REDUCTION OF SYMBOLIC SPARSE MATRICES

Indexing terms: Circuit design, Computer-aided design, Linear systems

A new general method is derived for reduction of linear systems of equations with symbolic coefficients suitable for using sparse matrix techniques. Its implementation in a program for symbolic analysis of circuits offers significant advantages over the existing reduction method.

In the symbolic analysis of linear time-invariant circuits, two classes of methods seem to offer the best results to date, namely: (i) reduction of the symbolic system of equations, and (ii) various combinations of various kinds of forward and backward substitution algorithms. Here, we restrict our discussion to (ii) since it is the most general of the two.

To overcome this difficulty several authors have made use of the method of 'deferred divisions'.

Focusing our attention on the first approach, we can see that direct application of the Gaussian reduction formula to a system of linear equations

In symbols, we have

where $A$, $B$, and $C$ are matrices, and $D$ is the determinant of $A$. The determinant of a matrix is the product of its eigenvalues.

To derive a different process of reduction unifying both features: exact divisions and the possibility of applying sparse matrix techniques.

When the coefficients of the system of equations are rational functions, we can express the Gaussian reduction formula as

where we have supposed that the process proceeds row by row, and the denominator of all the coefficients in row $i$ is $d_i^k$. In general, eqn. 2 must be accomplished with

where $k$ is the degree of freedom introduced, i.e., $D_i^k$. If we let $D_i^k = d_i^{k+1} - 1$, this is the same as the forward substitution formula.

we shall keep for every row $i$ a set $P_i^k$ of former pivots, such that its product $\prod P_i^k = d_i^k$ (we denote by $\prod A$ the product of all the elements belonging to the set $A$ and by agreement $\prod \{0\} = 1$). The selection of $D_i^k$ is the following:

(a) If $n_{ik}^k \neq 0$ (row coupled with current pivot) then

To satisfy eqn. 4, it is a sufficient condition that

because this implies that

and therefore

which is equivalent to eqn. 4.

(b) If $n_{ik}^k = 0$ (row uncoupled with current pivot) then

leading directly to eqn. 4.

We can see that, in case a, the division in eqn. 3 is carried out exactly, as a corollary of the proposition proved in Reference 7, namely, that if pivot $n_{i j}^k \in (P_i^k \cap P_i^k)$ then $n_{i j}^k = n_{i j}^k - n_{i k}^k n_{k j}^k$ contains the factor $n_{i k}^k$. In case b no division need be performed, as was our goal. Our proposed method can then be formulated as follows:

If $n_{ik}^k = 0$ then

If $n_{ik}^k \neq 0$

then $n_{i j}^k - n_{i k}^k n_{k j}^k = (n_{i j}^k - n_{i k}^k n_{k j}^k) \prod (P_i^k \cap P_i^k)$

with $P_i^0$ equal to the empty set, and $N$ as the initial order of the matrix.

Although this is the most general approach for taking full advantage of the sparsity of the system, in some instances the software implementation can be simplified applying a method requiring single divisors. In eqn. 3 we make a different choice of $D_i^k$, taking into account that on we keep for every row $i$ only one pivot $P_i^k$ such that $d_i^k = P_i^k$. The result is:

If $n_{ik}^k = 0$ and $i \neq k$ then

Otherwise,

we shall keep for every row $i$ a set $P_i^k$ of former pivots, such that its product $\prod P_i^k = d_i^k$ (we denote by $\prod A$ the product of all the elements belonging to the set $A$ and by agreement $\prod \{0\} = 1$). The selection of $D_i^k$ is the following:

(a) If $n_{ik}^k \neq 0$ (row coupled with current pivot) then

To satisfy eqn. 4, it is a sufficient condition that

because this implies that

and therefore

which is equivalent to eqn. 4.

(b) If $n_{ik}^k = 0$ (row uncoupled with current pivot) then

leading directly to eqn. 4.

We can see that, in case a, the division in eqn. 3 is carried out exactly, as a corollary of the proposition proved in Reference 7, namely, that if pivot $n_{i j}^k \in (P_i^k \cap P_i^k)$ then $n_{i j}^k = n_{i j}^k - n_{i k}^k n_{k j}^k$ contains the factor $n_{i k}^k$. In case b no division need be performed, as was our goal. Our proposed method can then be formulated as follows:

If $n_{ik}^k = 0$ then

If $n_{ik}^k \neq 0$

then $n_{i j}^k - n_{i k}^k n_{k j}^k = (n_{i j}^k - n_{i k}^k n_{k j}^k) \prod (P_i^k \cap P_i^k)$

with $P_i^0$ equal to the empty set, and $N$ as the initial order of the matrix.

Although this is the most general approach for taking full advantage of the sparsity of the system, in some instances the software implementation can be simplified applying a method requiring single divisors. In eqn. 3 we make a different choice of $D_i^k$, taking into account that on we keep for every row $i$ only one pivot $P_i^k$ such that $d_i^k = P_i^k$. The result is:

If $n_{ik}^k = 0$ and $i \neq k$ then

Otherwise,

we shall keep for every row $i$ a set $P_i^k$ of former pivots, such that its product $\prod P_i^k = d_i^k$ (we denote by $\prod A$ the product of all the elements belonging to the set $A$ and by agreement $\prod \{0\} = 1$). The selection of $D_i^k$ is the following:

(a) If $n_{ik}^k \neq 0$ (row coupled with current pivot) then

To satisfy eqn. 4, it is a sufficient condition that

because this implies that

and therefore

which is equivalent to eqn. 4.

(b) If $n_{ik}^k = 0$ (row uncoupled with current pivot) then

leading directly to eqn. 4.
account the uneven quantity of information of the nonzero coefficients, and efficient memory management for the process of reduction.

S. VERDU  
Department of Electrical Engineering  
University of Illinois at Urbana-Champaign  
Urbana, Ill. 61801, USA  

J. HERRANZ  
ETSI Telecommunicacion  
Arda, J. Girona-Salgado S/N  
Universidad Politecnica Barcelona  
Barcelona -34-, Spain

References

LONG WAVELENGTH PERFORMANCE OF SiO₂/GeO₂/P₂O₅ CORE FIBRES WITH DIFFERENT P₂O₅ LEVELS

*Indexing terms*: Optical fibres, Attenuation

The attenuation performance of SiO₂/GeO₂/P₂O₅ core graded index multimode fibres at 1.6 µm has been studied in relation to changing P₂O₅ concentrations. Loss is related to both P₂O₅ doping and OH contamination levels, and reduction of both has led to a minimum fibre attenuation of 0.23 dB/km at 1.62 µm.

Introduction: SiO₂/GeO₂/P₂O₅ core fibres have been reported with low losses and as suitable for operation in the 1.5-1.6 µm band. A previous letter demonstrated that hydroxyl radicals can be trapped at phosphorus sites in P₂O₅-doped silica, resulting in a shift of the fundamental Si-OH absorption at 2.74 µm to a broader, more intense absorption at ~3.05 µm, which was not observed with SiO₂/GeO₂ core fibres. The intensity of the 3.05 µm absorption was reported as being dependent on OH contamination levels, but independent of P₂O₅ concentrations, with no overtones absorptions of the 3.05 µm fundamental being observed. A subsequent letter, however, demonstrated the existence of a broad absorption in the vicinity of 1.6 µm, which was assigned as the first overtone of the 3.05 µm fundamental.

In our study, 50 µm core, 125 µm OD, nominal 0.2 NA graded index fibres were prepared with SiO₂/GeO₂/P₂O₅ cores using an internal vapour deposition technique. A 14 x 12 mm Heraulx WG silica tube was used as a substrate, which, after deposition of the optical cladding followed by core layers of SiO₂/GeO₂/P₂O₅, was collapsed into a rod preform prior to subsequent drawing into fibre. Typically 2-3 km lengths of fibre were in-line primary coated with silicone resin to a thickness of ~45 µm and wound onto 25 cm diameter aluminium reels. Spectral attenuation measurements used a cutback reference method with 40 µm spot 0.1 NA launch conditions.

Fig. 1 shows a spectral attenuation curve for 1.6 fibres with constant P₂O₅ level, but with changes in OH contamination resulting from changes in cladding thickness and from variables such as OH diffusion times during processing. This illustrates the existence of the broad 1.6 µm absorption assigned to P-OH.

![Figure 1](image)

**Fig. 1** Spectral attenuation of fibres with constant P₂O₅ concentration with changing OH levels

J: 2.1 km; K: 2.1 km; L: 2.4 km; M: 2.4 km

Fig. 2 shows a similar plot for two fibres having different P₂O₅ doping levels. Although the 1.24 µm Si-OH combination overtone absorption peak heights are similar, the attenuation at 1.6 µm is much higher in fibre T, which has a doping level of P₂O₅ ~6 times that of S. To study this effect, a series of fibres was fabricated with different P₂O₅ levels. This was achieved by changing the oxygen carrier gas flow rate through the POC1 source material whilst keeping the SiCl₄ and GeCl₄ carrier flows constant, with the changes in P₂O₅ concentration being insufficient to appreciably alter the nominal 0.2 NA value. Fig. 3 shows a plot of attenuation at 1.6 µm against OH peak height at 1.24 µm for this series of fibres, showing a clear dependence of loss on P₂O₅ concentration. To explain the different slopes in this plot, it is postulated that the increase in P₂O₅ concentration traps more of the OH impurity, and shifts the absorption from the Si-OH absorption to the more intense P-OH absorption at 1.6 µm. Thus, for any given OH contamination level, the loss at 1.6 µm will be higher for higher P₂O₅ concentrations, and both P₂O₅ and OH levels must be reduced in order to achieve minimum loss at 1.6 µm. An alternative postulate is that P-O phonon absorption could cause the excess loss.

![Figure 2](image)

**Fig. 2** Spectral attenuation of fibres with changing P₂O₅ concentrations

S: 2.3 km; T: 1.1 km

![Figure 3](image)

**Fig. 3** Attenuation at 1.6 µm against OH absorption peak height at 1.24 µm for different P₂O₅ doping levels
This has been suggested from studies of SiO₂-P₂O₅ core fibres with very low OH levels, where it was felt that low deposition temperatures and concomitant high P₂O₅ levels resulted in low OH diffusion rates and hence low OH concentrations in the core, a rising edge above 1.4 μm being assigned to P-O phonon absorption. The data would be consistent, however, with the higher P₂O₅ levels preferentially trapping OH into the 1.6 μm P-OH absorption. Recent data has also shown that the P-O phonon absorption can be as low as 0.1-0.3 dB/km at 1.6 μm in P₂O₅-doped cladding single-mode fibres, so this alternative postulate is not favoured. It may not be completely resolved, however, until an independent measurement of OH concentrations is possible, as the height of the Si-OH absorption peaks cannot be used to estimate OH levels in P₂O₅ containing fibres as used previously.

Following the results of this study, attempts were made to reduce attenuation further at 1.6 μm with SiO₂-GeO₂-P₂O₅ core fibres. Fibres were prepared with a minimum quantity of P₂O₅ consistent with reducing deposition temperatures by 100–100°C compared with binary SiO₂-GeO₂ fibres, (temperatures measured with a Williamson 4400 IR pyrometer.) SEM microprobe analysis indicated P₂O₅ levels of 0.2 mole % and from parallel studies, fluorine doping from a CCl₄-F₂ source was also incorporated in the core to a level which reduced the fibre NA from 0.2 with SiO₂-GeO₂-P₂O₅ to 0.15 with SiO₂-GeO₂-P₂O₅ with P₂O₅. F, and reduced deposition temperature by a further 50°C. Hydroxyl impurity was also reduced by careful attention to process operation, and by reduction of OH diffusion times during fabrication.

Fig. 4 shows the spectral loss characteristics of one such 2.4 km fibre in which the minimum attenuation at 1.62 μm has been reduced to 0.23 dB/km, measurement reproducibility during 4 measurements being within 0.05 dB/km.

Conclusions: Studies of P₂O₅ doping in SiO₂-GeO₂-P₂O₅ core fibres have shown a relationship between attenuation at 1.6 μm and P₂O₅ concentrations in addition to OH impurity levels. Both P₂O₅ and OH have to be reduced in order to achieve minimum loss in the 1.6 μm window.

Acknowledgments: The author would like to thank P. Tomlinson, L. Abraham and particularly C. R. Smith for assistance with fibre preparation and measurement, and the directors of STL Ltd. for permission to publish this letter.

I. IRVEN 12th November 1980
STL Ltd. Standard Telecommunication Laboratories Ltd.
London Road, Harlow, Essex, England
1980 STL Ltd. All rights reserved.

References

LONG WAVELENGTH PERFORMANCE OF OPTICAL FIBRES CO-DOPED WITH FLUORINE

Indexing terms: Optical fibres, Attenuation

Glasses in the SiO₂-GeO₂-F-P₂O₅ system have been investigated as materials for core and cladding of multimode and single-mode fibres, and the possibility of using F-doping whilst achieving low attenuation in the 1.3 and 1.55 μm low loss windows of interest for both multimode and single-mode fibres, except for reports of its use in cladings for single-mode fibres.

Introduction: Fluorine doping has been used for reducing the refractive index of silica, and thus obtaining lower refractive-index cladding material in optical fibres. As yet, little data has been available on its effect on attenuation levels in the 1.3 and 1.55 μm low loss windows of interest for both multimode and single-mode fibres, except for reports of its use in cladings for single-mode fibres.

The fundamental phonon absorption frequency of Si-F is at 725 cm⁻¹ (138 μm), and as this is at longer wavelength than Si-O (≈ 9 μm) Si-F should not contribute significantly to absorption loss above that due to silica itself. The other index reducing dopant, B₂O₅, has been found to significantly increase loss above ~1.2 μm due to the tail of the fundamental B-O absorption at ~1.2 μm, making it impractical for use at 1.3 or 1.55 μm. Fluorine is, therefore, a dopant of interest for such longer wavelengths.

A series of fibres was fabricated via vapour deposition inside a substrate tube, collapse into a rod preform and subsequent drawing into fibre. Source compounds for the vapour deposition were SiCl₄, GeCl₄, POCl₃, CCl₄, F₂, and oxygen, and both 50 and 125 μm graded index multimode and 125 μm OD single-mode fibres were made with cores and cladings based on the SiO₂-GeO₂-F-P₂O₅ glass system. From a parallel study, where P₂O₅ was incorporated either in the core or cladding, the concentration was kept below 0.2 mole % to minimise the effect of P-OH absorption at 1.5-1.6 μm. The optical cladding for all fibres was composed of SiO₂-GeO₂-F-P₂O₅ with low doping levels and a refractive index just less than that of silica, and was deposited at a temperature of ~1700°C.

Multimode fibre was fabricated with a numerical aperture value nominally of 0.2. When F was incorporated in the core this reduced the value to 0.15, and in other fibres additional GeO₂ doping was incorporated to increase the value to 0.2 with SiO₂-GeO₂-F and SiO₂-GeO₂-P₂O₅ cores to allow comparison of attenuation performance.

Fibre was pulled from preforms using a graphite resistance furnace and primary coated in-line with silicone resin to a thickness of ~45 μm prior to winding onto 25 cm diameter aluminium reels. Attenuation was measured by the cutback method using 40 μm spot and 0.1 NA launch for