Minimax Robust Discrete-Time Matched Filters

SERGIO VERDÚ, STUDENT MEMBER, IEEE, AND H. VINCENT POOR, SENIOR MEMBER, IEEE

Abstract—The problem of designing finite-length discrete-time matched filters is considered for situations in which exact knowledge of the input signal and/or noise characteristics is not available. Such situations arise in many applications due to channel distortion, interferences, nonlinear effects, and other modeling uncertainties. In such cases it is often of interest to design a minimax robust matched filter, i.e., a nonadaptive filter with an optimum level of worst-case performance for the expected uncertainty class. This problem is investigated here for three types of uncertainty models for the input signal, namely, the mean-absolute, mean-square, and maximum-absolute distortion classes, and for a wide generality of norm-deviation models for the noise covariance matrix. Some numerical examples illustrate the robustness properties of the proposed designs.

I. INTRODUCTION

THE linear system that has maximum output signal-to-noise ratio at some instant of time when the input is a deterministic signal embedded in additive random noise is known as the matched filter for this pair of signal and noise. If the noise is a Gaussian process, then the output of this filter in the instant in which the signal-to-noise ratio is maximized provides a sufficient statistic for any likelihood ratio test for detection of the presence of the input signal. Since the second-order statistics of the input noise determine the power of the noise at the output of the linear filter, a complete specification of the signal and second-order statistics of the noise is necessary in order to derive the corresponding matched filter. However, channel nonlinearities, timing jitter, and other nonideal effects tend to distort the signal in an unpredictable, or difficult to model, fashion. Also, due to nonstationarities, changing operating environments, or other modeling uncertainties, it is often the case that the second-order characterization of the noise is not exactly known to the designer. A simple way of modeling the available partial knowledge about the signal noise is through an uncertainty set, any member of which can be the actual input signal and noise pair. Within this framework, the matched filter must be designed for a set of possible input signal and noise pairs rather than for a fixed pair. Since, in general, no filter will be optimum for every member of the uncertainty set, it is of interest to design a robust filter, i.e., one whose performance is close to the optimum independently of which member of the uncertainty set is actually present. A well-established approach to the robust filtering problem is the minimax strategy, in which the design goal is to optimize the worst-case performance over the uncertainty class.

Several studies of minimax matched filter design to combat jamming (a special case of noise uncertainty) have been reported in the literature over the past two decades (see, for example, [1], [8], [14]). More recently, Kuznetsov [5] and Kassam et al. [3] have examined the minimax robust matched filtering problem for particular types of signal and noise uncertainty in continuous time and in two-dimensional discrete time, respectively. The second author [6] has treated this problem in a general Hilbert space setting and has given results, recently generalized by the first author [10], that are applicable to a wide variety of uncertainty classes.

In this paper we focus our attention on the design of minimax robust finite-length discrete-time matched filters (see [7] and [9] for discussions of the computation of such filters for known signal and noise covariance). We suppose that the input to the discrete-time linear filter with impulse response \( \{h_i, i = 0, \ldots, k - 1 \} \) is given by

\[
    x_i = s_i + n_i, \quad i = 0, 1, \ldots, k - 1
\]

where \( \{n_i, i = 0, \ldots, k - 1 \} \) is a zero-mean stochastic process with covariance \( E[n_i n_j] = (\Sigma)_{ij} \) and \( \{s_i, i = 0, \ldots, k - 1 \} \) is a deterministic signal. The ratio between the output power due to the signal and the output power due to the noise at the \((k - 1)\)-sampling instant is given by

\[
    \text{SNR} = \frac{\left( \sum_{i=0}^{k-1} s_i h_{k-1-i} \right)^2}{\sum_{i=0}^{k-1} \sum_{f=0}^{k-1-i} h_{k-1-i} h_{k-1-i-f} (\Sigma)_{ij}}
\]

and the above-mentioned objective is, thus, to select the filter \( \hat{h} \) such that the minimum value of (2) over the class of possible input signal vectors and noise covariance matrices is maximized. This problem fits within the general framework developed in [6] and [10], and thus, these earlier results can be applied to characterize solutions to this robust matched filtering problem.

In Section II a discussion of those results from [6] and [10] that are relevant to the robust discrete-time matched filtering problem are given. Those results characterize saddle-point solutions to the minimax game that defines the robust matched filter. Specific solutions to this game for the cases of signal and noise uncertainty for various types of useful uncertainty classes are first treated separately, in Sections III and IV, and then jointly in Section V. The signal uncertainty classes we consider are models for distortion of a nominal (or transmitted) signal. The distortion measures considered are...
mean-square, mean-absolute, and maximum-absolute distortion. To model noise uncertainty, we consider, primarily, classes which consist of those noise covariance matrices which differ from a nominal model (in matrix norm) by no more than some fixed amount. Finally, the performance of the proposed minimax filters is illustrated by means of an example in section VI.

II. PROBLEM FORMULATION

A general Hilbert space formulation of the matched filter design problem that allows the description of the input signal structure and noise correlation in various ways has been given in [6]. Before looking at the specific case of interest here, we describe this general formulation. Let \( s \in H \) be a signal quantity (in the time or frequency domain), let \( \Sigma \in P \) be a second-order characterization of the noise (e.g., covariance matrix, autocorrelation function, or spectrum), and let \( h \in \mathcal{H} \) be a linear filter quantity (impulse response or transfer function), where \( \mathcal{H} \) is a Hilbert space with inner product \((\cdot, \cdot)\) and \( P \) is a set of bounded, linear, self-adjoint, positive operators mapping \( \mathcal{H} \) into itself. The real-valued functional defined by

\[
\rho(h; s, \Sigma) = |\langle h, s \rangle|^2 / (h, \Sigma h)
\]

represents, for properly defined quantities, the power of the output of the filter due to the signal divided by the power of the output due to the noise (signal-to-noise ratio) at some time instant. By a well-known application of the Schwarz inequality, an optimal (matched) filter \( h^*(s, \Sigma) \) for \( (s, \Sigma) \) is any solution to the equation

\[
\Sigma h^*(s, \Sigma) = \kappa s
\]

for any nonzero real constant \( \kappa \) (henceforth assumed to be unity for convenience). Also, the optimal signal-to-noise ratio achievable with \( (s, \Sigma) \) is given by

\[
\max_{h \in \mathcal{H}} \rho(h; s, \Sigma) = \langle s, h^*(s, \Sigma) \rangle.
\]

If, as discussed above, the input signal and noise quantities are known only to belong to some uncertainty classes \( s \in H, \Sigma \in P \), independently of one another, then one possible optimal (maxima robust) filter design strategy is to choose the one that exhibits the best performance for the worst-case signal and noise pair, i.e., to choose \( h_R \) such that

\[
\mathcal{H}_R = \arg \max_{h \in \mathcal{H}} \inf_{(s, \Sigma) \in S \times N} \rho(h; s, \Sigma).
\]

The dual concept to (6) is that of a least favorable signal and noise pair, determined by the relationship

\[
\mathcal{H}_L = \arg \min_{(s, \Sigma) \in S \times N} \rho(h^*(s, \Sigma); s, \Sigma).
\]

\( (s_L, \Sigma_L) \) is the pair in the uncertainty class with minimum optimal signal-to-noise ratio. It is useful to consider this design problem as a game \((H, S \times N, \rho)\) in which the designer tries to maximize the function \( \rho \) by selecting a filter from \( \mathcal{H}_R \), and his opponent (nature) tries to minimize by choosing a signal and noise pair from \( S \times N \). Note that if a saddle point exists for this game, i.e., if there is a point \((h_L, (s_L, \Sigma_L))\) that satisfies

\[
\rho(h; s_L, \Sigma_L) \leq \rho(h_L; s_L, \Sigma_L) \leq \rho(h_L; s, \Sigma)
\]

for all \( h \in H, s \in S, \Sigma \in N \), then \( h_L \) has its worst performance at \((s_L, \Sigma_L)\), and any other filter has worse behavior at \((s_L, \Sigma_L)\), which implies that \( h_L \) is minimax robust. Furthermore, by definition of \( h^*(s, \Sigma) \), we have

\[
\rho(h_L; s, \Sigma) \leq \rho(h^*(s, \Sigma), s, \Sigma)
\]

which, together with (8), implies that \((s_L, \Sigma_L)\) is a least favorable pair. The existence and characterization of a saddle point for the robust matched filtering problem (6) can be derived straightforwardly from (3) and (8), and are summarized here in the following lemma (a proof is found in [10, Theorem 3.2]).

Lemma 1: \((h_L, (s_L, \Sigma_L))\) is a saddle point for \((H, S \times N, \rho)\) if and only if

1. \( \Sigma_L h_L = s_L \),

2. \( |\langle s_L, h_L \rangle| \leq |\langle s, h_L \rangle| \), \( \forall s \in S \),

and

3. \( 0 \leq \langle h_L, (\Sigma_L - \Sigma) h_L \rangle \), \( \forall \Sigma \in N \).

By means of this result, the robust filtering problem is reduced, whenever saddle points exist, to the recursive minimization of (10)–(12), which results in the least favorable pair for the uncertainty class, and its optimal filter (the sought-after minimax robust filter). Repeated use of these equations will be made in the following sections when dealing with specific uncertainty models for which analytical results are derived. Note, as well, that further results ([6, Lemma 2], [10, Theorem 3.1]) indicate that, under certain conditions (satisfied in the discrete-time case and for the uncertainty classes that we study below), if a least favorable pair exists, then it forms a saddle point with its matched filter.

We now apply this general framework of [6] to the particular case of finite-length discrete-time matched filtering. Resorting to the time-domain formulation, let \( H = \mathbb{R}^k; P = \{\Sigma \in \mathbb{R}^{k \times k}, \Sigma > 0\} \), \( s = [s_0, \ldots, s_{k-1}]^T \), and \( h = [h_0, \ldots, h_{k-1}]^T \), where \( h_i = h_{k-i-1} \), and \( s_i, h_i \) are the values of the signal and of the filter impulse response, respectively, at the \( i \)th sample. The inner product is defined as the usual scalar product: \( (a, b) = a^T b \), and \( \Sigma \) represents the covariance matrix of the additive zero-mean input noise. It is easy to verify that with these definitions (3) gives the power of the filter output in the absence of input noise divided by the variance of the filter output at the \((k-1)\)th sample, i.e., the signal-to-noise ratio of (2).

III. SIGNAL UNCERTAINTY

In this section we assume that the noise covariance is known, i.e., \( N = \{\Sigma_0\} \), but that the signal is known only to belong to some box uncertainty set \( S \). The problem is then to find the filter that minimizes the mean-square error for the worst-case signal-to-noise ratio. The design problem can be stated formally as:

\[
\min_{h \in \mathcal{H}} \max_{s \in S \times N} \rho(h; s, \Sigma_0)
\]

for all \( h \in H \), \( s \in S \), and \( \Sigma \in N \), where \( h_L \) has its worst performance at \((s_L, \Sigma_L)\), and any other filter has worse behavior at \((s_L, \Sigma_L)\), which implies that \( h_L \) is minimax robust. Furthermore, by definition of \( h^*(s, \Sigma) \), we have

\[
\rho(h_L; s, \Sigma) \leq \rho(h^*(s, \Sigma), s, \Sigma)
\]

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2. \( |\langle s_L, h_L \rangle| \leq |\langle s, h_L \rangle| \), \( \forall s \in S \),

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3. \( 0 \leq \langle h_L, (\Sigma_L - \Sigma) h_L \rangle \), \( \forall \Sigma \in N \).

By means of this result, the robust filtering problem is reduced, whenever saddle points exist, to the recursive minimization of (10)–(12), which results in the least favorable pair for the uncertainty class, and its optimal filter (the sought-after minimax robust filter). Repeated use of these equations will be made in the following sections when dealing with specific uncertainty models for which analytical results are derived. Note, as well, that further results ([6, Lemma 2], [10, Theorem 3.1]) indicate that, under certain conditions (satisfied in the discrete-time case and for the uncertainty classes that we study below), if a least favorable pair exists, then it forms a saddle point with its matched filter.

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long to a deviation class around a given nominal signal \( s_o \). This type of situation arises naturally in many applications since the noise is often receiver generated and, thus, its statistics are more easily measured, whereas the received signal is distorted in the channel by a number of factors (e.g., fading, timing jitter, nonlinearities, etc.) which cannot be easily modeled or measured. Thus, as discussed in [6], the nominal signal \( s_o \) can be thought of as a transmitted signal and the actual received signal \( s \) can be thought of as a version of \( s_o \) that has been distorted by nonideal effects in the channel. We focus our attention on the three most widely used measures of signal distortion; namely, mean-absolute, mean-square, and maximum-absolute distortion. These distortion measures generate the following uncertainty classes for the signal.

1) Mean-absolute distortion (\( L_1 \) norm)

\[
S_1 = \left\{ s \in \mathbb{R}^k, \sum_{i=0}^{k-1} |s_i - s_{oi}| \leq \Delta \right\}
\]

(13)

2) Mean-square distortion (\( L_2 \) norm)

\[
S_2 = \left\{ s \in \mathbb{R}^k, \sum_{i=0}^{k-1} |s_i - s_{oi}|^2 \leq \Delta^2 \right\}
\]

(14)

3) Maximum-absolute distortion (\( L_\infty \) norm)

\[
S_3 = \left\{ s \in \mathbb{R}^k, \max_{i=0,\cdots,k-1} |s_i - s_{oi}| \leq \Delta \right\}
\]

(15)

In the following paragraphs we derive necessary and sufficient conditions in order for a filter \( h_L \) and a pair of signal and noise covariances \((s_{L}, \Sigma_o)\) to form a saddle point of the game \((H, S \times N, \rho)\) where \( S \) is one of the distortion classes of (13)–(15). These, in turn, give sufficient conditions for the filter \( h_L \) to be minimax robust as discussed in Section II. In the case of interest here, in which only the input signal is uncertain, it is useful to translate (11) into a recursive equation giving \( s_{L} \) in terms of \( h_{L} = \Sigma_{L}^{-1} s_{L} \). If, as in our case \((H = \mathbb{R}^k)\), the previously defined inner product \((\cdot, \cdot)\) is real, it follows from the positivity of \( \Sigma_L \) that the condition

\[
\langle s_{L}, h_{L} \rangle = \langle s, h \rangle \quad \forall s \in S
\]

(16)

is sufficient in order for (11) to hold. Furthermore, if \( S \) is convex (as is the case with \( S_1, S_2, \) and \( S_3)\) it can be shown [10] that (11) and (16) are equivalent. This fact will be useful in the following development.

We now consider the solutions to the minimax robust filtering problems corresponding to the uncertainty classes \( S_1, S_2, \) and \( S_3. \)

A. Mean Absolute Distortion

The minimax filter for a mean-absolute distortion uncertainty class \( S_1 \) is characterized by the following proposition.

Proposition 1: \((h_{L}, (s_{L}, \Sigma_{L}))\) is a saddle point for \((H, S_1 \times \{\Sigma_o\}, \rho)\) if and only if \( h_{L} = \Sigma_{L}^{-1} s_{L} \) and

\[
s_{L} = s_{oi} - \delta_i \, sgn(h_{L})
\]

(17)

where \( \{\delta_i\}_{i=0}^{k-1} \) is such that

\[
\delta_i = 0 \quad \text{if}\ |h_{L_i}| < M, \quad \max_{i=0,\cdots,k-1} |h_{L_i}|
\]

and

\[
\sum_{i=0}^{k-1} \delta_i^2 = \Delta.
\]

(18)

Proof: To prove sufficiency, suppose a pair \((h_{L}, s_{L})\) satisfying the above conditions exists. Then, for every \( s \in S_1, \) we have

\[
\langle s - s_{L}, h_{L} \rangle = \langle s - s_{o}, h_{L} \rangle = \sum_{i=0}^{k-1} (s_i - s_{oi})h_{L_i} + \Delta M
\]

\[
\geq -M \sum_{i=0}^{k-1} |s_i - s_{oi}| + \Delta M \geq 0.
\]

(19)

Therefore, condition (16) holds and \((h_{L}, s_{L})\) is a saddle point. Conversely, to prove necessity, we note that, if \( s \) does not have the form given by (17), the inequality of (19) is strict; thus, if there exists no filter and signal pair with the above conditions, then there exists no saddle point, and the proposition is proved.

For example, suppose the noise is white with covariance matrix \( \Sigma_o = \sigma^2 I \) and the nominal signal is constant \( i.e., s_{oi} = \bar{V} \) for \( i = 0, \cdots, k - 1 \). Then, assuming, without loss of generality, that \( \Delta \leq k \bar{V} \), Proposition 1 implies that the least favorable signal is given by

\[
s_{L_i} = \bar{V} - \Delta/k, \quad i = 0, \cdots, k - 1
\]

(20)

as was to be expected. More generally, if the nominal signal is arbitrary and the noise samples are uncorrelated, \( i.e., \Sigma_o = \text{diag}(\sigma_0^2, \cdots, \sigma_{k-1}^2) \), then the robust matched filter is a clipped version of the nominal matched filter, given by

\[
h_{L_i} = \frac{h_{oi}}{C \text{ sgn}(h_{oi})} \quad \text{if} \quad |h_{oi}| \leq C
\]

\[
h_{L_i} = \frac{|h_{oi}| - C^+}{\text{s}} \quad \text{if} \quad |h_{oi}| > C
\]

(21)

where the positive constant \( C \) satisfies the equation

\[
\sum_{i=0}^{k-1} \sigma_i^2 (|h_{oi}| - C^+) = \Delta
\]

(22)

with \((x)^+ = \max\{0, x\}\). This fact can be verified by considering that \( s_{L_i} = \sigma_i^{-2} h_{L_i} \) satisfies (17) with

\[
\delta_i = 0 \quad \text{if} \quad |s_{oi}| \leq \sigma_i^2 C
\]

\[
\delta_i = \frac{|s_{oi} - \sigma_i^2 C}{\sigma_i^2 C} \quad \text{if} \quad |s_{oi}| > \sigma_i^2 C.
\]

B. Mean Square Distortion

For this distortion model \( S_2 \) of (14), the minimax solution can be characterized by the following result, part of which follows from [6, Theorem 1].

...
Proposition 2: \((h_L, (s_L, \Sigma_o))\) is a saddle point for \((H, S_2 \times \{\Sigma_o\}, \rho)\) if and only if
\[
s_l = s_o - \sigma^2 h_L
\]
with
\[
h_L = \Sigma_o^{-1} s_o
\]
and
\[
\sigma^2 \|h_L\| = \Delta.
\]

Proof: For every \(s \in S_2\), we have
\[
\langle s - s_L, h_L \rangle = \langle s - s_o, h_L \rangle + \sigma^2 \langle h_L, h_L \rangle
\]
\[
= \langle s - s_o, h_L \rangle + \Delta \|h_L\| \\
\geq -\|s - s_o, h_L\| + \Delta \|h_L\| \\
\geq -\|s - s_o\| \|h_L\| + \Delta \|h_L\| \geq 0
\]
where the second and third inequalities follow from the Schwarz inequality and the definition of the uncertainty class \(S_2\) (14), respectively. In order to complete the proof, it suffices to show that for \(0 < \Delta < \|s_o\|\) (outside this interval the minimax filtering problem is trivial) there exists a solution \((h_L, s_L)\) to (24)-(26). First, from (24) and (25) it is possible to obtain an alternative expression for the robust filter
\[
h_L = (\Sigma_o + \sigma^2 I)^{-1} s_o
\]
which together with (26) results in
\[
\sigma^2 \|h_L\| = \Delta.
\]
It can be shown that the function \(f(x) = x \| (\Sigma_o + xI)^{-1} s_o \|\)
is continuous for \(x \in [0, \infty)\) (because \(\Sigma_o\) is positive definite), and since \(f(0) = 0\), and \(\lim_{x \to \infty} f(x) = \|s_o\|\), (29) must have a solution. In fact, since it can also be shown that \(f(x)\) is monotone increasing, the solution to (24)-(26) is unique.

From (28) we can draw the interesting conclusion that the robust filter, in this case, is the matched filter to the nominal signal and a noise process that consists of the input noise (corresponding to \(\Sigma_o\)) with an added component of white noise whose level \(\sigma^2\) is proportional to the size of the uncertainties. Thus, in a sense, signal distortion of the type described by \(S_2\) is equivalent to adding a certain level of white noise and having no signal uncertainty. This phenomenon has also been noted earlier in spectral formulations of the robust matched filtering problem [3], [5] and in the general Hilbert space formulation of [6].

C. Maximum Absolute Distortion

The uncertainty class \(S_3\) of (15) admits an analytical characterization of its saddle point solution given by the following proposition.

Proposition 3: \((h_L, (s_L, \Sigma_o))\) is a saddle point for \((H, S_3 \times \{\Sigma_o\}, \rho)\) if and only if \(s_L \in S_3\) is such that
\[
s_{Li} = \begin{cases} 
  s_{oi} - \Delta & \text{if } h_{Li} > 0 \\
  s_{oi} + \Delta & \text{if } h_{Li} < 0
\end{cases}
\]
with \(h_L = \Sigma_o^{-1} s_L\).

Proof: The definition of the class \(S_3\), (15), and (30) imply that if there is a pair \((h_L, s_L)\) satisfying (30) then, for any \(s \in S_3\) and for \(i = 0, \ldots, k - 1\), we have
\[
h_{Li} s_{Li} \leq h_{Li} s_i
\]
which is sufficient in order for (16) to hold. Conversely, if a given filter and signal pair form, together with the nominal noise covariance, a saddle point, then they must satisfy (30), because otherwise there is a sample for which (31) is a strict inequality. Thus, the proposition is proved.

Proposition 3 gives the minimax robust filter for the maximum-absolute distortion model in a recursive form that, in general, requires further numerical computation; however, a nonrecursive expression for the least favorable signal and the robust filter can be given under the conditions of the following two results.

Corollary 1: If the maximum-absolute distortion is bounded by
\[
\Delta \leq \min \left|\frac{\partial f}{\partial x}\right|/\max \sum_{i=0}^{k-1} |(\Sigma_o^{-1})_{im}|
\]
where \(h_o = \Sigma_o^{-1} s_o\) is the nominal matched filter, then \((h_L, (s_L, \Sigma_o))\) is a saddle point given by \(h_L = \Sigma_o^{-1} s_L\) and
\[
s_{Li} = \begin{cases} 
  s_{oi} - \Delta & \text{if } h_{Li} > 0 \\
  s_{oi} + \Delta & \text{if } h_{Li} < 0
\end{cases}
\]

Proof: For \(i = 0, \ldots, k - 1\), we have
\[
|h_{Li} - s_{oi}| = |(\Sigma_o^{-1} (s_L - s_o))| \leq \Delta \sum_{i=0}^{k-1} |(\Sigma_o^{-1})_{im}|
\]
\[
\leq \min \left|\frac{\partial f}{\partial x}\right| \leq |h_{oi}|
\]
where the first two inequalities follow, respectively, from the fact that \(s_L \in S_3\) and from (32). Then, (34) is sufficient in order that either \(h_{Li} = 0\) or \(s_{Li} = s_{oi}\), which implies via Proposition 3 that \((h_L, (s_L, \Sigma_o))\) is a saddle point.

Corollary 2: If the noise samples are uncorrelated, then \((h_L, (s_L, \Sigma_o))\) is a saddle point given by \(h_L = \Sigma_o^{-1} s_L\) and
\[
s_{Li} = \begin{cases} 
  s_{oi} - \Delta & \Delta < s_{oi} \\
  0 & -\Delta \leq s_{oi} \leq \Delta \\
  s_{oi} + \Delta & s_{oi} < -\Delta
\end{cases}
\]

Proof: If the noise samples are uncorrelated, then \(\Sigma^{-1} = \text{diag}(I, \ldots, I, \Delta I)\) with \(\Delta > 0\) and \(h_L = l_{Li} s_L\), which
It can be conjectured that every possible signal and noise pair is equivalent to one with a diagonal noise covariance by appropriately choosing the basis of the space, which amounts to intercalating a prewhitener at the input of the filter. However, in this way the signal uncertainty class must be referred to the output of the prewhitener, limiting the practical significance of this approach.

IV. NOISE COVARIANCE UNCERTAINTY

In this section we discuss the robust discrete-time matched filtering problem for the case in which the received signal is known, but, due to modeling uncertainties, the noise covariance is only assured to belong to a set $N$ of positive-definite matrices, independent of the nominal signal $s_o$.

To study this problem, we first give a general result that is useful for different types of uncertainty models.

**Proposition 4**: $(\Sigma_L^{-1}s_o, (s_o, \Sigma_L))$ is a saddle point of $(H, \{s_o\} \times N, \rho)$ for every $s_o \in \mathbb{K}^k$ if and only if $\Sigma_L$ is a maximal element of $N$.

**Proof**: By definition, $\Sigma_L$ is a maximal element of $N$ if and only if

$$ \Sigma_L \succeq \Sigma \quad \forall \Sigma \in N. \quad (36) $$

Then, from the positivity of $\Sigma_L$, it follows that for every $h_L \in \mathbb{K}^k$ there exists $s_o \in \mathbb{K}^k$ such that $\Sigma_L h_L = s_o$ and vice versa. Therefore, (36) is equivalent to (12) holding for every signal $s_o \in \mathbb{K}^k$.

In order to quantify the "size" of a matrix, various types of matrix norms can be used according to the specific application. A general characterization of a valid matrix norm $\| \cdot \|: \mathbb{K}^{k \times k} \to \mathbb{K}$ is given by the following four conditions [13]:

1. If $A \neq 0$ then $\| A \| > 0$.
2. $\| \alpha A \| = |\alpha| \| A \|$, for $\alpha \in \mathbb{K}$.
3. $\| A + B \| \leq \| A \| + \| B \|.$
4. $\| AB \| \leq \| A \| \| B \|.$

If $\| I \| = 1$, then $\| \cdot \|$ is a unit matrix norm. Denoting by $\ell(A)$ the spectral radius of $A$, i.e., the maximum of the absolute values of its eigenvalues, it follows from the above conditions that for any matrix norm and matrix $A$

$$ \ell(A) \leq \| A \|. \quad (37) $$

Now, we consider several types of noise uncertainty classes described by the respective covariance matrix norms.

**Proposition 5**: Suppose that, among the constraints imposed on a noise uncertainty class $N$, we have that, for every $\Sigma \in N$,

$$ \| \Sigma \| \leq B. \quad (38) $$

Then $(s_o, (s_o, \Sigma_L))$ is a saddle point of $(H, \{s_o\} \times N, \rho)$ if and only if $s_o$ is an eigenvector of $\Sigma_L$ with eigenvalue $B$.

Proof: Since $s_o$ is an eigenvector of $\Sigma_L$, their matched filter is any scaled version of $s_o$. Also, for every $\Sigma \in N$ we have that

$$ \langle s_o, (\Sigma s_o) \rangle \leq \| s_o \|^2 \sup_{x \neq 0} \frac{\langle x, \Sigma x \rangle}{\| x \|^2} \leq \| \Sigma \| \| s_o \|^2 $$

$$ \leq B\| s_o \|^2 = \langle s_o, \Sigma_L s_o \rangle \quad (39) $$

where the second inequality follows from (37). Then (12) is satisfied and the saddle point property follows.

Note that in order for $\Sigma_L$ to have $B$ as an eigenvalue, it is necessary that the norm in condition (38) is a unit matrix norm. An important special case of this result is that in which $B1 \in N$ (white noise with maximum power). Then, it follows that for any signal $s_o, (s_o, (s_o, B1))$ is a saddle point.

Next, in analogy with the signal uncertainty classes presented in Section III, defined by a bound on the norm of the difference between the actual signal and the nominal, we study the noise covariance uncertainty class defined for a generic matrix norm by

$$ N_1 = \{ \Sigma \in \mathbb{K}^{k \times k}, \| \Sigma - \Sigma_o \| \leq \varepsilon, \Sigma \succ 0 \}. \quad (40) $$

**Proposition 6**: If $N_1$ is defined for a unit matrix norm, $(h_L, (s_o, \Sigma_L))$ is a saddle point of $(H, \{s_o\} \times N_1, \rho)$ if and only if $h_L = \Sigma_L^{-1}s_o$ and

$$ \Sigma_L = \Sigma_o + el. \quad (41) $$

**Proof**: First, note that if $N_1$ is defined for a unit matrix norm, then $\Sigma_L \in N_1$. Moreover, for every $\Sigma \in N_1$ and $x \in \mathbb{K}^k$, we have

$$ \langle x, (\Sigma - \Sigma_o) x \rangle = \langle x, (\Sigma_o + el - \Sigma) x \rangle $$

$$ \geq (\varepsilon - \| \Sigma_o - \Sigma \|) \| x \|^2 \geq 0 \quad (42) $$

where the last two inequalities follow, respectively, from (37) and the fact that $\Sigma \in N_1$. Therefore, $\Sigma_L$ is a maximal element of $N_1$, and the result follows from Proposition 4.

Perhaps the most important case of nonunit matrix norm is the Euclidean norm, defined by

$$ \| A \|^2 = \sum_{j=0}^{k-1} \sum_{j=0}^{k-1} |(A)_{ij}|^2. \quad (43) $$

For this type of norm the following result applies.

**Proposition 7**: If $N_1$ is defined for the Euclidean matrix norm, $(h_L, (s_o, \Sigma_L))$ is a saddle point of $(H, \{s_o\} \times N_1, \rho)$ if and only if

$$ \Sigma_L = \Sigma_o + \sigma_n^2 h_L h_L^T. \quad (44) $$

with

$$ \sigma_n^2 \| h_L \|^2 = \varepsilon \quad (45) $$

and

$$ h_L = (\Sigma_o + el)^{-1}s_o. \quad (46) $$
Proof: For every $\Sigma \in \mathbb{N}_1$ we have
\[ \langle h_L, (\Sigma_L - \Sigma)h_L \rangle = a_n^2 h_L^T \Sigma_L T h_L + \langle h_L, (\Sigma_o - \Sigma)h_L \rangle \]
\[ \geq ||h_L||^2 \epsilon - ||\Sigma_o - \Sigma||_2 \geq 0. \]  (47)

Furthermore, for the Euclidean norm, we have that
\[ ||h_L T h_L^T||_2 = h_L^T h_L \]  (48)

which implies that $\Sigma_L \in \mathbb{N}_1$, and that $h_L = \Sigma_L^{-1} s_o$. Similarly to the proof of Proposition 2, it remains to be shown that a pair $(h_L, \Sigma_L)$ satisfying (44)-(46) indeed exists, but this is obvious since in this case $a_n^2$ is given explicitly, via (45) and (46), in terms of $\epsilon$, $\Sigma_o$ and $s_o$.

Note that, in the above proof, (48) is the only property of the Euclidean norm that is used. Thus, this proposition is valid for any norm satisfying (48). It is also interesting to note that the minimax robust matched filter for uncertainties in the noise covariance given by the class (40) defined for any of the norms studied here is given by (46), even though $(\Sigma_o + \epsilon I)$ may not belong to the uncertainty class.

V. SIGNAL AND NOISE COVARIANCE UNCERTAINTY

The general case, in which the input signal and noise pair is only known to belong to the Cartesian product of independent uncertainty classes $\mathcal{S}$ and $\mathcal{N}$, is treated in this section. In general, if $\mathcal{S}$ and $\mathcal{N}$ are such that analytical solutions, $s_2(h_L)$ for (11) and $\Sigma_2(h_L)$ for (12), exist, then by making use of (10) a system of three equations in three unknowns (the robust filter and the least favorable pair) is obtained. Note that otherwise, (10)-(12) must be solved iteratively, and that whenever a saddle point exists (see comments on Lemma 1) the existence of the solution to such a system is assured. Expressions for $s_2(h_L)$ and $\Sigma_2(h_L)$ have been obtained in the previous two sections for specific uncertainty classes, and now we consider the solution to the combined system of equations for these cases. Further simplification of the problem is obtained when the equations give $s_2$ or $\Sigma_2$ directly, i.e., independently of $h_L$. In such a case the problem is reduced to that of only noise or only signal uncertainty, respectively. For the noise covariance uncertainty case, this property is characterized by Proposition 4; therefore, whenever there is a maximal element $\Sigma_L$ in the noise uncertainty class, the problem is simplified to that of signal uncertainty for a fixed covariance $\Sigma_L$, as studied in Section III.

A case of special interest here is the one in which the signal uncertainty is modeled by a bound on the mean-square distortion ($\mathcal{S}_p$) and the noise uncertainty is modeled by a bound on some matrix norm of the deviation from a nominal ($\mathcal{N}_1$).

Equations (24), (26), and (41) (for unit matrix norm) and (44) and (45) (for Euclidean norm) result in
\[ h_L = (\Sigma_o + (\epsilon + \sigma^2 o) I)^{-1} s_o \]  (49)

with $\sigma^2 o$ given by (26). Notice that the proof of existence of this filter is identical to that in Proposition 2. We emphasize again that this solution transfers all the uncertainties to the noise covariance in the form of an added component of white noise, a fact that leads to the following result.

Proposition 8: The nominal filter matched to $(s_o, \Sigma_o)$ is robust for deviations of the input and signal noise pair modeled by $\mathcal{S}_p$ and $\mathcal{N}_1$ if and only if $s_o$ is an eigenvector of $\Sigma_o$.

Proof: It is easy to show that $s_o$ is an eigenvector of $\Sigma_o$ if and only if it is an eigenvector of $(\Sigma_o + (\epsilon + \sigma^2 o) I)^{-1}$; therefore, $h_L$, $h_L$, and $s_o$ all differ by a constant and the result follows.

It is interesting to mention that the nominal signal is an eigenvector of the nominal noise covariance (and, therefore, the nominal matched filter is minimax robust) in two important cases, namely, when the nominal noise is white, and when the nominal signal is optimally designed (in the absence of uncertainties) to belong to the minimum eigenvalue eigenspace of the noise covariance matrix. This fact is exploited in [12], where the problem of optimum nominal signal selection for minimax matched filtering is studied.

VI. PERFORMANCE ANALYSIS: AN EXAMPLE

In assessing the utility of a minimax robust filter for a particular situation, two factors are of major interest, namely, the degree to which the performance of the nominal filter is degraded in the worst case relative to that of the minimax filter, and the degree to which the nominal filter outperforms the robust filter when the nominal model is actually present. Also, since one can rarely place an absolute maximum value of the possible degree of distortion present, it is of interest to assess the behavior of the proposed filters when the degree of distortion differs from that assumed in the design. This can provide useful information in deciding which robust filter achieves the best tradeoff of worst-case performance versus behavior close to the nominal model.

With these points in mind, a useful performance measure is the (minimax) robustness index of a filter $y \in \mathcal{H}$, in an uncertainty class $\mathcal{C}$, defined by
\[ \eta(y, C) = \min_{p \in \mathcal{C}} \rho(y, p) \max_{h \in \mathcal{H}} \min_{p \in \mathcal{C}} \rho(h, p). \]  (50)

Note that for the minimax robust filter for $C$, $h(C)$, we have $\eta(h(C), C) = 1$, and that with appropriate choices of $y$ and $C$ the robustness index gives the desired relative performance of the nominal and minimax filter in the nominal model and the corresponding uncertainty class noted above.

We study, now, a particular case from among those treated in the previous sections, namely, that of signal uncertainty modeled by a mean-square distortion class ($\mathcal{S}_p$) (parametrized with maximum RMS distortion $s = \Delta ||s_o||$) and known noise covariance matrix $\Sigma_o$. For the sake of clarity of presentation we choose to treat a two-dimensional example with
\[ s_o = \begin{bmatrix} 1 & \rho \\ 0 & 1 \end{bmatrix}, \quad \Sigma_o = \begin{bmatrix} 10 & \gamma \\ \gamma & 1 \end{bmatrix}. \]

Note that this corresponds to a case in which the nominal signal is far from being optimally designed (see [12]), which, as we shall see, enhances the utility of the robust filter relative to the nominal.
Fig. 1. Minimum signal projections on the nominal and robust filters.

Fig. 2. Robustness of nominal and minimax filters for δ = 0.1 and α = 30°.

Fig. 3. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 4. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 5. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 6. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 7. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 8. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 9. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 10. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 11. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 12. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 13. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 14. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 15. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 16. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 17. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 18. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 19. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 20. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 21. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 22. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 23. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 24. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 25. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 26. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 27. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 28. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 29. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 30. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 31. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 32. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 33. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 34. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 35. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 36. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 37. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 38. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.

Fig. 39. Robustness of nominal and minimax filters for δ = 0.1 and α = 60°.

Fig. 40. Robustness of nominal and minimax filters for δ = 0.1 and α = 71.6°.

Fig. 41. Robustness of nominal and minimax filters for δ = 0.25 and α = 71.6°.
making the signal less suited to the noise, can have a dramatic effect in the relative performances of the nominal and minimax robust matched filters.

VII. CONCLUSIONS

The solution of the minimax robust filtering problem, for several specific uncertainty classes in finite-length discrete-time processing, has been obtained through the application of previous general results in the area [6], [10]. The proposed solutions suggest a philosophy of design to make matched filters more robust against possible uncertainties in the received signal and in the statistical modeling of the noise. In this respect, we have seen that assuming an additional level of white noise desensitizes the design for mean-square signal distortion and covariance matrix deviations described by a wide variety of matrix norms. Also, in the uncorrelated noise case, we have observed the convenience of avoiding peaks in the matched filter, and of attenuating the filter samples corresponding to the lowest noise eigenvalues, in order to combat mean-absolute and maximum-absolute distortion of the signal, respectively. It has been shown that when the nominal signal is an eigenvector of the nominal noise (e.g., if the nominal noise is white), the nominal matched filter exhibits a certainty equivalence property that makes it minimax robust for any degree of mean-square signal distortion and covariance matrix deviation (a fact that has been confirmed in practice for the case of nominal white noise [4]).

With the purpose of evaluating the performance of these minimax robust matched filters, we have introduced a robustness index, which turns out to be a useful criterion in order to select a robust filter for a particular application and in order to compare performance to the nominal filter design. Note, as well, that it provides helpful information for a possible selection of tolerances (e.g., choice of \( \Delta \) in the signal uncertainty model) of the system to be designed. Finally, some numerical examples have indicated the noticeable advantages in using the proposed minimax robust matched filters. In a sequel to this paper ([12]) we discuss the selection of nominal signals for minimax robust filtering for noise uncertainty classes with maximal elements and for the \( l_1, l_2 \), and \( l_\infty \) signal uncertainty classes discussed above.

REFERENCES


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Sergio Verdú (S'79) was born in Barcelona, Catalonia, Spain, on August 15, 1958. He received the Telecommunication Engineer (first in his class) degree from the Polytechnic University of Barcelona in 1980, and the M.S. degree in electrical engineering from the University of Illinois at Urbana-Champaign in 1982, where he is currently working toward the Ph.D. degree.

In 1978-1979 he was with the Computer Systems Division of Hewlett-Packard E.S.A. In 1980 he was a Teaching Assistant in the Department of Electrical Engineering of the University of Illinois. Currently he is a Research Assistant in the Coordinated Science Laboratory.

Mr. Verdú has received the Prize for the best Scholar and the National University Prize, both from the Ministry of Education and Science of Spain. In 1980 he was awarded a Fulbright Scholarship by the U.S. International Communication Agency. He is a member of Tau Beta Pi.

H. Vincent Poor (S'72-M'77-SM'82) was born in Columbus, GA, on October 2, 1951. He received the B.E.E. and M.S. degrees in electrical engineering from Auburn University, Auburn, AL, in 1972 and 1974, respectively, and the M.A. and Ph.D. degrees in electrical engineering from Princeton University, Princeton, NJ, in 1976 and 1977, respectively.

Since August 1977 he has been with the University of Illinois at Urbana-Champaign where he is currently Associate Professor of Electrical Engineering and Research Associate Professor in the Coordinated Science Laboratory.

Mr. Poor is a member of the Delta Phi Epsilon. He is an Associate Professor of Electrical Engineering and Research Associate Professor in the Coordinated Science Laboratory.