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Computational Complexity of Optimum Multiuser Detection[1]

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Abstract. Optimum centralized demodulation of the independent data streams transmitted simultaneously by several users through a Code Division Multiple-Access channel is considered. Each user sends an arbitrary assigned signal waveform, which is linearly modulated by symbols drawn from a finite alphabet. If the users are asynchronous, the optimum multiuser detector can be implemented by a Viterbi algorithm whose time-complexity is linear in the number of symbols transmitted by each user and exponential in the number of users. It is shown that the combinatorial problem of selecting the most likely transmitted data stream given the sufficient statistics (sequence of matched filter outputs), and the signal energies and cross-correlations is nondeterministic polynomial-time hard (NP-hard) in the number of users. And it remains so even if the users are restricted to be symbol-synchronous.

The performance analysis of optimum multiuser detection in terms of the set of multiuser asymptotic efficiencies is equivalent to the computation of the minimum Euclidean distance between any pair of distinct multiuser signals. This problem is also shown to be NP-hard and a conjecture on a longstanding open problem in single user data communication theory is presented.

Key Words. NP-complete, Hypothesis testing, Code Division Multiple Access, Gaussian communication channels, Maximum-likelihood sequence detection.

1. Introduction. The purpose of hypothesis testing problems is to select a solution (decision) from among a *finite* set of possible solutions (hypotheses). Typically, the number of hypotheses is small, in which case the inherent combinatorial optimization nature of the problem does not play any role and the main question is to obtain the values of the likelihood function or other finite-dimensional set of sufficient statistics. In this paper we study a data demodulation problem where the reverse situation is encountered: it is straightforward to obtain a set of scalar sufficient statistics but the number of hypotheses is very large.

An important problem arising in multipoint-to-point digital communication networks (e.g., radio networks, local-area networks, and uplink satellite channels) is the optimum centralized demodulation of the information sent simultaneously by several users through a Gaussian multiple-access channel. Even though the users may not employ a protocol to coordinate their transmission epochs, effective sharing of the channel is possible because each user modulates a different signature signal waveform which is known by the intended receiver (Code Division Multiple Access (CDMA)). Recently [1], optimum multiuser detection has been shown to offer important gains in bit-error-rate performance over single-user detectors, which are conventionally used in practice and neglect the

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presence of interfering users. The optimum multiuser receiver can be viewed as a bank of single-user detectors followed by a common algorithm that selects the most likely transmitted symbols. The structure of this algorithm depends crucially on whether or not the users maintain symbol synchronism. In the synchronous case, it is enough to maximize a quadratic function, while in the asynchronous case the way to get rid of the interference among the users is to employ a version of the Viterbi forward dynamic programming algorithm [2], whose time-complexity is exponential in the number of active users. It is shown in this paper that the problem is nondeterministic polynomial-time hard (NP-hard) in the number of users, and hence there exists no polynomial-time algorithm for optimum multiuser detection unless such an algorithm is found for a large class of combinatorial problems, such as the traveling salesman and integer linear programming problems. It is shown that the problem remains NP-hard even in the synchronous case despite an earlier claim of existence of polynomial solutions [3] for that case.

In Section 2 the multiple-access channel model and the maximum-likelihood detection problem are formulated and it is shown that optimum multiuser detection is NP-hard in the number of users. In Section 3 it is shown that the performance analysis of optimum multiuser detectors is intrinsically difficult due to the fact that the computation of the minimum distance between any pair of distinct multiuser signals is also NP-hard. Finally, Section 4 summarizes the main points of the paper, discusses suboptimum alternatives, and presents a conjecture on a longstanding open problem in data communication theory.

2. Optimum Multiuser Detection.

Assume each of K users transmits independent symbols by modulating a preassigned waveform from a signal constellation $\{s_k(t),\ t\in[0, T],\ k=1,\ldots, K\}$. If the users cooperate to maintain symbol synchronism, the receiver observes the sum of the modulated signals imbedded in noise, i.e.,

$$r(t)=\sum_{k=1}^{K} b_k s_k(t)+n(t), \tag{1}$$

where the symbols b_k, $k=1,\ldots, K$, are drawn by each user from a finite alphabet A. A reasonable decision rule is to select the set of symbols corresponding to that signal among the possible ones that resembles most closely (in a mean-square sense) the received waveform. If the noise is Gaussian and white, then this rule is optimum in the maximum-likelihood sense. If, furthermore, all vectors $\mathbf{b}=(b_1,\ldots, b_K)\in A^k$ are *a priori* equiprobable, then the minimum distance rule gives the maximum-*a-posteriori* (MAP) decision. In the single-user case, this detector is implemented by comparing the output of a matched filter with a set of thresholds. Analogously, in the multiuser problem we have

$$\arg\min_{b\in A^K}\left\|r(t)-\sum_{k=1}^{K} b_k s_k(t)\right\| = \arg\max_{b\in A^K} 2\mathbf{b}^T\mathbf{y}-\mathbf{b}^T\mathbf{H}\mathbf{b}, \tag{2}$$

where $\mathbf{y}=(y_1,\ldots,y_K)$, $y_k=\int_0^T s_k(t)r(t)\,dt$, i.e., y_k is the output of the matched filter of the kth user signal and the entries of the nonnegative definite matrix $\mathbf{H}$ are given by

$$h_{ij}=\int_0^T s_i(t)s_j(t)\,dt. \tag{3}$$

If the signal waveforms are orthogonal, then $\mathbf{H}$ is a diagonal matrix and the maximization in (2) decouples into K single-user problems. Otherwise (in practice, there may be bandwidth or complexity constraints that prevent the designer from choosing an orthogonal signal set), a combinatorial algorithm is required to solve the quadratic optimization (2) over the finite set A^K, given the vector of sufficient statistics $\mathbf{y}$ and the signal cross-correlations $\mathbf{H}$. Since the set of quantities $\{\mathbf{b}^T\mathbf{y}, \mathbf{b}\in A^K\}$ can be computed in $O(|A|^K)$ operations, an upper bound on the time-complexity per bit (TCB)[3] required to solve (2) is $O(|A|^K/K\log|A|)$. This is the best available upper bound; in [3] it is claimed that a receiver whose complexity is polynomial in the number of users (basically, if $A=\{-1,+1\}$, select the sign of the components of $\mathbf{H}^{-1}\mathbf{y}$) is optimal. Unfortunately, this claim is erroneous; the mistake in the derivation of the detector is commited in equation 4 of [3] where it is implicitly assumed that the symbols put out by the detector are uncorrelated with the noise component of the matched filter outputs.

More significant in practical applications is the case where the users are mutually asynchronous, and indeed one of the chief advantages of CDMA over other channel sharing strategies is that no type of coordination among the users is required. Now, however, (1) is no longer a valid model. The delays $\{\tau_k, k=1,\ldots,K\}$ account for the offsets between the signaling epochs and (1) has to be generalized to

$$r(t)=\sum_{i=-M}^{M}\sum_{k=1}^{K} b_k(i)s_k(t-iT-\tau_k)+n(t), \tag{4}$$

where by convention $s_k(t)=0$ for $t\notin[0,T]$. Now we can no longer restrict our attention to the one-shot case because optimum decisions are based on the whole received waveform due to the interference between the symbols. The optimum receiver [1] for the asynchronous case in the sense of selecting the most likely sequence of symbols consists of a front-end of matched filters (just as in the synchronous case) followed by a Viterbi dynamic programming algorithm with $|A|^{K-1}$ states and a periodically time-varying branch metric. The TCB of this decision algorithm is $O(|A|^K/\log|A|)$, and hence the penalty in time-complexity due to the lack of synchronism between the users is slight. The usefulness and relevance of the computational complexity results proved in this paper stem from the fact that when the users are asynchronous, the cross-correlations between

[3] The time-complexity per bit is defined as the limit of the ratio of total time to the number of demodulated bits as this goes to infinity. Note that any preprocessing of the signal cross-correlations does not affect TCB.

their signals are unknown *a priori*, and the worst-case TCB over all possible mutual offsets is the complexity measure of interest since it determines the maximum achievable data rate in the absence of synchronism among the users. Actually, no family of signature signals is known to result in optimum demodulation with polynomial-in-K complexity for all possible signal offsets. So even if the designer of the signal constellation were to include in his design criterion the complexity of the optimum demodulator in addition to the bit-error-rate performance (which dictates signals with low cross-correlations), he would not be able to endow the signal set with any structure that would overcome the inherent intractability of the optimum asynchronous demodulation problem for all possible offsets. On the other hand, in the synchronous case, the designer of the signal constellation has more control on the cross-correlations (subject to constraints such as bandwidth or number of chips per symbol in Direct-Sequence Spread-Spectrum), and it is conceivable that there exist synchronous signal design constraints that result in families of signature signals whose structure can be exploited to result in optimum polynomial-time decision algorithms. This is the reason why the computational complexity results of this paper appear to be more relevant to asynchronous channels even though for the purposes of the proofs in our lower bound analysis we may restrict attention to the special case where all the delays coincide ($\tau_1 = \cdots = \tau_K$), because the optimum multiuser detector must be able to deal with any arbitrary set of delays.

In order to ascertain that the intractability of the optimum multiuser problem arises when the number of users is large and the alphabet size is kept constant, we first fix an arbitrary alphabet $A = \{a_1, \ldots, a_m\}$ (which is a set of integers satisfying $a_i < a_{i+1}$), and define a class of instances of the combinatorial optimization problem for that *fixed* A.[4]

MULTIUSER DETECTION
Instance: Given $K \in \mathbb{Z}^+$, $\mathbf{y} \in Q^K$, and a nonnegative definite matrix $\mathbf{H} \in Q^{K \times K}$
Find: $\{\mathbf{b}^* \in A^K\}$ that maximizes $2\mathbf{b}^T\mathbf{y} - \mathbf{b}^T\mathbf{H}\mathbf{b}$.

PROPOSITION 1. *If* $|A| > 1$, *then MULTIUSER DETECTION is* NP-*hard.*

PROOF. The proof of NP-hardness of MULTIUSER DETECTION can be carried out by transformation from PARTITION, an NP-complete recognition problem. Recall its definition [4]:

Instance: Given $L \in \mathbb{Z}^+$ and $\{l_i \in \mathbb{Z}^+, i = 1, \ldots, L\}$.
Question: Is there a subset $I \subset \{1, \ldots, L\}$ such that $\sum_{i \in I} l_i = \sum_{i \notin I} l_i$?

For each instance of PARTITION, we can find in polynomial time an instance of MULTIUSER DETECTION whose solution can in turn be processed in

[4] Note that since the alphabet A is not part of the instance, if a specific A is assumed, then the corresponding NP-hard result is a *corollary* to Proposition 1. Actually, for $A = \{-1, +1\}$, the proof of Proposition 1 can be simplified considerably by letting $h_{ij} = l_i l_j$ and $y_k = 0$ therein.

polynomial time to give an answer to PARTITION. Given $l_1, \ldots, l_L$, we choose the following instance of MULTIUSER DETECTION:

$$K = L,$$

$$h_{ij} = l_i l_j, \qquad i \neq j,$$

$$h_{ii} = \left\lceil l_i \max\left\{ l_i, (a_2 - a_1)^{-1}[2a_m - a_1 - a_2] \sum_{\substack{j=1 \\ j \neq i}}^{K} l_j \right\} \right\rceil, \qquad i = 1, \ldots, K,$$

$$y_k = \tfrac{1}{2}(a_1 + a_2)\left(h_{kk} + l_k \sum_{\substack{j=1 \\ j \neq k}}^{K} l_j \right), \qquad k = 1, \ldots, K.$$

Note that this is a valid instance of MULTIUSER DETECTION, because $\mathbf{H}$ is a nonnegative definite matrix. Once the solution to this instance of MULTIUSER DETECTION is found, we can find the solution to the original instance of PARTITION, because $\{l_1, \ldots, l_L\}$ is a *yes* instance of PARTITION if and only if

$$\max_{b \in A^K} 2\mathbf{b}^T\mathbf{y} - \mathbf{b}^T\mathbf{H}\mathbf{b} = \left[\tfrac{1}{2}(a_1 + a_2) \sum_{i=1}^{K} l_i \right]^2 + a_1 a_2 \sum_{i=1}^{K} (h_{ii} - l_i^2). \tag{5}$$

Equation (5) can be shown by changing the variable in the left-hand side of (5) $\mathbf{b} = \frac{1}{2}(a_2 - a_1)\mathbf{z} + \frac{1}{2}(a_1 + a_2)$ and proving that it is enough to restrict attention to the values $z_i = \pm 1$ in the maximization in (5) (see [5] for details). □

The foregoing proof shows that the same transformation works if the value of the diagonal elements of $\mathbf{H}$ is arbitrarily increased. Hence MULTIUSER DETECTION remains NP-hard if $\mathbf{H}$ is restricted to be strongly diagonal (an important special case in CDMA with equal-energy users). Note that MULTIUSER DETECTION was defined for a fixed arbitrary alphabet. Thus, Proposition 1 implies that the problem is inherently difficult when the number of users is large, regardless of the alphabet size (often a small integer). Conversely, in Section 2 we saw that the problem is polynomial in the alphabet size for fixed number of users.

3. NP-hardness of Multiuser Asymptotic Efficiency. In this section we examine the complexity of the *performance analysis* of optimum multiuser detection. The purpose of this analysis is to evaluate the effect of the energies and cross-correlations of the signal constellation on the bit-error-rate of the receiver for an arbitrary level of background noise. It has been shown [6] that the key performance measure is the multiuser asymptotic efficiency, or ratio between the exponential decay rate of the bit-error-rates with and without interfering users. This parameter effectively quantifies the degradation in bit-error-rate due to the presence of other users, in situations where the background noise is not dominant. The asymptotic efficiency of the kth user, η_k, is proportional to the Euclidean distance between any pair of transmitted signals whose kth symbols do not agree [6]. Specifically, assuming synchronous users and antipodal modulation (i.e., $A=\{-1,+1\}$), the kth user asymptotic efficiency can be expressed as

$$\eta_k=\frac{\min\limits_{b_k^1\neq b_k^2}\int_0^T\left[\sum_{i=1}^K(b_i^1-b_i^2)s_i(t)\right]^2dt}{4\int_0^T s_k^2(t)\,dt}=\frac{1}{h_{kk}}\min_{\substack{\varepsilon\in\{-1,0,1\}^K\\ \varepsilon_k\neq 0}}\boldsymbol{\varepsilon}^T\mathbf{H}\boldsymbol{\varepsilon}. \tag{6}$$

PROPOSITION 2. *The following problem is* NP-*hard*:

MULTIUSER ASYMPTOTIC EFFICIENCY
Instance: *Given* $K\in\mathbb{Z}^+$, $k\in\{1,\dots,K\}$, *and a nonnegative matrix* $\mathbf{H}\in\mathbb{Z}^{K\times K}$
Find: *The* kth *user maximum asymptotic efficiency,*

$$\eta_k=(1/h_{kk})\min_{\varepsilon\in\{-1,0,1\}^K,\varepsilon_k\neq 0}\boldsymbol{\varepsilon}^T\mathbf{H}\boldsymbol{\varepsilon}.$$

PROOF. The proof is divided in two steps, first $-1/0/1$ KNAPSACK is polynomially transformed to MULTIUSER ASYMPTOTIC EFFICIENCY, and then we show that $-1/0/1$ KNAPSACK is NP-complete. In analogy to the 0/1 KNAPSACK problem (e.g., [7]) we define

$-1/0/1$ KNAPSACK
Instance: $L\in\mathbb{Z}^+$, $G\in\mathbb{Z}^+$, and a family of not necessarily distinct positive integers $\{l_i\in\mathbb{Z}^+, i=1,\dots,L\}$.
Question: Are there integers $\varepsilon_i\in\{-1,0,1\}$, $i=1,\dots,L$, such that $\sum_{i=1}^L\varepsilon_i l_i=G$?

We transform $-1/0/1$ KNAPSACK to MULTIUSER ASYMPTOTIC EFFICIENCY by adding an additional user. Given $\{G,l_1,\dots,l_L\}$, denote $l_{L+1}=G$ and construct the following instance: $K=L+1$, $k=L+1$, $h_{ij}=l_il_j$, $1\le i,j\le K$.

The Kth-user asymptotic efficiency is equal to 0 if and only if $\{G,l_1,\dots,l_L\}$ is a *yes* instance of $-1/0/1$ KNAPSACK. To see this, note that we can fix $\varepsilon_k=-1$ in the right-hand side of (6) without loss of generality. Then,

$$\begin{aligned}\eta_K&=\frac{1}{G^2}\min_{\substack{\varepsilon_i\in\{-1,0,1\}\\ 1\le i\le K-1}}\left\{h_{KK}+\sum_{n=1}^{K-1}\varepsilon_n\left[-2h_{nK}+\sum_{m=1}^{K-1}\varepsilon_m h_{nm}\right]\right\}\\ &=\frac{1}{G^2}\min_{\substack{\varepsilon_i\in\{-1,0,1\}\\ 1\le i\le K-1}}\left(G-\sum_{n=1}^{K-1}\varepsilon_n l_n\right)^2.\end{aligned} \tag{7}$$

Now we show that $-1/0/1$ KNAPSACK is NP-complete. Its membership in NP is obvious. Note that it is easy to transform $-1/0/1$ KNAPSACK to 0/1 KNAPSACK ($\{G,l_1,\dots,l_L\}$ is a *yes* instance of $-1/0/1$ KNAPSACK if and only if $\{G,l_1,-l_1,\dots,l_L,-l_L\}$ is a *yes* instance of 0/1 KNAPSACK). However, we need to show the reverse transformation, namely, fixing any instance of 0/1

KNAPSACK obtain an equivalent instance of $-1/0/1$ KNAPSACK. The idea is reminiscent of the polynomial transformation of 0/1 KNAPSACK to POSITIVE INTEGER KNAPSACK (see p. 376 of [7]), and it consists of constructing an augmented instance of $-1/0/1$ KNAPSACK whose integers are large enough to force every coefficient to be either 0 or 1. Choose an instance $\{G, l_1, \ldots, l_L\}$ of 0/1 KNAPSACK, and construct the following instance of $-1/0/1$ KNAPSACK $\{D, p_1, \ldots, p_{2L}\}$:

$$D = G + \sum_{j=1}^{L} M^j,$$

$$p_j = \begin{cases} l_j + M^j & \text{for } 1 \le j \le L, \\ M^{j-L} & \text{for } L+1 \le j \le 2L, \end{cases}$$

$$M = 1 + \max\left\{3,\, G + \sum_{i=1}^{L} l_i\right\}.$$

Then it follows from this choice that for all $\{\varepsilon_i \in \{-1, 0, 1\}, 1 \le i \le 2L\}$

$$\sum_{i=1}^{2L} \varepsilon_i p_i - D = -G + \sum_{i=1}^{L} \varepsilon_i l_i + \sum_{i=1}^{L} M^i(\varepsilon_i + \varepsilon_{i+n} - 1). \tag{8}$$

It is straightforward to show that M is too large to be a root of any L-degree polynomial $\sum_{i=0}^{L} \beta_i x^i$, where $\beta_i \in \{-3, -2, -1, 0, 1\}$ for $1 \le i \le L$ and $\beta_0 \in \{-G + \sum_{i=1}^{L} \varepsilon_i l_i,\ \varepsilon_i \in \{-1, 0, 1\}, 1 \le i \le L\}$. Hence, (8) is equal to zero if and only if all the coefficients of the polynomial in M of the right-hand side are zero, i.e.,

$$\sum_{i=1}^{2L} \varepsilon_i p_i = D \quad \Leftrightarrow \quad \sum_{i=1}^{L} \varepsilon_i l_i = G \quad \text{and} \quad \{\varepsilon_i = 0, \varepsilon_{i+n} = 1 \text{ or } \varepsilon_i = 1, \varepsilon_{i+n} = 0, 1 \le i \le l\}.$$

Therefore, $\{G, l_1, \ldots, l_L\}$ is a *yes* instance of 0/1 KNAPSACK if and only if $\{D, p_1, \ldots, p_{2L}\}$ is a *yes* instance of $-1/0/1$ KNAPSACK. □

Note that the proof of Proposition 2 shows in fact that the problem of deciding whether a signal constellation is uniquely decodable, i.e., whether different transmitted bits result in the same waveform, is NP-complete.

4. Concluding Remarks, Suboptimum Algorithms, and Open Problems. It has been shown that the problem of optimum detection in Gaussian multiple-access channels is NP-hard in the number of users. This result holds for any nontrivial alphabet even if the channel is symbol-synchronous. Exponential-in-K optimum detectors for Poisson multiple-access channels with point-process observations were obtained in [8]. It can be shown that this problem is also NP-hard in both the additive-rate and additive-light models of the channel.

Not only is the optimum decision rule intrinsically difficult, but so is the analysis of its performance due to the NP-hardness of the computation of multiuser asymptotic efficiencies. It should be pointed out, however, that in many instances of signal constellations used in CDMA Direct Sequence Spread-Spectrum systems [9], the cross-correlations are low enough to pass sufficient conditions [6] ensuring unit asymptotic efficiency that are computable in quadratic time in K. So, unlike the situation we encountered in the optimum asynchronous demodulation problem, the worst-case complexity measure may be overly pessimistic for specific instances exhibiting low cross-correlations.

What alternatives, then, does the designer have when the number of users is large? The suboptimum solution currently employed in practice is the bank of single-user receivers (i.e., a matched filter for each user followed by a threshold). Unfortunately, this scheme achieves far from optimum bit-error-rate and its performance breaks down when the signal energies are dissimilar (the near–far problem) [1], [6], [10]. Therefore, the search for approximation algorithms that achieve near-optimum bit-error-rate with polynomial complexity appears to be an open research area with important consequences in practice. Numerical results indicate that performance extremely close to the single-user lower bound is achievable in the bit-error-rate region of usual interest (10^{-4} or less) by the maximum-likelihood multiuser receiver even if the cross-correlation qualities of signal constellations typically used in practice are considerably relaxed. This is a sign that the Viterbi-based optimum multiuser receiver possesses an important degree of redundancy in situations with good signal sets and low background noise and hence faster decision algorithms achieving similar performance are plausible. Furthermore, a linear multiuser demodulator whose TCB is linear in K is found in [10] to achieve the same worst-case asymptotic efficiency over the energies of the interfering users[5] as the optimum demodulator. While for specific values of the received energies, its asymptotic efficiency (and hence, its bit error rate) need not be close to the optimum one, its performance is guaranteed to exceed a high lower bound in all cases of practical interest, thus making this suboptimum demodulator an attractive choice from both the complexity and performance standpoints.

Finally, let us consider an interesting special case of the asynchronous optimum multiuser detection model in (4), namely the single-user intersymbol interference problem,

$$r(t)=\sum_{i=-M}^{M} b(i)s(t-iT)+n(t),$$

where the duration of $s(t)$ is greater than T. In general, there is no known efficient method to obtain the most likely sequence of transmitted symbols given the received waveform (TCB is exponential in the frame length M). However, if the number of signals that interfere at any given time is bounded by, say, L, then it

[5] This is called the *near–far resistance*, a key measure of the robustness of the system against variations in the received energies.

is well known [11] that maximum-likelihood sequence detection can be implemented by the Viterbi algorithm in TCB which although exponential in L is independent of M. Despite many efforts (e.g., [12]-[14]) motivated by the importance of this problem in the area of data transmission through bandlimited channels, no polynomial-in-L algorithm for maximum-likelihood sequence detection is known. This fact and the results of this paper lead us to suspect that we may be facing another NP-hard problem. In fact following the same steps as in Section 2, it can be seen that the most likely sequence of symbols corresponding to (13) is the one that maximizes $2\mathbf{b}^T\mathbf{y}-\mathbf{b}^T\mathbf{H}\mathbf{b}$, where $\mathbf{b}=(b(-M),\ldots,b(M))$, $\mathbf{y}=(y(-M),\ldots,y(M))$, $y(i)=\int s(t-iT)r(t)\,dt$, and $\mathbf{H}$ is the nonnegative definite Toeplitz matrix (i.e., constant along diagonals) with entries given by $h_{ij}=\int s(t-iT)s(t-jT)\,dt$. Hence if we specialize $L=2M+1=K$, the underlying combinatorial problem coincides with MULTIUSER DETECTION with an additional restriction on the data:

Conjecture 1. MULTIUSER DETECTION remains NP-hard if $\mathbf{H}$ is restricted to be Toeplitz.

Indeed, it appears that the Toeplitz condition imposes an analytically inconvenient restriction on the set of allowable instances. A possible route is to consider the following restricted version of the problem.

FIR
Instance: Given $L\in\mathbb{Z}^+$, $E\in\mathbb{Z}^+$, and the coefficients of a finite-impulse response (FIR) digital filter of length L ($h_i\in\mathbb{Z}$, $i=0,\ldots,L-1$).
Question: Does there exist an input sequence ($b_i\in\{-1,+1\}$, $i=0,\ldots,L-1$) such that the output energy of the FIR is less than E, i.e.,

$$\sum_{i=0}^{2L-2}\left(\sum_{j=0}^{L-1} b_j h_{i-j}\right)^2 \le E?$$

Conjecture 2. FIR is NP-complete.

Similarly, the problem of the performance analysis of the single-user intersymbol interference channel is equivalent to finding the minimum distance between any pair of transmitted data streams. This problem for which no polynomial algorithm in the length of the interference is known can be put as a special case of MULTIUSER ASYMPTOTIC EFFICIENCY and it is not known whether it is NP-hard.

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