf{s}_k^T.$$

If the symbol b_k were the result of quaternary phase-shift keying (QPSK) modulation (or other such symmetric constellations in the complex plane), then $E[b_k^2] = 0$. However, in the case of BPSK, $E[b_k^2] = 1$ and the pseudocovariance matrix of the multiaccess interferers, whose fading values are known at the receiver, is nonzero. As already noticed and as will be more clear in the next sections, the proposed receivers degenerate in the conventional decorrelating and MMSE receiver when constellations that are symmetric in the complex plane (like QPSK) are employed. Even though the leading standard proposals for third-generation wireless networks (UMTS in Europe and cdma2000 in North America) use mostly symetric signaling, the proposed approach extends also to the case of code division multiple access (CDMA) systems employing QPSK modulation with phase and quadrature components modulated by different signature. This type of signaling is used in many modes within UMTS [1]. Moreover, our approach extends also to CDMA systems employing BPSK with complex signatures and to more general CDMA system employing symmetric constellation with improper cochannel interference.

Following up on work of the first author in [3] and [7] in this paper, we exploit the nonzero conditional pseudocovariance functions to obtain alternative decorrelator and MMSE receivers. Since these linear transformations operate on both the received observable vector and its conjugate, we refer to them as *linear conjugate receivers*.

Our new performance results of the linear conjugate receivers use the tools developed in recent years [13], [5], [14] for the analysis of multiuser detectors in the wideband limit when the number of users goes to infinity while keeping the ratio β of users to spreading-gain constant. The performance gain afforded over the classical linear multiuser receivers by the linear conjugate receivers can be quite dramatic. For example, the near–far resistance of the decorrelator becomes $1-\beta/2$ instead of $1-\beta$ and the output signal-to-noise ratio (SNR) of

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[1]Unlike the covariance matrix which averages a vector times its Hermitian, the pseudocovariance matrix is the average of the vector times its transpose.

the MMSE depends on β and the input SNR in the same way as in the classical MMSE receiver (Tse–Hanly formula [5]) except that the effective number of interferers is halved and 3 dB is added to the input SNR.

Another contribution of this paper is to propose low-complexity approximations to the original linear-conjugate receivers, which would be of interest when the CDMA system employs long codes. Matrix inverses are approximated by polynomials whose coefficients can be computed in various ways. A particularly promising way is to obtain them through an asymptotic design which, like the asymptotic SNR analysis, uses recent results on the asymptotic eigenvalue distribution of random matrices.

II. System Model

We consider a synchronous BPSK-modulated direct sequence CDMA (DS/CDMA) system with K active users, employing long spreading codes and operating over a frequency-flat fading channel. The baseband equivalent of the received signal is

$$r(t) = \sum_{\ell=-\infty}^{\infty}\sum_{k=1}^{K}\alpha_k(\ell)\sqrt{\mathcal{E}_k}b_k(\ell)s_k^\ell(t-\ell T_b)+n(t) \qquad (1)$$

where

- $\mathcal{E}_k$ — energy of the kth user;
- $\{b_k(\ell)\}_{\ell=-\infty}^{+\infty}$ — bit stream of the k-th user, modeled as a sequence of independent and identically distributed binary variables, each taking on values in the set $\{-1, 1\}$;
- T_b — bit interval duration;
- $s_k^\ell(t)$ — signature waveform assigned to the kth user in the ℓth signaling interval, expressed as

$$s_k^\ell(t) = \sum_{n=0}^{N-1} s_{k,n}^\ell u_{T_c}(t-nT_c)$$

  with N the processing gain, $\boldsymbol{s}_k^\ell = [s_{k,0}^\ell, \ldots, s_{k,N-1}^\ell]$ the kth spreading code in the ℓth signaling interval, T_c the chip interval, and $u_{T_c}(\cdot)$ a chip waveform with zero autocorrelation at multiples of T_c and unit energy. The spreading code $\boldsymbol{s}_k^\ell$ is normalized to have unit norm and its elements[2] are $s_{k,n}^\ell \in \{-1/\sqrt{N}, 1/\sqrt{N}\}$ $n = 0, \ldots, N-1$;
- $\alpha_k(\ell)$ — complex channel gain. The complex-valued fading parameters $\alpha_k(\ell)$ are zero-mean independent from user to user and follow a common distribution (for all k and ℓ) such that $E[\alpha_k(\ell)\alpha_n^*(\ell)] = \delta_{nk}$ and $E[\alpha_k(\ell)\alpha_n(\ell)] = 0$ with $(\cdot)^*$ denoting conjugate.

Let us assume that we want to demodulate user "1", and that a decision about $b_1(\ell)$ is made by processing the observation in the interval $[\ell T_b, (\ell+1)T_b]$. Chip-matched filtering the received waveform in the interval $[\ell T_b, (\ell+1)T_b]$, we obtain the following N-dimensional vector sequence of observables

$$\begin{aligned}\boldsymbol{r}(\ell) &= b_1(\ell)\sqrt{\mathcal{E}_1}\alpha_1(\ell)\boldsymbol{s}_1^\ell \\ &\quad + \sum_{k=2}^{K} b_k(\ell)\sqrt{\mathcal{E}_k}\alpha_k(\ell)\boldsymbol{s}_k^\ell + \boldsymbol{n}(\ell) \\ &= b_1(\ell)\sqrt{\mathcal{E}_1}\alpha_1(\ell)\boldsymbol{s}_1^\ell + \boldsymbol{S}_1^\ell\boldsymbol{A}_1\boldsymbol{B}_1(\ell)\alpha_1(\ell) + \boldsymbol{n}(\ell) \qquad (2)\end{aligned}$$

where

- $\boldsymbol{A}_1$ — $\mathrm{diag}(\sqrt{\mathcal{E}_2}, \ldots, \sqrt{\mathcal{E}_K})$;
- $\boldsymbol{B}_1(\ell)$ — $\mathrm{diag}(b_2(\ell), \ldots, b_K(\ell))$;
- $\boldsymbol{S}_1^\ell$ — $N \times (K-1)$ dimensional matrix whose columns are the signatures $\{\boldsymbol{s}_k^\ell\}_{k=2}^K$;
- $\alpha_1(\ell) = [\alpha_2(\ell), \ldots, \alpha_K(\ell)]$ — $K-1$ dimensional fading coefficient vector.

In (2) the first term on the right-hand side (RHS) represents the contribution from the bit to be decoded, while the other terms represent the contributions from the MAI and the thermal noise, $\boldsymbol{n}(\ell)$, respectively. Finally, notice that $\boldsymbol{n}(\ell)$ is a white complex Gaussian vector with covariance matrix $2\mathcal{N}_0\boldsymbol{I}_N$, with $\boldsymbol{I}_N$ the identity matrix of order N. For future reference, we define

$$\mathbf{u}_1(\ell) \triangleq \alpha_1(\ell)\boldsymbol{s}_1^\ell, \quad \boldsymbol{U}_1^\ell \triangleq \boldsymbol{S}_1^\ell \mathrm{diag}(\alpha_1(\ell))$$
$$\boldsymbol{z}(\ell) = \boldsymbol{S}_1^\ell\boldsymbol{A}_1\boldsymbol{B}_1(\ell)\alpha_1(\ell)$$

additionally we denote by $\mathcal{R}(\boldsymbol{A})$ the column space of $\boldsymbol{A}$, by $\mathcal{C}^N$ the space of N-tuples on the complex field $\mathcal{C}$ with the usual internal and external operations, and by $\mathcal{S}(\ell) \triangleq \mathcal{R}(\boldsymbol{S}_1^\ell)$ the *interference subspace*, namely the subspace of $\mathcal{C}^N$ spanned by the MAI: $\{\boldsymbol{s}_k^\ell\}_{\ell,k}$ with $k = 2, \ldots, K$.

Any *linear* one-shot detector for user "1" implements a decision rule based on the projection of $\boldsymbol{r}(\ell)$ along a given vector, i.e.

$$\hat{b}_1(\ell) = \mathrm{sgn}\left[\Re\left\{\hat{b}_1^s(\ell)\right\}\right] \quad \text{with } \hat{b}_1^s(\ell) \triangleq \boldsymbol{c}_1^H(\ell)\boldsymbol{r}(\ell) \qquad (3)$$

where

- $\hat{b}_1^s(\ell)$ — complex-valued soft estimate of the bit $b_1(\ell)$;
- $\mathrm{sgn}(\cdot)$ — signum function;
- $\Re\{\cdot\}$ — real part;
- $(\cdot)^H$ — conjugate transpose;
- $\boldsymbol{c}_1(\ell) \in \mathcal{C}^N$ — suitable direction, to be determined based upon some optimality criterion, complexity constraints and information available at the receiver.

The most popular linear optimization criteria are the decorrelator and the MMSE techniques. If the desired signal does not belong to $\mathcal{S}(\ell)$, the decorrelator can be obtained as the unique solution to the following constrained maximization problem [13]

$$\tilde{\boldsymbol{c}}_1(\ell) = \arg \max_{\text{s.t. } \boldsymbol{c}_1^H\boldsymbol{z}(\ell)=0} \left\{\frac{\Re\{\boldsymbol{c}_1^H\mathbf{u}_1^\ell\}}{\|\boldsymbol{c}_1\|}\right\}. \qquad (4)$$

Let $\chi(\ell)$ be the vector of all parameters in $\boldsymbol{w}(\ell)$ which are known at the receiver, and with respect to which we do not take the

[2]Similar results can be obtained for nonbinary sequences.

expectation, the classical MMSE detector may be obtained as the solution to the problem [13]

$$\tilde{\boldsymbol{c}}_1(\ell) = \arg\min E\left[\left|b_1(\ell) - \boldsymbol{c}_1^H \boldsymbol{r}(\ell)\right|^2 \,\middle|\, \chi(\ell)\right] \qquad (5)$$

or, alternatively, to the following constrained minimization of the output interference plus thermal noise energy

$$\tilde{\boldsymbol{c}}_1(\ell) = \arg \min_{\text{s.t. } \boldsymbol{c}_1^H \boldsymbol{u}_1^\ell = 1} E\left[\left|\boldsymbol{c}_1^H(\boldsymbol{z}(\ell) + \boldsymbol{n}(\ell))\right|^2 \,\middle|\, \chi(\ell)\right]. \qquad (6)$$

As well known, the decorrelator (4) and MMSE receiver (6) can be subsumed under a single framework. In fact, both of them are solutions to following linear constrained-optimization problem, viz. [see (7) at the bottom of the page], for different choices of the vector $\boldsymbol{w}(\ell)$:

Decorrelator $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) \triangleq \boldsymbol{S}_1^\ell \boldsymbol{A}_1 \boldsymbol{B}_1(\ell) \alpha_1(\ell)$
MMSE $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$

Notice that, since the expectation in (7) is conditioned with respect to all parameters in $\boldsymbol{w}(\ell)$ known at the receiver, the solution to (7) depends on the prior information available at the receiver. In the following, we will assume that the channel state information (CSI) is available at the receiver (this might be the case of a base station in a cellular network); under this hypothesis, the expectation in (7) is conditioned on all fading coefficients of all other users (i.e., $\chi(\ell) = \alpha_1(\ell)$).

If $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell)$, the solution to (7) is the classical *conditional* decorrelator (CD)

$$\boldsymbol{c}_{\text{CD}}(\ell) = \gamma_{\text{CD}} \left(\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + \mathcal{E}_1 \mathbf{u}_1^\ell \mathbf{u}_1^{\ell H}\right)^+ \mathbf{u}_1^\ell \qquad (8)$$

where

$$\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) \triangleq E[\boldsymbol{z}(\ell)\boldsymbol{z}^H(\ell) \,|\, \alpha_1]$$

is the covariance matrix of the interference vector $\boldsymbol{z}(\ell), \gamma_{\text{CD}}^{-1} = \mathbf{u}_1^{\ell H}(\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + \mathcal{E}_1 \mathbf{u}_1^\ell \mathbf{u}_1^{\ell H})^+ \mathbf{u}_1^\ell$ and $(\cdot)^+$ denotes pseudoinverse [13].

Notice that when $\boldsymbol{s}_1^\ell \notin \mathcal{S}(\ell), \min E[|\boldsymbol{c}_1^H \boldsymbol{z}(\ell)|^2 \,|\, \chi(\ell)]$ is equal to zero; indeed solving the problem (7) for $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell)$ is equivalent to solving (4) and (8) coincides with the classical decorrelator detector given by (4). A block scheme is given in Fig. 6 where

$$\boldsymbol{P}_\perp\left(\boldsymbol{S}_1^\ell\right) = \boldsymbol{I}_N - \boldsymbol{S}_L^\ell \left(\boldsymbol{S}_L^{\ell T} \boldsymbol{S}_L^\ell\right)^{-1} \boldsymbol{S}_L^{\ell T} \qquad (9)$$

represents the projector onto the orthogonal complement of the the subspace of $\mathcal{C}^N$ spanned by the columns of the $N \times (K-1)$ dimensional matrix $\boldsymbol{S}_1^\ell$. In Eq. (9) $\boldsymbol{S}_L^\ell$ represents the $N \times L$ matrix whose columns are a linearly independent subset spanning the column space of matrix $\boldsymbol{S}_1^\ell$.If instead, $\boldsymbol{s}_1^\ell \in \mathcal{S}(\ell)$, then (8) depends also on the energy of the interferers and on the realizations of the fading coefficients.

When, instead, $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$, problem (7) admits as its solution the classical *conditional* MMSE (CM) receiver (6)

$$\boldsymbol{c}_{\text{CM}}(\ell) = \gamma_{\text{CM}} \left(\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0 \boldsymbol{I}_N\right)^{-1} \mathbf{u}_1^\ell \qquad (10)$$

where $\gamma_{\text{CM}}^{-1} = \mathbf{u}_1^{\ell H}(\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0 \boldsymbol{I}_N)^{-1} \mathbf{u}_1^\ell$.

However, the constrained minimization problem (7) does not exploit the structure of the modulation format of the MAI and of the desired user. To see this, notice that, since the interfering users employ BPSK modulation, the baseband equivalent of the MAI, conditioned on all fading variables, is an improper random process[3] as its nonzero pseudo-autocovariance function is nonzero when conditioned on the received fading coefficients. As a consequence, its correlation properties are specified by four real functions, or, equivalently, by two complex functions. Likewise, the projections of these complex envelopes onto an orthonormal system are specified by two nonzero complex matrices

- the conditional covariance matrix $\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ given by:

$$\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) = E[\boldsymbol{z}(\ell)\boldsymbol{z}^H(\ell) \,|\, \alpha_1(\ell)] = \sum_{k=2}^{K} \mathcal{E}_k |\alpha_k(\ell)|^2 \boldsymbol{s}_k^\ell \boldsymbol{s}_k^{\ell H} = \boldsymbol{U}_1^\ell A_1^2 \boldsymbol{U}_1^{\ell H} \qquad (11)$$

- and the conditional pseudocovariance matrix $\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ [8], given by:

$$\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) \triangleq E[\boldsymbol{z}(\ell)\boldsymbol{z}^T(\ell) \,|\, \alpha_1(\ell)] = \sum_{k=2}^{K} \mathcal{E}_k (\alpha_k(\ell))^2 \boldsymbol{s}_k^\ell \boldsymbol{s}_k^{\ell T} = \boldsymbol{U}_1^\ell A_1^2 \boldsymbol{U}_1^{\ell T} \qquad (12)$$

with $(\cdot)^T$ denoting transpose.

As a consequence an *optimum* approach in the sense of second-order moment statistics is to come up with a structure where both the conditional covariance matrix and the conditional pseudocovariance matrix are utilized. The classical decorrelator (8) and the classical MMSE receiver (10) depend on the conditional covariance matrix $\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ and indeed they take into account the information of perfect CSI. However, they do not depend on the conditional pseudocovariance matrix $\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ and, as a consequence, they do not take advantage of the additional information available when the interferers use BPSK modulation: in other words, they are suboptimum in the sense of second-order moment statistics. As a consequence, in order to exploit the BPSK modulation format of the interferers, we should define an optimization criterion such that $\boldsymbol{c}_1(\ell)$ in (3) is function of both $\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ and $\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$. Furthermore, in order to exploit the modulation format of the desired users

[3]According to [16], [8], a complex random process, $n(t)$ is said to be proper if its pseudoautocorrelation function $R_n(t, u) = E[n(t)n(u)]$ is zero $\forall t, u$.

$$\begin{cases} \tilde{\boldsymbol{c}}_1(\ell) = \arg \min_{\boldsymbol{c}_1 \in \mathcal{D}(\ell)} \|\boldsymbol{c}_1\| \\ \mathcal{D}(\ell) \triangleq \left\{\boldsymbol{c}_1 \in \mathcal{C}^N : E\left[\left|\boldsymbol{c}_1^H \boldsymbol{w}(\ell)\right|^2 \,\middle|\, \chi(\ell)\right] = \min, \text{ s.t. } \boldsymbol{c}_1^H \mathbf{u}_1^\ell = 1\right\} \end{cases} \qquad (7)$$

we can constrain the soft estimate $\hat{b}_1^s(\ell)$ in (3) to be real. Thus, we obtain the following nonlinearly constrained-optimization criterion

$$\hat{b}_1^s(\ell) = \Re\{\tilde{\boldsymbol{c}}_1^H(\ell)\boldsymbol{r}(\ell)\} \tag{13}$$

with (14) (found at the bottom of the page). Notice that the proposed estimate (13) is a *nonlinear* estimate. In fact $\Re\{\boldsymbol{x}^H\boldsymbol{y}\}$ (with $\boldsymbol{x},\boldsymbol{y} \in \mathcal{C}^N$) is not linear in $\boldsymbol{x}$ and the suitable direction $\tilde{\boldsymbol{c}}_1(\ell)$ is the solution to a minimum norm problem on a constrained set, $\mathcal{D}(\ell)$, defined by a nonlinear operator; in this sense, we can say that (14) is a nonlinearly constrained minimization problem. Therefore, in order to solve (14), we cannot apply the orthogonality principle. Thus, unlike the conventional optimization criteria (7), the constrained minimization problem (14) is nonlinearly constrained, and hence, its solution requires special attention.

III. Linear Conjugate Receivers

As already noticed, the problem is that no *linear* operator on $\mathcal{C}^N$ exists fulfilling the constraint in (14). As a consequence, to proving the existence and uniqueness of the solution to (14) is not straightforward and the structure of a receiver fulfilling the nonlinearly constrained minimization problem (14) is not trivial unless the problem is reformulated on suitable subspace wherein such a constraint is linear. To be more definite, let us define the subset of $\mathcal{C}^{2N}$:

$$\mathcal{V} \stackrel{\triangle}{=} \left\{\boldsymbol{x}_a : \boldsymbol{x}_a = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}^*\end{pmatrix}, \boldsymbol{x} \in \mathcal{C}^N\right\}.$$

Notice that there is a one-to-one correspondence between the vector space $\mathcal{C}^N$ and the subset $\mathcal{V} \subset \mathcal{C}^{2N}$; furthermore

$$\begin{aligned} d^2(\boldsymbol{x}_a,\boldsymbol{y}_a) &\stackrel{\triangle}{=} \|\boldsymbol{x}_a - \boldsymbol{y}_a\|^2 = 2d^2(\boldsymbol{x},\boldsymbol{y})\\ \boldsymbol{x}_a^H\boldsymbol{y}_a &= \boldsymbol{x}^H\boldsymbol{y} + \boldsymbol{x}^T\boldsymbol{y}^* = 2\Re\{\boldsymbol{x}^H\boldsymbol{y}\} \end{aligned} \tag{15}$$

where the left sides of (15) are the usual distance and inner product defined in $\mathcal{C}^{2N}$. Thus, solving the problem in $\mathcal{C}^N$ is equivalent to rewriting the nonlinearly constrained minimization problem (14) in terms of the elements of $\mathcal{V}$ and solving it in the subset $\mathcal{V} \subset \mathcal{C}^{2N}$. It is shown in Appendix A that rewriting the nonlinearly constrained minimization problem (14) in terms of the *augmented* version of the following vectors

$$\begin{aligned} \boldsymbol{s}_{1a}^\ell &= \begin{bmatrix}\boldsymbol{s}_1^\ell\\ \boldsymbol{s}_1^{\ell*}\end{bmatrix}, \quad \mathbf{u}_{1a}^\ell = \begin{bmatrix}\mathbf{u}_1^\ell\\ \mathbf{u}_1^{\ell*}\end{bmatrix} = \alpha_1 \cdot \boldsymbol{s}_{1a}^\ell,\\ \boldsymbol{c}_{1a} &= \begin{bmatrix}\boldsymbol{c}_1\\ \boldsymbol{c}_1^*\end{bmatrix}, \quad \boldsymbol{w}_a(\ell) = \begin{bmatrix}\boldsymbol{w}(\ell)\\ \boldsymbol{w}^*(\ell)\end{bmatrix} \end{aligned} \tag{16}$$

we can work with a linear constraint and obtain the following results.

Proposition 1: Assuming that $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell)$ define the $N \times 2N$-dimensional matrices $\boldsymbol{F} = [\boldsymbol{I}_N\ \mathbf{0}]$ and $\boldsymbol{F}' = [\mathbf{0}\ \boldsymbol{I}_N]$; then the solution to the constrained problem (14) is given by

$$\boldsymbol{c}_{\rm LCD}(\ell) = \gamma_{\rm LCD}\boldsymbol{F}\left(\mathbf{Q}_{\alpha_1}(\ell)\boldsymbol{F}^T\mathbf{u}_1^\ell + \mathbf{Q}_{\alpha_1}^*(\ell)\boldsymbol{F}'^T\mathbf{u}_1^{\ell*}\right) \tag{17}$$

wherein

$$\mathbf{Q}_{\alpha_1}(\ell) = \begin{pmatrix} \left(\boldsymbol{M}_{\boldsymbol{zz}|\alpha_1}(\ell) + \mathcal{E}_1\mathbf{u}_1^\ell\mathbf{u}_1^{\ell H}\right) & \left(\boldsymbol{M}'_{\boldsymbol{zz}|\alpha_1}(\ell) + \mathcal{E}_1\mathbf{u}_1^\ell\mathbf{u}_1^{\ell T}\right)\\ \left(\boldsymbol{M}'^*_{\boldsymbol{zz}|\alpha_1}(\ell) + \mathcal{E}_1\mathbf{u}_1^{\ell*}\mathbf{u}_1^{\ell H}\right) & \left(\boldsymbol{M}^*_{\boldsymbol{zz}|\alpha_1}(\ell) + \mathcal{E}_1\mathbf{u}_1^{\ell*}\mathbf{u}_1^{\ell T}\right) \end{pmatrix}^+$$

and $\gamma_{\rm LCD}$ ensures that $\Re\{\boldsymbol{c}_{\rm LCD}^H\mathbf{u}_1^\ell\} = 1$.

Define the matrix

$$\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\alpha_1}(\ell) \stackrel{\triangle}{=} \begin{pmatrix}\boldsymbol{M}_{\boldsymbol{zz}|\alpha_1}(\ell) & \boldsymbol{M}'_{\boldsymbol{zz}|\alpha_1}(\ell)\\ \boldsymbol{M}'^*_{\boldsymbol{zz}|\alpha_1}(\ell) & \boldsymbol{M}^*_{\boldsymbol{zz}|\alpha_1}(\ell)\end{pmatrix}.$$

It is shown in Appendix A that if

$$\text{rank}\left(\mathbf{Q}_{\alpha_1}(\ell)\right) = \text{rank}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\alpha_1}(\ell)\right) + 1$$

the proposed receiver $\boldsymbol{c}_{\rm LCD}(\ell)$ admits a block scheme as depicted in Fig. 6 where $\boldsymbol{P}_\perp(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\alpha_1})$ denotes the projector onto the orthogonal complement of the subspace of $\mathcal{C}^{2N}$ spanned by the columns of the $2N$-dimensional matrix $\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\alpha_1}(\ell)$. Since we assume that all users employ a BPSK modulation with $E[b_k^2] = E[|b_k|^2] = 1$, it is shown in Appendix A that the proposed receiver $\boldsymbol{c}_{\rm LCD}(\ell)$ can be rewritten as

$$\boldsymbol{c}_{\rm LCD}(\ell) = \mathbf{u}_1^\ell + \boldsymbol{U}_1^\ell\left(\Re\left\{\boldsymbol{U}_1^{\ell H}\boldsymbol{U}_1^\ell\right\}\right)^{-1}\left(\Re\left\{\boldsymbol{U}_1^{\ell H}\mathbf{u}_1^\ell\right\}\right). \tag{18}$$

Proposition 2: Assuming $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$, the solution to the constrained minimization problem (14) is given by

$$\begin{aligned} \boldsymbol{c}_{\rm LCM}(\ell) = {}&\gamma_{\rm LCM}\boldsymbol{H}^{-1}(\ell)\mathbf{u}_1^\ell - \left(\boldsymbol{M}_{\boldsymbol{zz}|\alpha_1} + 2\mathcal{N}_0\boldsymbol{I}_N\right)^{-1}(\ell)\\ &\times \boldsymbol{M}'_{\boldsymbol{zz}|\alpha_1}(\ell)(\boldsymbol{H}^*(\ell))^{-1}\mathbf{u}_1^{\ell*} \end{aligned} \tag{19}$$

with

$$\begin{aligned} \boldsymbol{H}(\ell) = \Big(&\boldsymbol{M}_{\boldsymbol{zz}|\alpha_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_N - \boldsymbol{M}'_{\boldsymbol{zz}|\alpha_1}(\ell)\\ &\times \left(\boldsymbol{M}^*_{\boldsymbol{zz}|\alpha_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_N\right)^{-1}\boldsymbol{M}'^*_{\boldsymbol{zz}|\alpha_1}(\ell)\Big) \end{aligned}$$

and $\gamma_{\rm LCM}$ such that $\Re\{\boldsymbol{c}_{\rm LCM}^H\mathbf{u}_1^\ell\} = 1$.

We refer to (17) and to (19) as linear-conjugate decorrelator (LCD) and linear-conjugate MMSE (LCM) receiver, respectively. The proofs of the Propositions 1 and 2 are given in Appendix A.

The receivers defined by (17) and (19) outperform the CD and MMSE receiver (8) and (10) respectively; the reason is that (17) and (19) fully exploit the information about the fading coefficients and the modulation format resorting to linear/conjugate processing, wherein not only the received vector but also

$$\begin{cases} \tilde{\boldsymbol{c}}_1(\ell) = \arg\min_{\boldsymbol{c}_1 \in \mathcal{D}(\ell)} \|\boldsymbol{c}_1\|\\ \mathcal{D}(\ell) \stackrel{\triangle}{=} \left\{\boldsymbol{c}_1 \in \mathcal{C}^N : E\left[\Re^2\left\{\boldsymbol{c}_1^H\boldsymbol{w}(\ell)\right\}\middle|\, \chi(\ell)\right] = \min,\ \text{s.t. } \Re\{\boldsymbol{c}_1^H\mathbf{u}_1^\ell\} = 1\right\} \end{cases} \tag{14}$$

its conjugate is processed. In order to underline the difference between the usual MMSE and the proposed receiver (19) that fully exploits the information about the modulation format of all the users, let us consider the case that all users have the same fading coefficients: i.e.,

$$\alpha_k(\ell) = \alpha(\ell) \quad \forall k = 1, \dots, K \tag{20}$$

(this is the case for a downlink CDMA system) and let us specialize for this case the proposed linear-conjugate structures. In this case, $\boldsymbol{U}_1^\ell = \alpha(\ell)\boldsymbol{S}_1^\ell$ and

$$\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) = |\alpha(\ell)|^2 \boldsymbol{S}_1^\ell \boldsymbol{S}_1^{\ell T} \tag{21}$$

$$\boldsymbol{M}'_{\boldsymbol{zz}|\alpha_1}(\ell) = (\alpha(\ell))^2 \boldsymbol{S}_1^\ell \boldsymbol{S}_1^{\ell T} \tag{22}$$

Plugging (21) and (22) in (19), it follows that the proposed LCM receivers

$$\boldsymbol{c}_{\mathrm{LCM}}(\ell) = \gamma_{\mathrm{LCM}} \left(\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + \mathcal{N}_0 \boldsymbol{I}_N\right)^{-1} \mathbf{u}_1(\ell) \tag{23}$$

in contrast to the classical MMSE receiver is given by (10). From these results, we can immediately say that with respect the classical structure (10) the proposed receiver (23) gains 3 dB in terms of output SNR as will be shown in the next section for the general case. As regards as linear-conjugate decorrelator, notice that under (20) the LCD receiver $\boldsymbol{c}_{\mathrm{LCD}}(\ell)$ coincides with the classical CD as can be immediately checked substituting the expressions of $\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell), \boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ and $\boldsymbol{U}_1^\ell$ in (17) and (18), respectively.

Before proceeding further in our discussion, we provide some insight on the proposed linear conjugate receivers. Let us assume that the interfering users employ QPSK modulation format, while the desired user employs BPSK modulation. In this case, the conditional pseudocovariance matrix, $\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ of the interference vector is zero and the nonlinearly constrained minimization problem (14) degenerates to the linearly constrained minimization problem (7). As a consequence, the soft estimate of the bit $b_1(\ell)$ is still given by $\hat{b}_1^s(\ell) = \Re\{\boldsymbol{c}_1^H(\ell)\boldsymbol{r}(\ell)\}$ but the direction $\boldsymbol{c}_1^H(\ell)$ is, now, the solution to (7). More in general, whenever the conditional pseudocovariance matrix $\boldsymbol{M}'_{\boldsymbol{ww}\,|\,\chi}(\ell)$ is zero, (this is the case of interfering users employing a modulation format symmetric in the complex plane $\mathcal{C}^N$ or of CSI not available at the receiver), the set $\mathcal{D}(\ell)$ defined in (14) degenerates to the subspace of $\mathcal{C}^N$ defined in (7) and the proposed receivers (17) and (19) degenerate in the classical CD and MMSE receiver (8) and (10) respectively as can be easy checked substituting $\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) = \boldsymbol{0}$ in (17) and (19). Notice also that (18) does not contain the case of $\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) = \boldsymbol{0}$ as a particular case since it has been obtained using the hypothesis that all users employ BPSK modulation.

IV. Analysis

The uncoded performance analysis of DS/CDMA communication systems is usually assessed by evaluating their bit-error-rate (BER), near–far resistance and SNR at the output of the receivers. As regards the newly proposed conditional receivers (17) and (19), the BER can be computed as in the case of any linear receiver [13] and is given by

$$P_{\mathrm{LCD}}(e) = E_{\mathbf{M}_1^\ell,\,\alpha_1(\ell)}\left[\frac{1}{2}\,\mathrm{erfc}\left(\frac{\mathcal{E}_1}{\sqrt{2\mathcal{N}_0}\|\boldsymbol{c}_{\mathrm{LCD}}\|}\right)\right]$$

$$P_{\mathrm{LCM}}(e) = E_{\boldsymbol{z}(\ell),\alpha_1(\ell)} \times \left[\frac{1}{2}\,\mathrm{erfc}\left(\frac{\Re\{\mathcal{E}_1 + \boldsymbol{c}_{\mathrm{LCM}}^H \boldsymbol{z}(\ell)\}}{\sqrt{2\mathcal{N}_0}\|\boldsymbol{c}_{\mathrm{LCM}}\|}\right)\right] \tag{24}$$

The statistical average with respect to the interference term $\boldsymbol{z}(\ell)$ (i.e., the average with respect to the MAI fading coefficients and bit pattern realizations) cannot be carried out in closed form, and numerical evaluation is called required.

In Appendix A, it is shown that the near–far resistance of (17) and (19) for a given set of fading coefficients $\{\alpha_k(\ell)\}_{\ell,k}$, can be written as

$$\eta_1(\ell) = 1 - \frac{1}{2|\alpha_1(\ell)|^2}\mathbf{u}_{1a}^{\ell H}\boldsymbol{U}_J^\ell\left(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^\ell\right)^{-1}\boldsymbol{U}_J^{\ell H}\mathbf{u}_{1a}^\ell \tag{25}$$

where $\boldsymbol{U}_J^\ell$ denotes the $2N \times J$ matrix whose columns form a linearly independent subset spanning the column space of the $2N \times K$-dimensional matrix $\boldsymbol{U}_{1a}^\ell$ defined as the augmented version of the interference matrix $\boldsymbol{U}_1^\ell$, i.e.

$$\boldsymbol{U}_{1a}^\ell \triangleq \begin{bmatrix}\boldsymbol{U}_1^\ell \\ \boldsymbol{U}_1^{\ell *}\end{bmatrix} = \begin{bmatrix}\boldsymbol{S}_1^\ell \mathrm{diag}(\alpha_1(\ell)) \\ \boldsymbol{S}_1^\ell \mathrm{diag}(\alpha_1^*(\ell))\end{bmatrix}$$

On the other hand, it is well known that the classical CD (8) and the conditional MMSE detector (10) have the same near–far resistance given by [13]

$$\eta_1^C(\ell) = 1 - \boldsymbol{s}_1^{\ell H}\boldsymbol{S}_L^\ell\left(\boldsymbol{S}_L^{\ell H}\boldsymbol{S}_L^\ell\right)^{-1}\boldsymbol{S}_L^{\ell H}\boldsymbol{s}_1^\ell \tag{26}$$

with $\boldsymbol{S}_L^\ell$ the $N \times L$ matrix whose columns are a linearly independent subset spanning the column space of matrix $\boldsymbol{S}_1^\ell$. It can be shown that $\eta_1(\ell) \geq \eta_1^C(\ell)$, but the general proof is very long and tedious; in the following, we will be prove the superiority of the proposed receivers (17) and (19) in the asymptotic regime, i.e., in the case that the number of users and the number of dimensions go to infinity but the ratio is keeped constant.

As regards the SNR, based on some results given in Appendix A, is also easy to see that the SNR for the LCM receiver (19) can be written as

$$\mathsf{SNR}_{\mathrm{LCM}}(\ell) = \mathcal{E}_1 \mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}\right)^{-1}\mathbf{u}_{1a}^\ell \tag{27}$$

while, for the classical linear detection structures (10), we have

$$\mathsf{SNR}_{\mathrm{CM}}(\ell) = \mathcal{E}_1 \mathbf{u}_1^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}\right)^{-1}\mathbf{u}_1^\ell. \tag{28}$$

Equations (27) and (28) for the SNR and (25) and (26) for the near–far resistance describe the performance of the proposed receivers, (17) and (19), and of the conventional receivers (8) and (10). However, they have to be computed for each specific choices of the signature sequences and of the bit interval and, as a consequence, it is not easy to obtain a qualitative insight directly from those formulas. Indeed, it should be desirable to come out with some expression of performance measures which do not depend on the choice of the signature waveforms. To obtain this, the key point is to notice that in the case of long sequences (and in other cases, e.g., with sequences distorted by random multipath, or in a network with random access) it is reasonable to assume that the spreading sequences are randomly and independently chosen. In this case, the performance measures of the receivers can be modeled as random variables, since they are functions of the spreading sequences; however, we can average the performance measures (27) and (25) with respect to the random sequences and additionally show that in the limiting regime $K \to \infty$, the random performance measures converge to deterministic quantities. The ratio of the number of users to the number of dimensions is denoted by

$$\beta \triangleq K/N.$$

The elements of the signatures $s_{k,n}^{\ell} \in \{-1/\sqrt{N}, 1/\sqrt{N}\}$ $n = 0, \ldots, N-1$ are real-valued equally likely and independently distributed random variables with zero mean and variance $1/N$ for all (k, n, ℓ). The normalization ensures that $E[\|\boldsymbol{s}_k^{\ell}\|] = 1$; additionally we need also the assumption $E[|s_{k,n}^{\ell}|^4] < \infty$.

Let us focus now on near–far resistance and on the (finite SNR) multiuser efficiency of the proposed receivers evaluated in limiting regime.

It is well known [13], [14] that the near–far resistance $\eta_1^C(\ell)$ of detectors (8) and (10) converges almost surely as $K \to \infty$ to

$$\lim_{K\to\infty} \eta_1^C(\ell) = [1-\beta]^+. \tag{29}$$

Additionally, define a nonnegative random variable $|A|^2$ whose distribution is the limit distribution of $\{(\mathcal{E}_k/\mathcal{E}_1)|\alpha_k|^2, k = 1, \ldots, K\}$. For the conventional MMSE structure (10), the multiuser efficiency converges almost surely to the solution $\eta_{\mathrm{CM}}^{\infty}(\mathsf{SNR})$ of [5], [10]

$$\eta + \beta E\left[\frac{|A|^2, \mathsf{SNR}\eta}{1+|A|^2\,\mathsf{SNR}\eta}\right] = 1 \tag{30}$$

with $\mathsf{SNR} = (\mathcal{E}_1)/(2\mathcal{N}_0)$ the average received SNR.

Two of our main results are the following.

Proposition 3: The average of the near–far resistance (25), $\eta_1(\ell)$, w.r.t. the signatures of all users is lower bounded by

$$E[\eta_1(\ell)] \geq \left[1 - \frac{K-1}{2N}\right]^+.$$

Furthermore, $\eta_1(\ell)$ for any $0 \leq \beta \leq \infty$ converges almost surely (a.s.) to

$$\eta_1^{\infty} = \lim_{K\to\infty} \eta_1(\ell) \overset{\text{a.s.}}{=} \left[1 - \frac{\beta}{2}\right]^+. \tag{31}$$

Proposition 4: The multiuser efficiency of the LCM receiver (19) converges almost surely to the solution $\eta_{\mathrm{LCM}}^{\infty}(\mathsf{SNR})$ of

$$\eta + \frac{\beta}{2} E\left[\frac{2\mathsf{SNR}|A|^2\eta}{1+2\mathsf{SNR}|A|^2\eta}\right] = 1. \tag{32}$$

The main conclusions from Propositions 3 and 4 are 1) the linear conjugate filters halve the number of effective interferers and 2) they add 3 dB to the input SNR. Effect 1) can be explained by noticing that the conventional decorrelator combats each interfering user as if it were quadrature-modulated which has the same noise-enhancement effect as having two BPSK-modulated users (for high SNR). To explain effect 2), it is best to consider the single-user case; whereas the conventional MMSE filter minimizes the averaged magnitude squared at its output, the linear conjugate MMSE detector targets its resources to minimize the averaged real part squared (to exploit the BPSK modulation format) which has the same effect as turning off half of the input noise power. The proofs are given in Appendix B. So far, we have considered the case of fading channels with real signatures. Notice that the same results can be obtained for the case of fading channel with with complex signatures; additionally, it is easy to see that these results still hold for the case of no fading and complex signatures whose entries are circularly symmetric. Finally, notice that the case of (20) and, as a consequence, the case of no fading (i.e., $\alpha_k(\ell) = \alpha(\ell) = 1$) are not encompassed by Proposition 3 and 4.

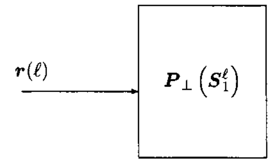

Fig. 1. Conventional decorrelator.

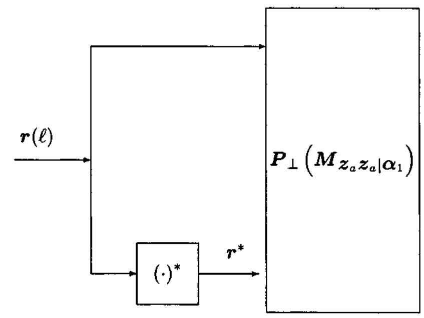

Fig. 2. Linear conjugate decorrelator.

V. Suboptimal Receivers

Notice that both proposed receivers (17) and (19) depend on the eigenvalues and eigenvectors of the covariance and pseudocovariance matrices of the interference vector $\boldsymbol{z}(\ell)$ conditioned on $\{\alpha_k(\ell)\}_{\ell,k}$. However, in long-code CDMA systems, the spreading codes as well as the corresponding matrices $\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$ and $\boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell)$, change in every symbol interval. This implies that the proposed optimal solutions (17) and (19) have to be computed and updated on a symbol-by-symbol basis, thus leading to a prohibitive computational effort in most cases. As a consequence, a less complex approach is desirable.

Using some results shown in Appendix A, we can rewrite the solutions (17) and (19) to the constrained optimization problem (14), except for an irrelevant positive factor, as follows:

$$\boldsymbol{c}_{\mathrm{LCD}} = \boldsymbol{F}\left(\boldsymbol{I}_{2N} - \boldsymbol{U}_J^{\ell}\left(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^{\ell}\right)^{-1}\boldsymbol{U}_J^{\ell H}\right)\mathbf{u}_{1a}^{\ell} \tag{33}$$

$$\boldsymbol{c}_{\mathrm{LCM}} = \boldsymbol{F}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_{2N}\right)^{-1}\mathbf{u}_{1a}^{\ell}. \tag{34}$$

The suboptimum approximation that we advocate consists of choosing $\tilde{\boldsymbol{c}}_1(\ell)$ as corresponding to the minimum of (14) over a feasible constrained set. Specifically, based on (33) and (34), the vectors $\boldsymbol{c}_{\rm LCD}$ and the $\boldsymbol{c}_{\rm LCM}$ can be expanded in an infinite power series of the conditional covariance matrix of the $2N$-dimensional vector $\boldsymbol{z}_a(\ell)$. As a consequence, in order to simplify the optimum structure, we will resort to a polynomial approximation of the matrix inverses in (33) and (34). In this case, after choosing the order of the polynomial approximation, the integer D, depending on the allowable complexity, we obtain the following parametric solutions:

$$\boldsymbol{c}_{\rm SCD}(\ell) = \boldsymbol{U}_{1a}^{\ell} - \sum_{m=0}^{D-1} w_{\rm SCD}^{m}(\ell)\boldsymbol{U}_{1a}^{\ell}A_1\left(\boldsymbol{A}_1 U_{1a}^{\ell H}\boldsymbol{U}_{1a}^{\ell}\boldsymbol{A}_1\right)^m \times \boldsymbol{U}_{1a}^{\ell}\mathbf{u}_{1a}^{\ell} \tag{35}$$

$$\boldsymbol{c}_{\rm SCM}(\ell) = \sum_{m=0}^{D-1} w_{\rm SCM}^{m}(\ell) \times \left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_{2N}\right)^m \mathbf{u}_{1a}^{\ell} \tag{36}$$

where the weight vector $\boldsymbol{w}_{(\cdot)}(\ell) = [w_{(\cdot)}^0(\ell), \ldots, w_{(\cdot)}^{D-1}(\ell)]$ is obtained by solving the constrained minimization problem (14). The optimal weight vectors are given by

$$\boldsymbol{w}_{\rm SCM}(\ell) = \begin{pmatrix} \mathcal{H}_1^{\ell} & ,\ldots, & \mathcal{H}_D^{\ell} \\ \vdots & ,\ldots, & \vdots \\ \mathcal{H}_D^{\ell} & ,\ldots, & \mathcal{H}_{(2D-1)}^{\ell} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{H}_1^{\ell} \\ \vdots \\ \mathcal{H}_D^{\ell} \end{pmatrix}$$

$$\boldsymbol{w}_{\rm SCD}(\ell) = \begin{pmatrix} \mathcal{Q}_1^{\ell} & ,\ldots, & \mathcal{Q}_D^{\ell} \\ \vdots & ,\ldots, & \vdots \\ \mathcal{Q}_D^{\ell} & ,\ldots, & \mathcal{Q}_{(2D-1)}^{\ell} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{Q}_1^{\ell} \\ \vdots \\ \mathcal{Q}_D^{\ell} \end{pmatrix} \tag{37}$$

where

$$\mathcal{H}_m^{\ell} \triangleq \mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_{2N}\right)^m \mathbf{u}_{1a}^{\ell}$$
$$\mathcal{Q}_m^{\ell} \triangleq \mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{U}_{1a}^{\ell}\boldsymbol{A}_1^2\boldsymbol{U}_{1a}^{\ell}\right)^{m+2}\mathbf{u}_{1a}^{\ell}. \tag{38}$$

We refer to (35) and (36) as the suboptimal linear-conjugate decorrelator (SCD) and the suboptimal linear-conjugate MMSE detector (SCM) respectively. Equation (37) implies that in order to compute the optimal weights [i.e., those solving the constrained minimization problem (14)] we have to invert a $D \times D$ Hankel matrix. The computational effort for on line inversion of the matrix in (37) is feasible when D is very small (this is equivalent, in order to obtain good performance, to saying that the number of users is not too large). If D is not too small, the weight optimization approach cannot be adopted since the optimal weights (37) can be computationally intensive, but we can resort to the following design guidelines. Specifically, we propose an asymptotic weighting where at the design stage of the suboptimal structures (35) and (36) we assume to operate in a limiting regime (i.e., the number of users and processing gain are large with fixed ratio and random spreading sequences). Using Proposition 7, given in Appendix B, we have

$$\lim_{K\to\infty} \boldsymbol{s}_{1a}^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1 a}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_{2N}\right)^m \boldsymbol{s}_{1a}^{\ell} \overset{\text{a.s.}}{=} \lim_{N\to\infty}\frac{1}{N}E\left[\text{trace}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1} + 2\mathcal{N}_0\boldsymbol{I}_{2N}\right)^m\right]$$
$$\lim_{K\to\infty} \boldsymbol{s}_{1a}^{\ell H}\left(\boldsymbol{U}_{1a}^{\ell}\boldsymbol{A}_1^{\ell}\boldsymbol{U}_{1a}^{\ell H}\right)^m \boldsymbol{s}_{1a}^{\ell} \overset{\text{a.s.}}{=} \lim_{N\to\infty}\frac{1}{N}E\left[\text{trace}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}\right)^m\right]. \tag{39}$$

Based on (39) and on (38), the parametric suboptimal structures in the limit regime are given by (35) and (36) where the weight vectors $\boldsymbol{w}_{(\cdot)}^{\infty}(\ell) = [w_{(\cdot)}^0(\ell), \ldots, w_{(\cdot)}^{D-1}(\ell)]$, are (40) and on (41), found at the bottom of the page, and the expectations are with respect to Λ, whose distribution is the asymptotic eigenvalue distribution of $\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}$. The entries of the matrices given in (40) and (41) can be computed by applying some tools of combinatorial theory [17]. In fact, it can be shown that [17]

$$E[\Lambda^m] = \mathcal{E}_1^m \sum_{n=1}^{m} \beta^{m-n} \sum_{(j_1,\ldots,j_n)\in Z_n^{(m)}} c(j_1,\ldots,j_n) \cdot (E[(|\mathsf{A}|^2)])^{j_1} \ldots (E[(|\mathsf{A}|^2)^n])^{j_n}$$

where the sum is over all possible $j_1, \ldots, j_n > 0$ such that

$$j_1 + j_2 + \ldots j_n = m - n + 1,$$
$$j_1 + 2j_2 + \ldots nj_n = m$$

for all $1 \le n \le m$ and

$$c(j_1,\ldots,j_n) = \frac{m!}{j_1!\ldots j_n! n!}.$$

We refer to (35) and (36) [where the weights are given by (40) and (41)] as the asymptotic suboptimal linear-conjugate decorrelator (ASD) and asymptotic suboptimal linear-conjugate MMSE detector (ASM).

$$\boldsymbol{w}_{\rm SCD}^{\infty} = \begin{pmatrix} E[\Lambda^3] & E[\Lambda^4] & ,\ldots, & E[\Lambda^{D+2}] \\ E[\Lambda^4] & E[\Lambda^5] & ,\ldots, & E[\Lambda^{D+3}] \\ \vdots & \vdots & ,\ldots, & \vdots \\ E[\Lambda^{D+2}] & E[\Lambda^{D+1}] & ,\ldots, & E[\Lambda^{2D+1}] \end{pmatrix}^{-1} \begin{pmatrix} E[\Lambda^3] \\ E[\Lambda^4] \\ \vdots \\ E[\Lambda^{D+2}] \end{pmatrix} \tag{40}$$

$$\boldsymbol{w}_{\rm SCM}^{\infty} = E\begin{pmatrix} E[\Lambda + 2\mathcal{N}_0] & ,\ldots, & E[(\Lambda + 2\mathcal{N}_0)^D] \\ E[(\Lambda + 2\mathcal{N}_0)^2] & ,\ldots, & E[(\Lambda + 2\mathcal{N}_0)^{D+1}] \\ \vdots & ,\ldots, & \vdots \\ E[(\Lambda + 2\mathcal{N}_0)^D] & ,\ldots, & E[(\Lambda + 2\mathcal{N}_0)^{2D-1}] \end{pmatrix}^{-1} \begin{pmatrix} E[(\Lambda + 2\mathcal{N}_0)] \\ E[(\Lambda + 2\mathcal{N}_0)^2] \\ \vdots \\ E[(\Lambda + 2\mathcal{N}_0)^D] \end{pmatrix} \tag{41}$$

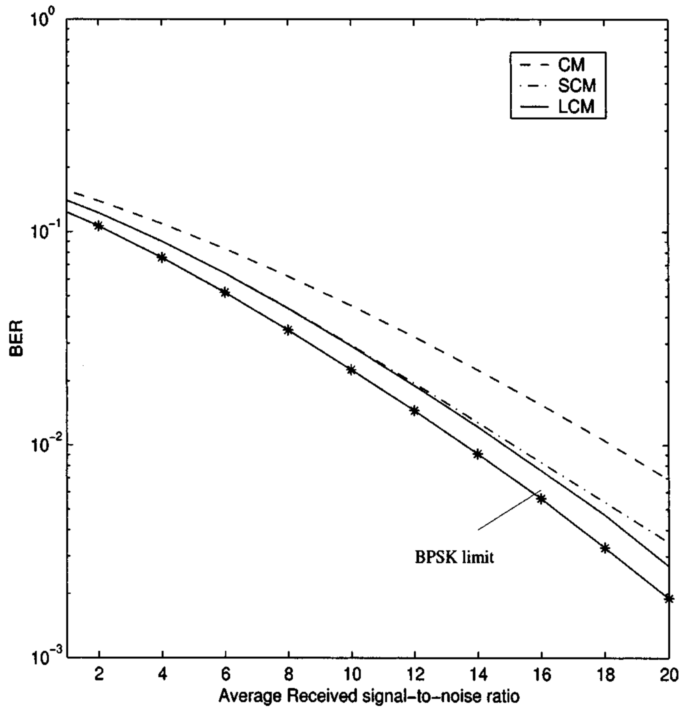

Fig. 3. BER of the conventional and of the conjugate MMSE detectors LCM and SCM for $K = 25$.

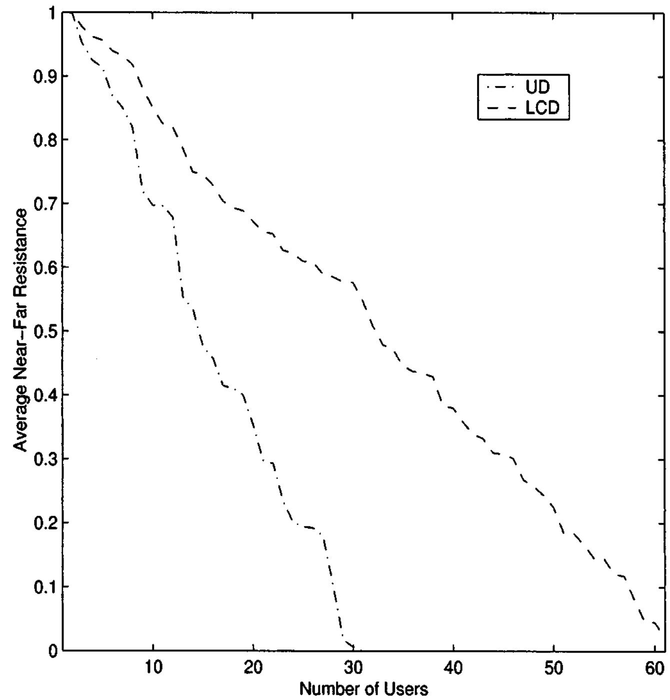

Fig. 4. Near–far resistance of the conventional decorrelator and of the linear-conjugate decorrelator.

VI. Numerical Results

In order to numerically evaluate the performance of the proposed receivers, (17) and (19), and to carry out comparisons with the classical linear multiuser receivers, our numerical results focus on a CDMA system employing Gold codes with spreading length $N = 31$ operating in Rayleigh fading. The statistical average of the error probability (24) with respect to the interference term $z(\ell)$ (i.e., the average with respect to the MAI fading coefficients and bit pattern realizations) has been evaluated via Monte-Carlo. In Fig. 3, we have represented the error probability versus the average received SNR [$\mathsf{SNR} = (\mathcal{E}_1)/(2\mathcal{N}_0)$], for the conditional MMSE receiver (10), labeled as "CM", for the newly proposed MMSE receiver (19), labeled as "LCM", and for the suboptimal linear-conjugate MMSE detector (36), labeled as "SCM" with $D = 10$. We assume 24 interfering users with a power level of 5 dB above the desired signal. The results clearly show the superiority of the new strategy, which largely outperforms the conditional MMSE receiver (10). For comparison purposes, we also report the error probability corresponding to an uncoded BPSK transmission over a MAI-free flat-fading channel [2].

Fig. 4 contrasts the classical linear detection structures (26) with the newly proposed receivers (25) in terms of near–far resistance, which is represented versus K. Monte-Carlo evaluation is used to average (25) with respect to the fading coefficients. Again, it is seen that the new approach yields a noticeable performance improvement.

In Fig. 5, we show the asymptotic multiuser efficiency solution $\eta^{\infty}_{\rm LCM}(\mathsf{SNR})$ of the linear-conjugate MMSE receiver, solution of (32),versus the ratio $\beta = K/N$ for a fixed average re-

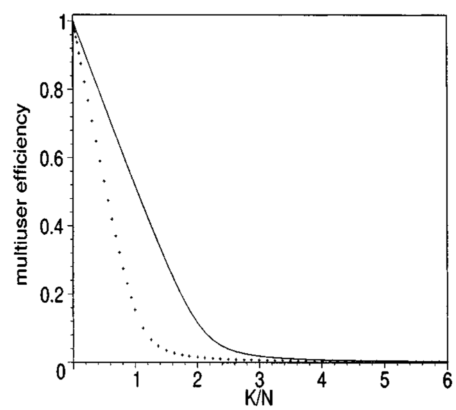

Fig. 5. LCM multiuser efficiency for $\mathsf{SNR} = 10$ dB.

ceived SNR ($\mathsf{SNR} = 10$ dB) and for equal powers. For comparison purposes, the spectral efficiency of the conditional MMSE receiver (CM) for equal powers [10] is also given.

In Fig. 6, we have represented the error probability versus the average received SNR [$\mathsf{SNR} = (\mathcal{E}_1)/(2\mathcal{N}_0)$] for the newly proposed MMSE receiver (19), labeled as "LCM," and for the asymptotic suboptimal linear-conjugate MMSE detector (36), labeled as "ASM" with $D = 15$. We assume $N = 150$ and $K = 30$ and equal-power interfering users. Again, the statistical average of error probability with respect to the fading coefficients and bit pattern realizations cannot be carried out in closed form and numerical evaluation is required. For comparison purposes, we also report the error probability corresponding to an uncoded BPSK transmission over an MAI-free flat-fading channel.

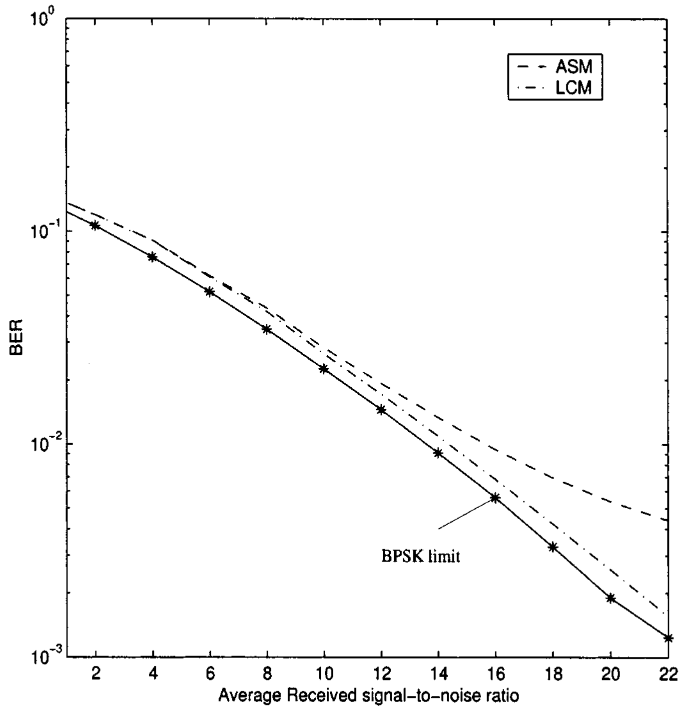

Fig. 6. BER of the conventional and of the conjugate MMSE detectors LCM and ASM for $K = 30$ and $N = 150$.

APPENDIX A

At first, we have to show that the solution to the nonlinearly constrained minimization problem (14) exists and it is unique. To this end, with the augmented vectors $\mathbf{u}_{1a}^\ell, \boldsymbol{c}_{1a}$ and $\boldsymbol{w}_a(\ell)$ given in (16) the nonlinearly constrained minimization problem (14) can be rewritten as (42) found at the bottom of the page. Notice that $\mathcal{D}_a(\ell)$ can be obtained as

$$\mathcal{D}_a(\ell) = \mathcal{D}'(\ell) \bigcap \mathcal{V} \tag{43}$$

where $\mathcal{D}'(\ell)$ is the subset of $\mathcal{C}^{2N}$ whose elements minimize $E[|\boldsymbol{x}_a^H \boldsymbol{w}_a(\ell)|^2 \,|\, \chi(\ell)]$ without the constrain that $\boldsymbol{v} \in \mathcal{V}$::

$$\mathcal{D}'(\ell) \triangleq \left\{ \boldsymbol{v} \in \mathcal{C}^{2N} : E[|\boldsymbol{v}^H \boldsymbol{w}_a(\ell)|^2 | \chi(\ell)] = \min, \; \frac{1}{2}\boldsymbol{v}^H \mathbf{u}_{1a}^\ell = 1 \right\}.$$

Thus, the set $\mathcal{D}'(\ell)$ can be found applying standard techniques. It can be shown that the intersection of the set $\mathcal{D}'(\ell)$ with $\mathcal{V}$ is not empty. As a consequence the set $\mathcal{D}_a(\ell)$ is a nonempty set and this implies the the existence of the solution. As regards the uniqueness, notice that $\mathcal{D}'(\ell)$ is a closed convex set of $\mathcal{V} \subset \mathcal{C}^{2N}$ Thus, the vector with minimum norm is unique. Let now solve the new linearly constrained minimization problem (42). If we assume that $\boldsymbol{w}_a(\ell) = \boldsymbol{z}_a(\ell)$, the solution to (42) is given by the following expression

$$\boldsymbol{c}_{a\,\mathrm{LCD}}(\ell) = \gamma_{\mathrm{LCD}} \left(\boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a \,|\, \alpha_1}(\ell) + \mathcal{E}_1 \mathbf{u}_{1a}^\ell \mathbf{u}_{1a}^{\ell H} \right)^+ \mathbf{u}_{1a}^\ell \tag{44}$$

with $\gamma_{\mathrm{LCD}} = 1/(\mathbf{u}_{1a}^{\ell H}(\boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a \,|\, \alpha_1}(\ell) + \mathcal{E}_1 \mathbf{u}_{1a}^\ell \mathbf{u}_{1a}^{\ell H})^+ \mathbf{u}_{1a}^\ell)$. In the above equation

$$\begin{aligned} \boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a \,|\, \alpha_1}(\ell) &\triangleq E\left[\boldsymbol{z}_a(\ell) \boldsymbol{z}_a^H(\ell) \,|\, \alpha_1(\ell) \right] \\ &= \begin{pmatrix} \boldsymbol{M}_{\boldsymbol{z}\boldsymbol{z}|\alpha_1}(\ell) & \boldsymbol{M}'_{\boldsymbol{z}\boldsymbol{z}|\alpha_1}(\ell) \\ \boldsymbol{M}'^*_{\boldsymbol{z}\boldsymbol{z}|\alpha_1}(\ell) & \boldsymbol{M}^*_{\boldsymbol{z}\boldsymbol{z}|\alpha_1}(\ell) \end{pmatrix} \\ &= \boldsymbol{U}_{1a}^\ell \boldsymbol{A}_1^2 \boldsymbol{U}_{1a}^{\ell H} \end{aligned} \tag{45}$$

is the covariance matrix of the augmented interference vector $\boldsymbol{z}_a(\ell)$, wherein

$$\boldsymbol{U}_{1a}^\ell = \begin{bmatrix} \boldsymbol{U}_1^\ell \\ \boldsymbol{U}_1^\ell \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_1^\ell \mathrm{diag}(\alpha_1(\ell)) \\ \boldsymbol{S}_1^\ell \mathrm{diag}(\alpha_1^*(\ell)) \end{bmatrix}.$$

Notice that when $\mathbf{u}_{1a}^\ell \notin \mathcal{R}(\boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a \,|\, \alpha_1}), \min E[\boldsymbol{c}_1^H \boldsymbol{z}(\ell)]$ is equal to zero. As a consequence the solution (45) to the constrained minimization problem (42) is, except for an irrelevant positive factor, the projection of $\mathbf{u}_{1a}$ onto the orthogonal complement of the range space of the matrix $\boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a \,|\, \alpha_1}(\ell)$. Denote by $\boldsymbol{U}_J^\ell$ the $2N \times J$ matrix whose columns form a linearly independent subset spanning the column space of matrix $\boldsymbol{U}_{1a}^\ell$. The solution (44) can be rewritten as

$$\boldsymbol{c}_{a\,\mathrm{LCD}}(\ell) = \gamma_{\mathrm{LCD}} \left(\boldsymbol{I}_{2N} - \boldsymbol{U}_J^\ell (\boldsymbol{U}_J^{\ell H} \boldsymbol{U}_J^\ell)^{-1} \boldsymbol{U}_J^{\ell H} \right) \mathbf{u}_{1a}^\ell \tag{46}$$

with $\gamma_{\mathrm{LCD}} = \mathbf{u}_{1a}^{\ell H}(\boldsymbol{I}_{2N} - \boldsymbol{U}_J^\ell(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^\ell)^{-1}\boldsymbol{U}_J^{\ell H})\mathbf{u}_{1a}^\ell$.

If $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$, the solution to the problem (42) admits the following expression

$$\boldsymbol{c}_{a\,\mathrm{LCM}}(\ell) = \gamma_{\mathrm{LCM}} \left(\boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a \,|\, \alpha_1} + 2\mathcal{N}_0 \boldsymbol{I}_{2N} \right)^{-1} (\ell) \mathbf{u}_{1a}^\ell \tag{47}$$

where $\gamma_{\mathrm{LCM}} = 1/(\mathbf{u}_{1a}^{\ell H}(\boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a \,|\, \alpha_1}(\ell) + 2\mathcal{N}_0 \boldsymbol{I}_{2N})^{-1} \mathbf{u}_{1a}^\ell)$ and $\boldsymbol{I}_{2N}$ is the $2N$-dimensional identity matrix.

So far, we have found an explicit expression of the solution to the linearly constrained minimization problem (42). It is easy to see that, based on (16), the solution to the nonlinearly constrained minimization problem (14) for $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell)$ and for $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$ can be obtained by considering the first N entries of the vectors $\boldsymbol{c}_{a\,\mathrm{LCD}}(\ell)$ and $\boldsymbol{c}_{a\,\mathrm{LCM}}(\ell)$ respectively. Specifically applying the inversion lemma to (47), we have an explicit expression for $\boldsymbol{c}_{\mathrm{LCM}}(\ell)$:

$$\begin{aligned} \boldsymbol{c}_{\mathrm{LCM}}(\ell) = \gamma_{\mathrm{LCM}} \alpha_1(\ell) \boldsymbol{H}^{-1}(\ell) \boldsymbol{s}_1^\ell - \alpha_1^*(\ell) \\ \times \left(\boldsymbol{M}_{\boldsymbol{z}\boldsymbol{z} \,|\, \alpha_1} + 2\mathcal{N}_0 \boldsymbol{I}_N \right)^{-1} (\ell) \boldsymbol{M}'_{\boldsymbol{z}\boldsymbol{z} \,|\, \alpha_1}(\ell) \\ \times (\boldsymbol{H}^*(\ell))^{-1} \boldsymbol{s}_1^\ell \end{aligned}$$

$$\begin{cases} \tilde{\boldsymbol{c}}_{1a}(\ell) = \arg \min_{\boldsymbol{c}_{1a} \in \mathcal{D}(\ell)} \|\boldsymbol{c}_{1a}\| \\ \mathcal{D}_a(\ell) \triangleq \left\{ \boldsymbol{c}_{1a} \in \mathcal{V} \subset \mathcal{C}^{2N} : E\left[\left(\boldsymbol{c}_{1a}^H \boldsymbol{w}_a(\ell) \right)^2 \,\middle|\, \chi(\ell) \right] = \min, \frac{1}{2} \boldsymbol{c}_{1a}^H \mathbf{u}_{1a}^\ell = 1 \right\} \end{cases} \tag{42}$$

with

$$\boldsymbol{H}(\ell) = \left(\boldsymbol{M}_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0 \boldsymbol{I}_N - \boldsymbol{M}'_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) \times \left(\boldsymbol{M}^*_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) + 2\mathcal{N}_0 \boldsymbol{I}_N \right)^{-1} \boldsymbol{M}'^*_{\boldsymbol{zz}\,|\,\alpha_1}(\ell) \right).$$

To obtain an explicit expression of $\boldsymbol{c}_{\rm LCD}(\ell)$, since the singularity of the matrix $\boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a\,|\,\alpha_1}$, we cannot apply the inversion lemma. Thus, define $\boldsymbol{F} = [\boldsymbol{I}_N \;\; \boldsymbol{0}]$ and $\mathbf{Q}^a_{\alpha_1}(\ell) = \boldsymbol{M}_{\boldsymbol{z}_a \boldsymbol{z}_a\,|\,\alpha_1} + \mathcal{E}_1 \mathbf{u}^\ell_{1a} \mathbf{u}^\ell_{1a}$, we obtain

$$\boldsymbol{c}_{\rm LCD}(\ell) = \gamma_{\rm LCD} \boldsymbol{F} \left(\mathbf{Q}^a_{\alpha_1}(\ell) \boldsymbol{F}^T \mathbf{u}^\ell_1 + \mathbf{Q}^{a*}_{\alpha_1}(\ell) \boldsymbol{F}'^T \mathbf{u}^{\ell*}_1 \right). \quad (48)$$

Let us now find an explicit expression of the near–far resistance of the receivers. Based on (44) and (47) and applying standard multiuser analysis techniques [13], it can be shown that the LCD and MMSE receiver have the same near–far resistance $\eta_1(\ell)$. Using (24) and on (46), it is easy to see that $\eta_1(\ell)$ is given by

$$\eta_1(\ell) = \frac{\mathbf{u}^{\ell H}_{1a} \left(\boldsymbol{I}_{2N} - \boldsymbol{U}^\ell_J \left(\boldsymbol{U}^{\ell H}_J \boldsymbol{U}^\ell_J \right)^{-1} \boldsymbol{U}^{\ell H}_J \right) \mathbf{u}^\ell_{1a}}{2 \left\| \mathbf{u}^\ell_1 \right\|^2} = 1 - \frac{1}{2|\alpha_1(\ell)|^2} \mathbf{u}^{\ell H}_{1a} \boldsymbol{U}^\ell_J (\boldsymbol{U}^{\ell H}_J \boldsymbol{U}^\ell_J)^{-1} \boldsymbol{U}^{\ell H}_J \mathbf{u}^\ell_{1a}.$$

APPENDIX B

For further use, notice that

$$\boldsymbol{U}^{\ell H}_{1a} \boldsymbol{U}^\ell_{1a} = \boldsymbol{\Phi}^*(\ell) \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma}(\ell) \boldsymbol{S}^\ell_1 \boldsymbol{\Phi}(\ell) + \boldsymbol{\Phi}(\ell) \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma}(\ell) \boldsymbol{S}^\ell_1 \boldsymbol{\Phi}(\ell)^*$$

where $\boldsymbol{\Phi}(\ell) \triangleq \mathrm{diag}(\arg(\alpha_2(\ell)), \ldots, \arg(\alpha_K(\ell)))$ and $\boldsymbol{\Gamma}(\ell) \triangleq \mathrm{diag}(|\alpha_2(\ell)|^2, \ldots, |\alpha_K(\ell)|^2)$.

Thus. $\forall m$

$$E\left[\mathrm{trace}\left(\boldsymbol{U}^\ell_{1a} \boldsymbol{U}^{\ell H}_{1a} \right)^m\right] = E\left[\mathrm{trace}\left(\boldsymbol{U}^{\ell H}_{1a} \boldsymbol{U}^\ell_{1a} \right)^m\right] = E\left[\mathrm{trace}\left(\boldsymbol{\Phi}^* \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma} \boldsymbol{S}^\ell_1 \boldsymbol{\Phi} + \boldsymbol{\Phi} \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma} \boldsymbol{S}^\ell_1 \boldsymbol{\Phi}^* \right)^m\right].$$

Furthermore, we recall here the following proposition [13]:

Proposition 5: Let $\boldsymbol{A}$ be a $N \times K$ dimensional matrix of independent identically distributed (i.i.d.) complex random variables with zero mean and variance $1/N$. Then, the empirical distribution function of the eigenvalues of the K-dimensional random matrix $\boldsymbol{A}^H \boldsymbol{A}$, converges almost surely, as $K \to \infty$, to the cumulative distribution function of the probability density function (pdf):

$$f_\beta(x) = \left[1 - \frac{1}{\beta}\right]^+ \delta(x) + \frac{\sqrt{(x-a)^+(b-x)^+}}{2\pi\beta x} \qquad (49)$$

where $z^+ = \max\{0, z\}$ and $a = (1-\sqrt{\beta})^2$ and $b = (1+\sqrt{\beta})^2$.

A generalization of Proposition 5 yields the following result [15].

Proposition 6: Let $\boldsymbol{A}$ be a $N \times K$ dimensional matrix defined as in Proposition 5. Let $\boldsymbol{Z}$ be a $K \times K$ random Hermitian nonnegative-definite matrix independent of $\boldsymbol{A}$ such that G^K, the empirical distribution of its eigenvalues, converges almost surely to a nonrandom limit G, as $K \to \infty$. Then, the empirical distribution of the eigenvalues of the matrix $\boldsymbol{AZA}^H$ converges almost surely to a nonrandom limit F whose Stieltjes transform $\mathcal{F}(z)$ with $z \in \mathcal{C}^+ = \{z \in \mathcal{C} : \mathrm{Im}(z) > 0\}$ satisfies

$$\mathcal{F}(z) = \frac{1}{-z + \beta \int \frac{\nu dG(\nu)}{1 + \nu \mathcal{F}(z)}}. \qquad (50)$$

Propositions 5 and 6 allow the following propositions to be derived.

Proposition 7: Let $\boldsymbol{AZA}^H$ have a bounded spectral radius and let $\boldsymbol{s}$ be a N-dimensional vector with i.i.d. entries with zero means and finite variances, independent of $\boldsymbol{AZA}^H$ and such that $E[||\boldsymbol{s}||^2] = 1$. Then

$$\boldsymbol{s}^H (\boldsymbol{AZA}^H)^m \boldsymbol{s} \overset{\text{a.s.}}{\to} \lim_{N\to\infty} \frac{1}{N} E\left[\mathrm{trace}(\boldsymbol{AZA}^H)^m\right] \triangleq \int_0^\infty \lambda^m \, dF(\lambda)$$

where $F(\lambda)$ its cumulative distribution function given in Proposition 6.

Proposition 8: The empirical distribution function of the eigenvalues of the K-dimensional random matrix $(\boldsymbol{\Phi}^* \boldsymbol{S}^{\ell T}_1 \boldsymbol{S}^\ell_1 \boldsymbol{\Phi} + \boldsymbol{\Phi} \boldsymbol{S}^{\ell T}_1 \boldsymbol{S}^\ell_1 \boldsymbol{\Phi}^*)$ converges almost surely, as $N \to \infty$ and $K \to \infty$ and $K/N = \beta$, to the cumulative distribution function of the probability density

$$f_\beta(x) = \left[1 - \frac{2}{\beta}\right]^+ \delta(x) + \frac{\sqrt{(x-2a)^+(2b-x)^+}}{\pi\beta x} \qquad (51)$$

with $a = (1-\sqrt{\beta/2})^2$ and $b = (1+\sqrt{\beta/2})^2$.

Proof: In order to prove Proposition 8, it is sufficient to show that as $N \to \infty$ and $K \to \infty$ but $K/N = \beta$

$$\frac{1}{K} \mathrm{trace}\left(\boldsymbol{\Phi}^* \boldsymbol{S}^{\ell T}_1 \boldsymbol{S}^\ell_1 \boldsymbol{\Phi} + \boldsymbol{\Phi} \boldsymbol{S}^{\ell T}_1 \boldsymbol{S}^\ell_1 \boldsymbol{\Phi}^* \right)^m \overset{\text{a.s.}}{\to} \lim_{K\to\infty} \frac{1}{K} E[\mathrm{trace}((\mathbf{T}^H \mathbf{T})^m)] \quad \forall m$$

with $\mathbf{T}$ a $2N \times (K-1)$ dimensional matrix whose entries are i.i.d. complex-valued variables with variance $1/\sqrt{N}$ and zero mean. The proof can be obtained by applying some standard combinatorial techniques [9], but the general proof for every m is very long and tedious. Another proof relies upon an application of the following Lemma from free-probability theory [15], [4] as applied to random matrices

Lemma 1: Let $\boldsymbol{A}$ and $\{\mathbf{Q}_1, \boldsymbol{O}_1, \mathbf{Q}_2, \boldsymbol{O}_2\}$ be a set of n-dimensional Hermitian random matrices with $\boldsymbol{A}$ independent of $\{\mathbf{Q}_1, \boldsymbol{O}_1, \mathbf{Q}_2, \boldsymbol{O}_2\}$ such that

$$\mathbf{Q}_j \boldsymbol{O}_j = \boldsymbol{I} \quad \text{and} \quad E[\mathrm{trace}(\mathbf{Q}_j \boldsymbol{O}_i)] = 0 \quad i \neq j$$

then, as $n \to \infty$, for any $m = 2, 3, \ldots$

$$\lim_{n\to\infty} \frac{1}{n} E[\mathrm{trace}(\mathbf{Q}_1 \boldsymbol{A} \boldsymbol{O}_1 + \mathbf{Q}_2 \boldsymbol{A} \boldsymbol{O}_2)^m]$$

is specified only by the moments $(1/n)E[\text{trace}(\mathbf{Q}_1\boldsymbol{AO}_1)^p]$ and $(1/n)E[\text{trace}(\mathbf{Q}_2\boldsymbol{AO}_2)^p]$ with $0 \le p \le m$. Additionally

$$\lim_{n\to\infty}\frac{1}{n}\,\text{trace}(\mathbf{Q}_1\boldsymbol{AO}_1+\mathbf{Q}_2\boldsymbol{AO}_2)^m \overset{\text{a.s}}{\to} \lim_{n\to\infty}\frac{1}{n}E[\text{trace}(\mathbf{Q}_1\boldsymbol{AO}_1+\mathbf{Q}_2\boldsymbol{AO}_2)^m].$$

Since $\mathbf{\Phi\Phi}^* = \boldsymbol{I}_k$ and $E[\text{trace}(\mathbf{\Phi\Phi})] = 0$ from Lemma 1, the moments $E[\text{trace}(\boldsymbol{U}_{1a}^{\ell H}\boldsymbol{U}_{1a}^{\ell})^m]$ depend only on

$$E\left[\text{trace}\left(\mathbf{\Phi}^*\boldsymbol{S}_1^{\ell T}\mathbf{\Gamma}\boldsymbol{S}_1^{\ell}\mathbf{\Phi}\right)^m\right] = E\left[\text{trace}\left(\boldsymbol{S}_1^{\ell H}\mathbf{\Gamma}\boldsymbol{S}_1^{\ell}\right)^m\right] = E\left[\text{trace}\left(\mathbf{\Sigma}_1^H\mathbf{\Gamma}\mathbf{\Sigma}_1\right)^m\right]$$

and on

$$E\left[\text{trace}\left(\mathbf{\Phi}\boldsymbol{S}_1^{\ell T}\mathbf{\Gamma}\boldsymbol{S}_1^{\ell}\Phi^*\right)^m\right] = E\left[\text{trace}\left(\boldsymbol{S}_1^{\ell H}\mathbf{\Gamma}\boldsymbol{S}_1^{\ell}\right)^m\right] = E\left[\text{trace}\left(\mathbf{\Sigma}_2^H\mathbf{\Gamma}\mathbf{\Sigma}_2\right)^m\right]$$

with $\mathbf{\Sigma}_i$ a $N \times (K-1)$ dimensional matrix with i.i.d. entries with zero mean and variance $1/N$. As a consequence

$$E\left[\text{trace}\left(\boldsymbol{U}_{1a}^{\ell H}\boldsymbol{U}_{1a}^{\ell}\right)^m\right] = E\left[\text{trace}\left(\mathbf{\Sigma}_1^H\mathbf{\Gamma}\mathbf{\Sigma}_1+\mathbf{\Sigma}_2^H\mathbf{\Gamma}\mathbf{\Sigma}_2\right)^m\right].$$

For $\mathbf{\Gamma} = \boldsymbol{I}_K$ Proposition 5 concludes the proof of Proposition 8. □

Proof of the Proposition 3: We will adapt the proof in ([13], p. 198–202) for our purposes. Based on (25), we have

$$\begin{aligned} E[\eta_1(\ell)] &= E\left[1-\frac{1}{2|\alpha_1(\ell)|^2}\mathbf{u}_{1a}^{\ell H}\boldsymbol{U}_J^{\ell}(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^{\ell})^{-1}\boldsymbol{U}_J^{\ell H}\mathbf{u}_{1a}^{\ell}\right] \\ &= 1-E\left[\frac{1}{2N}\text{trace}\left(\boldsymbol{U}_J^{\ell}(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^{\ell})^{-1}\boldsymbol{U}_J^{\ell H}\right)\right] \\ &= 1-\frac{E[L_a]}{2N} \end{aligned} \tag{52}$$

where L_a is the number of nonzero eigenvalues of the matrix $\boldsymbol{U}_{1a}^{\ell H}\boldsymbol{U}_{1a}^{\ell}$. Since the signatures are random, L_a is a random variable itself. Since $E[L_a] \le \min\{K-1, 2N\}$, we have that

$$E[\eta_1(\ell)] \ge \left[1-\frac{K-1}{2N}\right]^+.$$

Using Propositions 7 and 8 and since $\mathbf{u}_{1a}$ is independent of $\boldsymbol{U}_J^{\ell}$, we obtain

$$\eta_1^\infty = \lim_{K\to\infty}\eta_1(\ell) = \lim_{N\to\infty} 1-\frac{L_a}{2N} \tag{53}$$

where $\lim_{N\to\infty}(L_a/2N) = 1-\lim_{N\to\infty}F_\lambda^N(0) \overset{\text{a.s.}}{=} (\beta/2)$ is equal to the proportion of the K eigenvalues of the matrix $\boldsymbol{U}_{1a}^{\ell H}\boldsymbol{U}_{1a}^{\ell}$ that lie above zero, where $F_\lambda^N(x)$ is the empirical distribution function of the eigenvalues of the $2N$-dimensional random matrix $\boldsymbol{U}_{1a}^{\ell}\boldsymbol{U}_{1a}^{\ell H}$. Thus, (31) follows from Lemma 1. □

Proof of the Proposition 4: Based on (8), we have that the pdf of the eigenvalues of the matrix $(\mathbf{\Phi}^*\boldsymbol{S}_1^{\ell T}\boldsymbol{S}_1^{\ell}\mathbf{\Phi}+\mathbf{\Phi}\boldsymbol{S}_1^{\ell T}\boldsymbol{S}_1^{\ell}\mathbf{\Phi}^*)$ is given by

$$f_\beta(x) = \left[1-\frac{2}{\beta}\right]^+\delta(x)+\frac{\sqrt{(x-2a)^+(2b-x)^+}}{\pi\beta x} \tag{54}$$

with $a = (1-\sqrt{\beta/2})^2$ and $b = (1+\sqrt{\beta/2})^2$. Using (27), we additionally know that for a given set of fading coefficients $\{\alpha_k(\ell)\}_{\ell,k}$:

$$\begin{aligned} \mathsf{SNR}_{\text{LCM}}^\infty &= \lim_{K\to\infty}\mathsf{SNR}_{\text{LCM}}(\ell) \\ &= \mathcal{E}_1\lim_{N\to\infty}\mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}(\ell)+2\mathcal{N}_0\boldsymbol{I}\right)^{-1}\mathbf{u}_{1a}^{\ell} \\ &\overset{\text{a.s.}}{=} |\alpha_1(\ell)|^2\mathcal{E}_1 \\ &\quad\times\lim_{N\to\infty}\frac{1}{N}\,\text{trace}\left(\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a\,|\,\alpha_1}(\ell)+2\mathcal{N}_0\boldsymbol{I}\right)^{-1}\right) \\ &= 2|\alpha_1(\ell)|^2\mathcal{E}_1\int_0^\infty\frac{1}{\lambda+2\mathcal{N}_0}\,dG(\lambda) \\ &\triangleq |\alpha_1(\ell)|^2\mathcal{E}_1 m_G(-\mathcal{N}_0) \\ &= \frac{|\alpha_1(\ell)|^2\mathcal{E}_1}{\mathcal{N}_0+\frac{\beta}{2}E\left[\frac{|A|^2\mathcal{E}_1}{1+|A|^2\mathcal{E}_1 m_G(-2\mathcal{N}_0)}\right]} \end{aligned}$$

where

- $G(\lambda)$ cumulative distribution function of the eigenvalues of the matrix $(\boldsymbol{U}_{1a}^{\ell H}\boldsymbol{U}_{1a}^{\ell})$;
- m_G Stieltjes transform if $G(\lambda)$;
- $|A|^2$ nonnegative random variable whose distribution is the limit distribution of

$$\left\{\frac{\mathcal{E}_k}{\mathcal{E}_1}|\alpha_k|^2, k = 1,\ldots,K\right\}.$$

Notice that, for the last equality, we have used the Proposition 6. Applying the definition of multiuser efficiency [13] and the expression of the Tse–Hanly formula given in [14] we obtain that the multiuser efficiency of the LCM receivers (19) converges almost surely to the solution $\eta_{\text{LCM}}^\infty(\mathsf{SNR})$ of

$$\eta+\frac{\beta}{2}E\left[\frac{2\,\mathsf{SNR}|A|^2\eta}{1+2\mathsf{SNR}|A|^2\eta}\right] = 1. \tag{55}$$

□

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