

Optimum Multiuser Asymptotic Efficiency

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Abstract—The degradation in bit error rate due to the presence of multiple-access interference in a white Gaussian channel can be measured by the multiuser asymptotic efficiency, defined as the ratio between the SNR required to achieve the same uncoded bit error rate in the absence of interfering users and the actual SNR. In this paper, the asymptotic efficiency of the optimum multiuser demodulator (a bank of matched filters followed by a Viterbi algorithm) is investigated and compared to that of the conventional single-user matched filter receiver. The computation of the optimum asymptotic efficiency of any given user is equivalent to the minimization of the Euclidean distance between any pair of multiuser signals which differ in at least one of the symbols of that user. It is shown that the optimum multiuser efficiency of asynchronous systems is nonzero with probability 1, and therefore the optimum demodulator does not become multiple-access limited in contrast to the single-user receiver. A class of signal constellations with moderate cross-correlation requirements is shown to achieve unit optimum multiuser efficiencies and, hence, to be equivalent to orthogonal signal sets from the viewpoint of performance of the optimum multiuser detector.

I. INTRODUCTION

A COMMON strategy to share an additive white Gaussian multiple-access channel among several users is to let each user modulate a given signal waveform, without using a multiple-access protocol to coordinate transmission periods with the rest of the users. If symbol-epoch synchronism is maintained and the assigned waveforms are mutually orthogonal, then the channel decouples into single-user channels and a bank of conventional single-user matched filter receivers achieves optimum demodulation. Due to lack of symbol synchronism among the users or bandwidth limitations, orthogonality is usually not feasible and other users' signals are not transparent to each matched filter. In these circumstances, the usual strategy in practice (i.e., in code-division spread-spectrum multiple-access) is to design signals with low cross correlations for all possible mutual offsets and to employ single-user signal detectors, thus neglecting the lack of orthogonality of the signal set. Recently [1] (see also [2, sect. 5.3]), an optimum multiuser detector has been obtained and analyzed, and it has been shown that unless the background Gaussian noise is dominant, the conventional single-user detector is far from optimum.

The purpose of this paper is to study the performance dependence of the optimum multiuser detector on the energies and cross correlations of the signal constellation. Performance degradation due to the existence of other active users in the channel is measured by the efficiency or ratio of effective to actual SNR's of each user—the effective SNR being that required to achieve the same bit error rate in the absence of interfering users. The asymptotic efficiencies provide, as the Gaussian noise level goes to zero, an intuitive and manageable

characterization of uncoded performance which is equivalent to the bit error rate curve in the region of usual interest (rates lower than approximately 10^{-3}), and they determine the multiple-access capability of the optimum detector, i.e., the degree of quasi-orthogonality required to accommodate a certain number of users with given energies and bit error rates. Analogous measures for characterizing the performance of error control coded multiuser systems can be obtained by defining the effective SNR of a user of rate R as the SNR of a single-user channel whose capacity is equal to R .¹

To fix ideas, consider the antipodal *bit-synchronous* K -user problem. This is equivalent to a 2^K hypothesis test corresponding to the multiuser signals given by $\sum_{j=1}^K b_j s_j(t)$, $b_j \in \{-1, 1\}$, $t \in [0, T]$. Asymptotically as the SNR increases, the probability that the k th user's output of the maximum likelihood detector is erroneous coincides with the error probability of a binary hypothesis test between the closest pair of multiuser signals differing in the k th user's bit. Then, it is a simple exercise to show that the effective energy of the k th user is equal to the minimum of the quadratic form $x^T H x$ over all K -vectors whose components belong to $\{-1, 0, 1\}$ and satisfy $x_k \neq 0$, and where H is the cross-correlation matrix, $H_{ij} = \int s_i(t)s_j(t) dt$. If the restriction of bit-synchronism is dropped, the one-shot approach is no longer feasible and the effective energies are given by the minimum Euclidean distances between signals modulated by *sequences* of K -vectors.

The organization of the rest of the paper is as follows. In Section II, the asymptotic efficiencies achieved by the optimum and single-user receivers are obtained for antipodal systems with arbitrary signal waveforms and symbol offsets. It is shown that in completely asynchronous systems, the optimum asymptotic efficiencies are nonzero with probability 1—a property not shared by the single-user receiver which becomes multiple-access limited, i.e., its bit error rate may be nonzero in the absence of background Gaussian noise. Section II concludes with the derivation of an explicit formula of the two-user optimum asymptotic efficiency. Section III derives sufficient conditions on the energies and cross correlations of the signal constellation that guarantee unit optimum asymptotic efficiencies, i.e., no performance degradation due to the presence of other active users in the channel. Interestingly, a large class of signal designs (with mild cross-correlation requirements) is shown to be equivalent, from the viewpoint of optimum performance, to orthogonal signal sets. The computation of the optimum asymptotic efficiency is a combinatorial optimization problem for which an algorithm is presented in Section IV, along with a numerical example contrasting the behaviors of the optimum and conventional single-user receivers in a spread-spectrum application.

II. ASYMPTOTIC EFFICIENCIES FOR ASYNCHRONOUS MULTIPLE-ACCESS CHANNELS

Suppose that K users share an asynchronous Gaussian multiple-access channel, modulating antipodally a set of signal

¹ Note that this appears to be a much more fundamental measure of the multiple-access capability of error control coded CDMA systems than a single-user analysis [3] (via the Gaussian approximation of the multiple-access interference) of the maximum rate in bits/chip.

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waveforms $\{s_k(t), t \in [0, T], k = 1, \dots, K\}$. The receiver observes the signal

$$S_i(\mathbf{b}) = \sum_{i=-\infty}^{\infty} \sum_{k=1}^K b_k(i) s_k(t - iT - \tau_k) \quad (1)$$

imbedded in additive white Gaussian noise whose power spectral level is denoted by σ^2 . Without loss of generality, it is assumed that the users are numbered such that the delay parameters accounting for the symbol asynchronism satisfy $0 \leq \tau_1 \leq \dots \leq \tau_K < T$. In (1) \mathbf{b} denotes the sequence of bits transmitted by all users, i.e., $\mathbf{b} = \{b(i) = (b_1(i), \dots, b_K(i))^T \in \{-1, 1\}^K\}_i$. Since the noise is white and Gaussian, the performance of any receiver that generates a sequence of statistics by matched filtering is determined by σ and the signal energies and cross correlations:

$$H_{kj}(i) = \int_0^T s_k(t) s_j(t + iT - \tau_j + \tau_k) dt, \quad i \in \{-1, 0, 1\}. \quad (2)$$

In addition to (2), the following notation will be used in the sequel:

$$w_j = H_{jj}(0) \quad (3)$$

$$G_{ij} = \begin{cases} H_{i+jj}(-1) & \text{if } i+j \leq K \\ H_{i+j-Kj}(0) & \text{if } i+j > K \end{cases} \quad (4)$$

$$R_{ij} = G_{K-i\kappa(i+j)} \quad (5)$$

for $i = 1, \dots, K-1, j = 1, \dots, K$, and where $\kappa(n) \in \{1, \dots, K\}$ is the remainder of the modulo- K decomposition of the integer $n = \eta(n)K + \kappa(n)$. Note that G_{ij} and R_{ij} represent the cross correlations of the j th user's signal with the i th preceding and succeeding signals, respectively, in the order of arrival at the receiver.

The k th user asymptotic efficiency of a detector whose k th user bit error rate is equal to P_k is defined formally as²

$$\eta_k = \sup \left\{ 0 \leq r \leq 1; \lim_{\sigma \rightarrow 0} P_k(\sigma) / Q\left(\frac{\sqrt{r}w_k}{\sigma}\right) < +\infty \right\}, \quad (6)$$

i.e., the log bit error rate of the k th user goes to zero with the same slope as that of a single user with energy $\eta_k w_k$. Since in the absence of other users the minimum bit error rate is equal to $Q(\sqrt{w_k}/\sigma)$, η_k is equal to the limit as $\sigma \rightarrow 0$ of the ratio between the effective energy (that required by a single user to achieve the same error probability) and w_k , the actual energy of the k th user. Therefore, in the region of error probabilities of interest, it quantifies the performance loss due to the existence of other active users in the channel.

It is shown in [1] that for any $\delta > 0$, there exists $\sigma_0 > 0$ such that for all $\sigma < \sigma_0$, the minimum error probability of the k th user satisfies

$$C_k^L Q(d_{k, \min}/\sigma) \leq P_k \leq C_k^U (1 + \delta) Q(d_{k, \min}/\sigma) \quad (7)$$

where C_k^L and C_k^U are positive constants and $d_{k, \min}$ is the minimum distance of the k th user, i.e.,

$$d_{k, \min}^2 = \min_{\epsilon \in Z_k} \|S(\epsilon)\|^2 = \min_{\epsilon \in Z_k} \int S_t^2(\epsilon) dt \quad (8)$$

and Z_k is the set of finite-length error sequences $\epsilon = \{\epsilon(i) \in$

$\{-1, 0, 1\}^K\}_i$ that affect the k th user, i.e., $\epsilon_k(j) \neq 0$ for some j and only a finite number of $\epsilon(i)$ are nonzero.

Therefore, it follows that the optimum asymptotic efficiency is given by

$$\eta_k = d_{k, \min}^2 / w_k. \quad (9)$$

The optimum asymptotic efficiency is achieved not only by the minimum error probability detector, but by the maximum likelihood sequence detector, which consists of a bank of matched filters followed by a Viterbi algorithm whose metric is a function of the signal cross correlations [1].

In this paper we will compare the optimum asymptotic efficiencies with those attained by suboptimum detectors, in particular the conventional single-user receiver (a matched filter followed by a threshold comparison). The k th user error probability of this receiver is given by [1]

$$P_k = E \left[Q \left(\left\{ w_k + \sum_{j=1}^{K-1} (\alpha_j G_{jk} + \beta_j R_{jk}) \right\} / \sqrt{w_k} \sigma \right) \right] \quad (10)$$

where the expectation is over the ensemble of independent, uniformly distributed $\alpha_j \in \{-1, 1\}, \beta_j \in \{-1, 1\}, j = 1, \dots, K-1$. In the low-noise region the right-hand side of (10) is dominated by the summand with the smallest argument:

$$z_k = w_k - \sum_{j=1}^{K-1} \{|G_{jk}| + |R_{jk}|\} \quad (11)$$

which is the term corresponding to $\alpha_j = -\text{sgn}(G_{jk})$ and $\beta_j = -\text{sgn}(R_{jk})$. If $z_k \leq 0$, then the probability of error of the conventional single-user receiver has a nonzero limit as $\sigma \rightarrow 0$ and the asymptotic efficiency is equal to zero. If $z_k > 0$, then the low-noise effective energy is equal to z_k^2/w_k , and therefore the k th user asymptotic efficiency of the single user detector is given by³

$$\eta_k^c = \max^2 \{0, z_k/w_k\}. \quad (12)$$

The phenomenon of multiple-access limitation or nonzero limit of the bit error rate that plagues the single-user receiver occurs even if the cross-correlation properties of the signal set are good, provided that the number and relative power of the interfering users are large enough (e.g., [5]). This is in sharp contrast with the behavior of the multiuser maximum likelihood sequence detector, or any other detector achieving maximum multiuser asymptotic efficiencies. For $\eta_k = 0$ or equivalently $d_{k, \min} = 0$ it is necessary that two different transmitted sequences result in the same received noiseless signal, and this can only happen for signal constellations with extremely poor cross correlations. Moreover, even if for a set of delays, carrier phases, and received energies the minimum distance of the k th user is 0, $d_{k, \min} = 0$ occurs with zero probability if the delays are uniformly distributed. This fact is a corollary of the following result.

Proposition 1:⁴ Suppose that

- i) τ_k is a continuous random variable,
- ii) $\{\tau_1, \dots, \tau_K\}$ are independent random variables, and
- iii) $w_k \neq 0$.

Then, $d_{k, \min} \neq 0$ almost surely. ▽

The simplest illustration of Proposition 1 is the case where two equal-energy users employ identical signals. If they are synchronous (i.e., $\tau_1 = \tau_2$), then for any σ the minimum error

³ Notice the discrepancy between η_k^c , which is based on an exact analysis of the single-user receiver, and previously proposed measures [4] of SNR degradation based on the Gaussian approximation of the multiple-access interference.

⁴ Proofs of the results in this paper can be found in the Appendix.

² $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du$.

probability is greater than 1/4 and the asymptotic efficiencies are equal to zero. If the users do not cooperate to maintain bit-synchronism, then their delays are independent and uniformly distributed, and the asymptotic efficiencies are nonzero with probability 1.

The asymptotic efficiencies of the optimum and the single-user detectors depend exclusively on the energies and cross correlations of the signal constellation. While the asymptotic efficiency of the single-user detector admits a simple closed-form expression (12), the computation of the optimum efficiency (9) is an NP-hard combinatorial optimization problem [6]. However, even though general closed-form expressions do not exist, the two-user asymptotic efficiencies are given by the explicit formula of the following result.

Proposition 2: Let $K = 2$. Denote $\rho_{12} = |H_{12}(0)| = |G_{12}|$ and $\rho_{21} = |H_{12}(1)| = |G_{11}|$. If $(i, k) \in \{(1, 2), (2, 1)\}$, then

$$\eta_k = 1 - \max \{0, 2\rho_{12} - w_i, 2\rho_{21} - w_i, 2\rho_{12} + 2\rho_{21} - 2w_i\} / w_k. \tag{13}$$

Proposition 2 is illustrated in the example of Fig. 1, which shows the dependence of the error signal achieving $d_{1, \min}$ on the offset between both users. The asymptotic efficiency of user 1 is equal to 1, 5/8, 15/16, and 1 for offsets equal to 0, $T/4$, $T/2$, and $3T/4$, respectively, and it is linear between those values. Notice from (13) that in order for η_k to equal zero, it is necessary and sufficient that $w_1 = w_2 = \rho_{ij}$ and $\rho_{ji} = 0$; i.e., both waveforms are identical modulo a circular shift. We can also see from (13) and the Schwarz inequality that in the "stationary" case in which $w_1 = w_2$ and $\rho_{12} = \rho_{21}$, which corresponds to single-user intersymbol interference with two overlapping symbols, we obtain $\eta_k = 1$ (cf. [7, ex. 6.7.2]). In fact, Proposition 2 implies that $\max\{\rho_{12}, \rho_{21}\} \leq w_i/2$ suffices for $\eta_k = 1$, i.e., in the two-user case any pair of waveforms with mediocre cross-correlation properties will result in unit asymptotic efficiencies, provided the received energies are not overly dissimilar.

III. SUFFICIENT CONDITIONS FOR UNIT EFFICIENCY

If $\eta_k = 1$, then the effective k th-user SNR suffers no degradation due to the existence of other active users in the channel, and P_k approaches asymptotically the single user bit error rate. In asynchronous systems, performance insensitivity with respect to *a priori* unknown delays and carrier phases can be achieved by employing signal constellations that attain unit asymptotic efficiencies for the whole range of unknown quantities. Once unit efficiencies are achieved, further improvements of the signal cross correlations yield negligible gains in bit error rate. Hence, an important question is to determine what values of signal energies and cross correlations ensure unit efficiencies. Equation (12) implies that the k th-user asymptotic efficiency of the single-user receiver is equal to 1 only if the k th signal is orthogonal to the signals of the other users. Fortunately, the requirements for unit optimum efficiency are much less stringent. In Section II we saw that in the two-user case, if the magnitude of the partial cross correlations do not exceed half of the energy of one of the users, then the efficiency of the other user is equal to 1. Hence, unless the energies are very dissimilar, very simple signal constellations achieve unit efficiencies for both users. The sufficient conditions for $\eta_k = 1$ obtained in the remainder of this section imply that similar conclusions hold for any number of users.

It is instructive to consider first the synchronous problem.

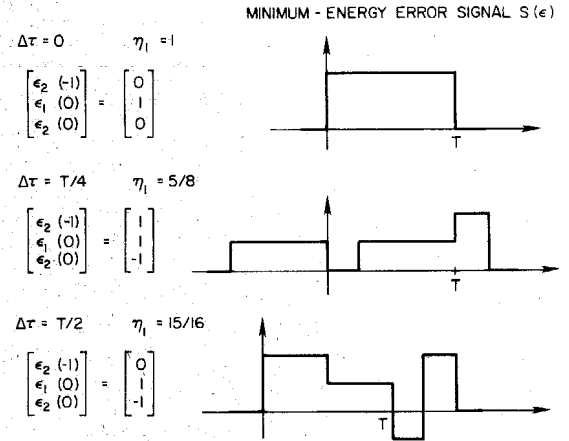
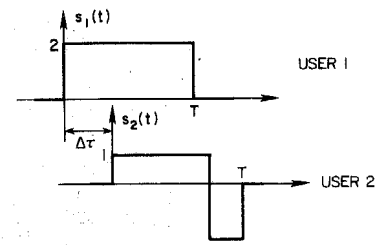


Fig. 1. Dependence of user 1 asymptotic efficiency and error signal achieving $d_{1, \min}$ on the relative delay.

In this case, $\eta_k = 1$ if and only if

$$2 \sum_{i \neq k} \epsilon_i H_{ik} + \sum_{i \neq k} \sum_{j \neq k} \epsilon_i \epsilon_j H_{ij} \geq 0 \tag{14}$$

for all $\epsilon_i \in \{-1, 0, 1\}$, $i \neq k$. If we take the worst-case approach of replacing the cross correlations with the negative of their magnitude, then it suffices to evaluate (14) for coefficients drawn from $\{0, 1\}$. Since the left-hand side of (14) is greater than or equal to $\sum_{i \neq k} |\epsilon_i| [w_{ii} - \sum_{j \neq i} |H_{ij}| - |H_{ik}|]$, a simple sufficient condition that implies that the $2^{K-1} - 1$ inequalities resulting from (14) are satisfied is that $z_i \geq |H_{ik}|$ for $i \neq k$ [cf. (11)]. The following results generalize this sufficient condition to the asynchronous problem.

Proposition 3: Fix $k \in \{1, \dots, K\}$. Suppose that

- i) $z_i \geq |H_{ik}(0)|$ for $i \neq k$
- ii) $z_i \geq |H_{ik}(1)|$ for $i < k$ and $z_i \geq |H_{ik}(-1)|$ for $i > k$
- iii) $w_k + \sum_{i=1, i \neq k}^K \min \left\{ z_i - \sum_{r=-1}^1 |H_{ik}(r)|, 0 \right\} \geq 0$.

Then $\eta_k = 1$.

Corollary: If $z_i \geq \sum_{r=-1}^1 |H_{ik}(r)|$, for $i \neq k$, then $\eta_k = 1$.

One noteworthy conclusion that can be drawn from (14) and Proposition 3 is that as long as the signals of the interfering users are linearly independent, the asymptotic efficiency of any given user is equal to unity, provided his signal is sufficiently *weak* compared to the rest (cf. Section IV).

As shown in the next result, generally less restrictive conditions are sufficient when the energies of all users coincide.

Proposition 4: Suppose that the energy of all users is the same ($w_1 = \dots = w_K = w$). Then any of the following

conditions implies that $\eta_1 = \dots = \eta_K = 1$:

$$i) \quad w - 2 \sum_{l=1}^{K-1} |G_{lk}| \geq 0, \quad k=1, \dots, K$$

$$ii) \quad w - 2 \sum_{l=1}^{K-1} |R_{lk}| \geq 0, \quad k=1, \dots, K$$

iii) $z_k \geq 0, \quad k=1, \dots, K$ and, for $i, j=1, \dots, K$

$$w - \sum_{l=1}^{K-1} \max \left\{ |G_{lk(i-l)}| - \sum_{n=1}^{l-1} |G_{nk(i-l)}|, 0 \right\} - \sum_{l=1}^{K-1} \max \left\{ |R_{lk(j-l)}| - \sum_{n=l+1}^{K-1} |R_{nk(j-l)}|, 0 \right\} \geq 0.$$

▽

The sufficient conditions of Propositions 3 and 4 are slightly stronger than $z_k \geq 0$ (recall that the asymptotic efficiency of the single-user receiver is nonzero if and only if $z_k > 0$). In fact, when particularized to the intersymbol interference problem ($R_{ik} = R_{ij} = G_{k-ij} = G_{K-ik}$, for $i = 1, \dots, K-1$; $j, k = 1, \dots, K$), each of the conditions in Proposition 4 reduces to $z_k \geq 0$, i.e., peak distortion of the matched filter output less than unity (open eye diagram)—a sufficient condition shown by Ungerboeck [8]. In the general case, however, although the second condition in Proposition 4-iii) is frequently less restrictive than $z_k \geq 0$ for all $k = 1, \dots, K$ (and, hence, it is not common to find cases where $\eta_k^c \neq 0$ and $\eta_k \neq 1$), counterexamples can be found where this condition is satisfied and the maximum asymptotic efficiency is not equal to unity (e.g., if $K = 2$; $w_1 = w_2 = 1$; $\rho_{12} = 2/3$, $\rho_{21} = 0$, then $z_1 = z_2 = 1/3$; $\eta_1 = \eta_2 = 2/3$; $\eta_1^c = \eta_2^c = 1/9$).

IV. COMPUTATION OF OPTIMUM ASYMPTOTIC EFFICIENCIES

If $K > 2$ and the signal cross correlations are not good enough to satisfy any of the above sufficient conditions, one has to resort to numerical computation of the multiuser asymptotic efficiencies. This combinatorial optimization problem is somewhat akin to the computation of minimum distance for intersymbol interference channels which has received ample attention in the literature; we can cite the Viterbi-Forney symbolic transfer function approach [9], the branch-and-bound technique of Fredricsson [10], the exhaustive search over a reduced subset of Anderson and Foschini [11], the convergent bounds of Messerschmitt [12], and the search for bounds to the minimum distance of signal space codes and continuous phase modulations of Aulin and Sundberg [13] and Seshadri and Anderson [14]. The main difficulty in applying any of these ideas towards an efficient solution of the multiuser problem is the time-varying periodic nature [1] of the transition costs and state spaces. This could be circumvented at the expense of a notoriously inefficient solution, using a vectorial approach, e.g., computing min-cost cycles in a fully connected directed graph on $\{-1, 0, 1\}^K$ where the weight of an arc from y to x is equal to $x^T[H(0)x + 2H(1)y]$; since weights may be negative and no sparsity can be exploited, the most efficient solution known to date is the Floyd-Warshall algorithm [15], resulting in an overall complexity of $O(27^K)$. Although the computation of the multiuser asymptotic efficiencies is an NP-hard problem in the number of users, a much more efficient algorithm can be shown to exist. Our approach is, first, to show that the computation of the multiuser minimum energies, $d_{k, \min}^2$, is equivalent to a min-cost cycle problem in a cyclic layered network, and second, to derive an efficient solution to this combinatorial optimization problem

by exploiting the special structure of the graph. The first step is given by the following result.

Proposition 5: Define the following weighted directed graph. The set of nodes is equal to the union of K copies⁵ of $\{-1, 0, 1\}^{K-1}/R - \{0\}$ (denoted by L_1, \dots, L_K , respectively) and the ground node $\{0\}$. There exists an arc from ground to each node or state $z \in L_n, n = 1, \dots, K$, and vice versa with respective costs,

$$c(0, z) = h_n(z) = \sum_{i=1}^{K-1} \left[|z^i| w_{\kappa(n+i)} + 2 \sum_{j=1}^{i-1} z^i z^j G_{K-i+j\kappa(n+i)} \right],$$

$$c(z, 0) = 0.$$

There exists an arc from $z \in L_n$ to $z^1 \in L_{\kappa(n+1)}, n = 1, \dots, K$ if there exists $j = 0, \dots, K-2$ such that

$$\sum_{i=j+1}^{K-1} z^i G_{i-j\kappa(n+j)} \neq 0,$$

and $z^1 = Rf(\bar{z}, a)$ where $a \in \{-1, 0, 1\}$ and \bar{z} is a representative of z . The cost of the arc from $z \in L_n$ to $z \in L_{\kappa(n+1)}$ is given by

$$c(z, z^1) = g_n(z, z^1) = |a| w_n + 2a \sum_{i=1}^{K-1} \bar{z}_i G_{in}$$

where $|a|$ and $a\bar{z}$ are defined uniquely by (z, z^1) . Then the following two statements are true.

- i) There are not negative-cost cycles in the digraph.
- ii) The minimum cost of a cycle that includes ground and at least one state $z \in L_m, m \neq k$, such that $z^{\kappa(k-m)} \neq 0$, is equal to $d_{k, \min}^2$.

▽

Proposition 5 implies that the asymptotic efficiencies of all users can be obtained by solving a single-source and a single-sink problem (i.e., find the shortest paths from ground to all nodes, and from all nodes to ground). Apparently, to date, the most efficient algorithm for the single-source problem in a network (without negative-cost cycles) with n nodes and e arcs has a running time of $O(\min(n^{2+e} + ne, n^2 \log n + ne \log n))$ [16], [17]. Applied to our case, this results in a time complexity of $O(9^K K^3 \log K)$. It turns out that we can improve upon this complexity by taking advantage of the special structure (cyclic and layered) of our graph (Fig. 2). The formal statement of the problem is the following (the single-sink problem is the directional dual).

Problem (P): Single-source shortest paths in a cyclic layered network. Given a cyclic network with K layers: L_1, \dots, L_K , such that the cost of an arc from $z \in L_n$ to $z^1 \in L_{\kappa(n+1)}$ is equal to $g_n(z, z^1)$ (costs may be negative, but there are no negative-cost cycles), and a ground node connected to all nodes $z \in L_n$ through arcs of cost $h_n(z)$, find the cost of a shortest path from ground to $z \in L_n, J_n(z)$, for all $z \in L_n, n = 1, \dots, K$.

The above single-source shortest-path problem in a layered network readily invites a dynamic programming solution if it were not for the fact that all nodes are directly connected to the source and the network is cyclic. To overcome this difficulty, first, instead of running a sequential dynamic programming recursion, we carry out K recursions in parallel, one for each layer-transition. Second, the tentative distance of each node is updated only if the value computed by the dynamic program-

⁵ The set $\{-1, 0, 1\}^{K-1}/R$ denotes the quotient with respect to the equivalence relation xRy iff $y = \pm x$.

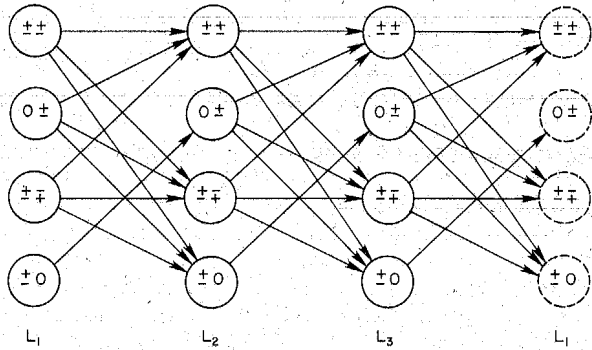


Fig. 2. Cyclic layered directed graph for the computation of multiuser asymptotic efficiencies ($K = 3$). Ground node (connected to all nodes) not shown.

ming iteration is lower than the current tentative distance. Otherwise, not only the tentative distance is unchanged but the node is ignored (or deactivated) during the next iteration. Although nodes can become active after having been deactivated, eventually all of them become deactivated (and the algorithm terminates) because there are no negative-cost cycles. The formalization of this idea and its correctness appear in Fig. 3 and Proposition 6, respectively.⁶

Proposition 6: The algorithm of Fig. 3 correctly solves Problem (P) in $O(IKD|L|)$ time, where I and $|L|$ are upper bounds to the indegree⁷ of each node and the cardinality of each layer, respectively, $D = \max_{n \in L\{1, \dots, K\}} \max_{z \in L_n} l_n(z)$, and $l_n(z)$ is the minimum number of arcs of a shortest path from ground to $z \in L_n$.

For the computation of the multiuser asymptotic efficiencies, the indegree of each node in the layered network (not counting arcs from ground) is equal to either 2 or 3, and the number of nodes in each layer is equal to $(3^{K-1} - 1)/2$. Hence, the overall asymptotic time complexity of finding the multiuser asymptotic efficiencies is equal to $O(DK3^K)$. The maximum length of a shortest path, D , depends on the cross correlations of the signal constellation and is usually a small positive integer. For example, suppose that

$$w_n - 2 \sum_{i=1}^{K-1} |R_{in}| \geq 0.$$

Then it can be shown that $h_n(x) \leq h_{\kappa(n-1)}(z) + g_{\kappa(n-1)}(z, x)$ for all $(z, x) \in L_{\kappa(n-1)} \times L_n$, and hence $D = 1$.

We turn to the study of a numerical example of a direct-sequence spread-spectrum signal constellation, namely the three maximum-length signature sequences of length 31 obtained in [19]. The error probability achieved with this signal set by the conventional single-user receiver and the optimum multiuser receiver has been obtained in [20], [21], and [1], respectively. The main goal of the present example is to investigate the effects of unequal received signal energies (the near-far problem) on the single-user and optimum efficiencies. These efficiencies are shown for user 2 in Fig. 4, as a function of the relative energy of the interfering user $n = 1, 3$. Both the best and worst cases with respect to the mutual bit-epoch delays are shown. As should be expected, the conventional detector approaches the optimum efficiency as the interfering user becomes weaker, and decreases monotonically with the energy ratio until it reaches zero (multiple-access limitation) for relative energies in the intervals $(+6.3, +30$ dB (users 2 and 1) and $(+9, +30$ dB) (users 2 and 3). For equal-energy users, the efficiency of the single-user

⁶ Notice that this algorithm is unrelated to Luenberger's cyclic dynamic programming [18].

⁷ The indegree of $z \in L_n$ in the layered network is equal to the number of arcs of the form $(z', z) \in L_{\kappa(n-1)} \times L_n$.

input: $h_n(z), g_n(z, z'), p_n(z); z \in L_n, z' \in L_{\kappa(n-1)}, n=1, \dots, K$
 (comment: $p_n(z) = \{z' \in L_{\kappa(n-1)}, g_{\kappa(n-1)}(z', z) < \infty\}$)
 output: $J_n(z), z \in L_n, n=1, \dots, K$
 (comment: minimum distance from ground to z)

```

begin
  for all  $n \in \{1, \dots, K\}$  do
    begin
      for all  $z \in L_n$  do
        begin
           $J_n(z) = h_n(z);$ 
           $a_n(z) = Y$ 
        end
      end
    end
  up = Y;
  while up = Y do
    begin
      up = N;
      for all  $n \in \{1, \dots, K\}$  do DP ( $Q_n, J_{\kappa(n-1)}, a_{\kappa(n-1)}, n$ );
      for all  $n \in \{1, \dots, K\}$  do
        for all  $z \in L_n$  do
          if  $Q_n(z) < J_n(z)$  then do
            begin
               $J_n(z) = Q_n(z);$ 
               $a_n(z) = Y;$ 
              up = Y;
            end
          else  $a_n(z) = N$ 
        end
      end
    end
  end
end

```

procedure DP (Q, J, a, n)

```

begin
  for all  $z' \in L_n$  do
     $Q(z') = \min\{J(z) + g_{\kappa(n-1)}(z, z'); z \in p_n(z') \text{ and } a(z) = Y\}$ 
  end
end

```

Fig. 3. Single-source shortest paths algorithm for Problem (P).

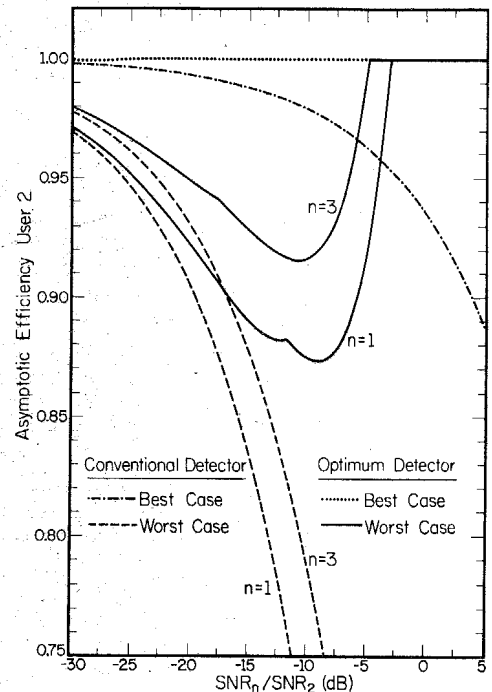


Fig. 4. Asymptotic efficiency as a function of the relative energy of the interfering user.

receiver is extremely dependent on the mutual delays ranging from 0.93 to 0.24. The behavior of the optimum asymptotic efficiency is radically different; the most striking discrepancy is that the optimum efficiency is not monotonic in the energy of the interfering users, and in fact it is identically equal to 1 for sufficiently powerful interference (in particular for equal-energy users). Note that while the efficiency computed for most-favorable delays is indistinguishable from unity for all cases, the least-favorable efficiency achieves its minimum

(around 0.9) when the interferer is 10 dB weaker. Another interesting observation is that the points of discontinuity of the derivative of the optimum asymptotic efficiency curves signal the fact that the minimum distance is achieved by different error sequences, depending on the energy of the interfering user. (Specifically, these sequences are zero everywhere except $(\epsilon_1(0), \epsilon_2(0), \epsilon_1(1)) = \{(1, 1, -1), (0, 1, -1), (0, 1, 0)\}$, respectively, from left to right in Figure 4.) Notice also that the optimum detector is near-far resistant, in the sense that it guarantees that performance degradation is no higher than 0.6 dB regardless of the energy of the interferers.

The remarkable behavior of the optimum asymptotic efficiency as the energy of the interference increases resembles that of the capacity of the Gaussian *interference channel*. Carleial [22] solved the two-user case showing that for sufficiently high interference, the capacity coincides with that of a single-user channel. Both phenomena find the same explanation. Noise, rather than the randomness of the information of the interfering users, is the primary source of the errors committed in the optimum demodulation of the user of interest, if the interfering users are sufficiently powerful. Since in this case there is little uncertainty about the information sent by the interfering users, one can subtract their noiseless transmitted signals from the received process, resulting in a single-user channel. Interestingly, we can do this in the foregoing example even if the interferer is 5 dB weaker than the user of interest.

APPENDIX

Proof of Proposition 1: Since there exists at least one error sequence attaining the minimum in (8), it is sufficient to show that $\|S(\epsilon)\| \neq 0$ a.s. for all $\epsilon \in Z_k$. To that end, fix ϵ and denote

$$V_\tau(t) = \sum_{i=-\infty}^{\infty} \epsilon_k(i) s_k(t - iT - \tau). \tag{A.1}$$

If $\|S(\epsilon; \tau_1, \dots, \tau_k)\| > 0$ for all $(\tau_1, \dots, \tau_k) \in [0, T)^k$, then we obtain the desired result; otherwise suppose that for a certain set of delays $(\tau_1^*, \dots, \tau_k^*)$ we have $\|S(\epsilon; \tau_1^*, \dots, \tau_k^*)\| = 0$. Then, using the triangle inequality, we obtain

$$\begin{aligned} & \|S(\epsilon; \tau_1^*, \dots, \tau, \dots, \tau_k^*)\|^2 \\ &= \|S(\epsilon; \tau_1^*, \dots, \tau, \dots, \tau_k^*)\|^2 \\ &+ \|S(\epsilon; \tau_1^*, \dots, \tau_k^*, \dots, \tau_k^*)\|^2 \\ &\geq \|V_\tau - V_{\tau_k^*}\|^2 = 2(\|V_{\tau_k^*}\|^2 - \langle V_\tau, V_{\tau_k^*} \rangle) \geq 0 \end{aligned} \tag{A.2}$$

for any $\tau \in [0, T]$. If $w_k \neq 0$ then the last inequality reduces to equality only if $\tau_k^* = \tau$ because $V_\tau(t)$ has finite support. Hence, if the delays of the rest of the users are $\tau_1^*, \dots, \tau_{k-1}^*, \tau_{k+1}^*, \dots, \tau_k^*$, the only possible k th user delay which results in $\|S(\epsilon)\| = 0$ is τ_k^* . Therefore, the result follows from the independence assumptions and the fact that τ_k is a continuous random variable. ∇

Proof of Proposition 2: For the sake of notational simplicity, the proof is given for $(i, k) = (2, 1)$. The main step is to prove that $d_{1, \min}$ is attained by an error sequence that satisfies

$$\epsilon^T(j) \in \begin{cases} \{[0 \ 1], [0 \ 0], [0 \ -1]\} & j = -1 \\ \{[1 \ 1], [1 \ 0], [1 \ -1]\} & j = 0 \\ \{[0 \ 0]\} & \text{otherwise.} \end{cases} \tag{A.3}$$

To that end we show that the energy of any sequence that does not fulfill (A.3) is greater than or equal to the energy of one of its subsequences belonging to Z_1 . Note first that since $\|S(\epsilon)\| = \|S(-\epsilon)\|$ and at least one of the components of the first user has to be nonzero, attention can be restricted to sequences

that satisfy $\epsilon_1(0) = 1$. For any error sequence ϵ having nonzero vectors beyond the origin, i.e., there exists an integer s such that $\epsilon(s) \neq 0$ and $\epsilon(j) = 0$ for $0 < s < j$, we find a subsequence $\epsilon^1 \in Z_1$ such that $\|S(\epsilon^1)\| \leq \|S(\epsilon)\|$ in the following way.

i) If $\epsilon_1(s)\epsilon_2(s-1) = 0$, then let $\epsilon^1(j) = \epsilon(j)$ for $j \neq s$ and $\epsilon^1(s) = 0$. It is straightforward to check that $\|S(\epsilon)\|^2 = \|S(\epsilon^1)\|^2 + \|S(\epsilon - \epsilon^1)\|^2$.

ii) If $\epsilon_1(s) \neq 0$ and $\epsilon_2(s) \neq 0$, then select ϵ^1 as in i). We have

$$\begin{aligned} & \|S(\epsilon)\|^2 - \|S(\epsilon^1)\|^2 \\ &= w_1 + w_2 + 2H_{12}(1)\epsilon_2(s-1)\epsilon_1(s) + 2H_{12}(0)\epsilon_2(s)\epsilon_1(s) \\ &\geq w_1 + w_2 - 2\rho_{21} - 2\rho_{12} \geq 0 \end{aligned} \tag{A.4}$$

where the last inequality is a generalization of $\int_0^T [s_1(t) \pm s_2(t)]^2 dt \geq 0$ to the asynchronous case, which follows by noticing that

$$\int_0^T (s_1(t) - \hat{s}_2(t))^2 dt = w_1 + w_2 - 2\rho_{12} - 2\rho_{21} \tag{A.5}$$

where

$$\hat{s}_2(t) = \begin{cases} \text{sgn}(H_{12}(1))s_2(t + T - \tau_2 + \tau_1) & 0 \leq t \leq \tau_2 - \tau_1 \\ \text{sgn}(H_{12}(0))s_2(t - \tau_2 + \tau_1) & \tau_2 - \tau_1 < t \leq T. \end{cases} \tag{A.6}$$

iii) If $\epsilon_1(s) \neq 0$, $\epsilon_2(s-1) \neq 0$, and $\epsilon_2(s) = 0$, then let $\epsilon^1(s) = \epsilon^1_2(s-1) = 0$ and $\epsilon^1_n(j) = \epsilon_n(j)$ for all other j and n . In this case,

$$\begin{aligned} & \|S(\epsilon)\|^2 - \|S(\epsilon^1)\|^2 \\ &= w_1 + w_2 + 2H_{12}(1)\epsilon_1(s)\epsilon_2(s-1) \\ &+ 2H_{12}(0)\epsilon_2(s-1)\epsilon_1(s-1) \\ &\geq w_1 + w_2 - 2\rho_{21} - 2\rho_{12} \geq 0. \end{aligned} \tag{A.7}$$

An entirely parallel argument shows that if ϵ is such that $\epsilon(j) \neq 0$ for $j < -1$ or $\epsilon_1(-1) \neq 0$, then a subsequence of ϵ whose energy is not greater than $\|S(\epsilon)\|^2$ can be found. Thus, at least one of the sequences that satisfy (A.3) achieves $d_{1, \min}$. The energy of those sequences is given by

$$\begin{aligned} \|S(\epsilon)\|^2 = & w_1 + |\epsilon_2(0)|[w_2 + 2H_{12}(0)b_2(0)] \\ & + |\epsilon_2(-1)|[w_2 + 2H_{12}(1)b_2(-1)] \end{aligned} \tag{A.8}$$

and (13) readily follows from the minimization of the right-hand side of (A.8) with respect to $\epsilon_2(-1)$ and $\epsilon_2(0)$. ∇

Proof of Proposition 3: Select $\epsilon \in Z_k$ such that $\epsilon_k(0) = 1$ and let $\epsilon_j(i) = u_{j+iK}$. The energy of an error sequence admits the following decomposition, which is the basis for the dynamic programming implementation of the optimum multiuser receiver [1]:

$$\|S(\epsilon)\|^2 = \sum_{i=-\infty}^{\infty} |u_i| w_{\kappa(i)} + 2u_i \sum_{l=1}^{K-1} u_{i+l-K} G_{l\kappa(i)}. \tag{A.9}$$

An analogous decomposition using the forward correlation matrix R is possible:

$$\|S(\epsilon)\|^2 = \sum_{i=-\infty}^{\infty} |u_i| w_{\kappa(i)} + 2u_i \sum_{l=1}^{K-1} u_{i+l} R_{l\kappa(i)}. \tag{A.10}$$

Consequently, we can write the energy of the sequence ϵ as the

average of (A.9) and (A.10):

$$\begin{aligned} \|S(\epsilon)\|^2 &= \sum_{i=-\infty}^{\infty} |u_i| w_{\kappa(i)} + u_i \sum_{l=1}^{K-1} u_{i+l-K} G_{l\kappa(i)} + u_{i+l} R_{l\kappa(i)} \\ &\geq \sum_{m=-\infty}^{\infty} |\epsilon_k(m)| w_k + A_k(m) \end{aligned} \quad (\text{A.11})$$

where

$$A_k(m) = \sum_{l=1}^{K-1} |u_{mK+k+l}| \cdot [z_{\kappa(k+l)} - |\epsilon_k(m)| |R_{lk}| - |\epsilon_k(m+1)| |G_{lk}|].$$

The right-hand side of (A.11) is greater than or equal to w_k , and hence $\eta_k = 1$, if

$$A_k(m) + |\epsilon_k(m)| w_k \geq 0, \quad m > 0 \quad (\text{A.12a})$$

$$A_k(m) + |\epsilon_k(m-1)| w_k \geq 0, \quad m < 0. \quad (\text{A.12b})$$

In order to show (A.12a), consider first the case where $|\epsilon_k(m)| = 1$. Then,

$$\begin{aligned} A_k(m) + |\epsilon_k(m)| w_k &\geq w_k + \sum_{l=1}^{K-1} |u_{mK+k+l}| [z_{\kappa(k+l)} - |R_{lk}| - |G_{lk}|] \\ &= w_k + \sum_{l=1}^{K-1} |u_{mK+k+l}| \cdot \left[z_{\kappa(k+l)} - \sum_{r=-1}^1 |H_{\kappa(k+l)k}(r)| \right] \\ &\geq 0 \quad [\text{assumption iii}]. \end{aligned} \quad (\text{A.13})$$

Otherwise, if $\epsilon_k(m) = 0$, then

$$\begin{aligned} A_k(m) + |\epsilon_k(m)| w_k &\geq \sum_{l=1}^{K-1} |u_{mK+k+l}| [z_{\kappa(k+l)} - |G_{lk}|] \\ &\geq 0 \quad [\text{assumptions i) and ii}]. \end{aligned} \quad (\text{A.14})$$

The proof of (A.12b) is identical and the proposition follows. ∇

Proof of Proposition 4: Fix any error sequence having at least two nonzero components. Denote $\epsilon_j(i) = \hat{u}_{j+iK}$, and let $q < t$ be such that $|u_q| = |u_t| = 1$ and $u_i = 0$ if $i < q$ or $i > t$. Since $|u_i| |u_{i+l-K}| \leq |u_i|$ for $i = q, \dots, t$ and $l = 1, \dots, K-1$ with strict inequality for $i = q$, (A.9) results in

$$\|S(\epsilon)\|^2 \geq w + \sum_{i=q+1}^t |u_i| \left(w - 2 \sum_{l=1}^{K-1} |G_{l\kappa(i)}| \right). \quad (\text{A.15})$$

This shows that i) is sufficient condition for $\eta_k = 1$, $k = 1, \dots, K$. Analogously, (A.10) and the above reasoning show that if the energies coincide, ii) is also a sufficient condition for unit asymptotic efficiencies. To show the sufficiency of condition iii), we can write following the first equation in (A.11):

$$\|S(\epsilon)\|^2 \geq \sum_{i=q}^t |u_i| \left[w - \sum_{l=1}^{K-1} |u_{i+l-K}| |G_{l\kappa(i)}| + |u_{i+l}| |R_{l\kappa(i)}| \right]. \quad (\text{A.16})$$

Recalling that $u_i = 0$ for $i < q$ and $i > t$, and $|u_q| = |u_t| = 1$, the right-hand side of the last equation is equal to

$$\begin{aligned} &\sum_{i=q+1}^{t-1} |u_i| \left[w - \sum_{l=1}^{K-1} |u_{i+l-K}| |G_{l\kappa(i)}| + |u_{i+l}| |R_{l\kappa(i)}| \right] \\ &\quad + 2w - \sum_{l=1}^{K-1} [|u_{q+l}| |R_{l\kappa(q)}| + |u_{t+l-K}| |G_{l\kappa(t)}|] \\ &\geq \sum_{i=q+1}^{t-1} |u_i| z_{\kappa(i)} + \sum_{i=q+1}^{q+K-2} |u_i| \sum_{l=1}^{q-i+K-1} |G_{l\kappa(i)}| \\ &\quad + \sum_{i=t-K+2}^{t-1} |u_i| \sum_{l=t-i+1}^{K-1} |R_{l\kappa(i)}| \\ &\quad + 2w - \sum_{l=1}^{K-1} [|u_{q+l}| |R_{l\kappa(q)}| + |u_{t+l-K}| |G_{l\kappa(t)}|] \\ &= \sum_{i=q+1}^{t-1} |u_i| z_{\kappa(i)} + 2w \\ &\quad - \sum_{l=1}^{K-1} \left[|u_{q+l}| \left(|R_{l\kappa(q)}| - \sum_{n=1}^{K-1-l} |G_{n\kappa(q+l)}| \right) \right. \\ &\quad \left. + |u_{t+l-K}| \left(|G_{l\kappa(t)}| - \sum_{n=K-l+1}^{K-1} |R_{n\kappa(t+l)}| \right) \right] \end{aligned} \quad (\text{A.17})$$

where the inequality follows by retrieving from $z_{q+1}, \dots, z_{q+K-2}$ and $z_{t-K+2}, \dots, z_{t-1}$ the null terms that correspond to u_i , $i < q$, $i > t$. Recalling (5), the right-hand side of (A.17) is equal to

$$\begin{aligned} &2w - \sum_{l=1}^{K-1} \left[|u_{q+K-l}| \left(|G_{l\kappa(q-l)}| - \sum_{n=1}^{l-1} |G_{n\kappa(q-l)}| \right) \right. \\ &\quad \left. + |u_{t-l}| \left(|R_{l\kappa(t-l)}| - \sum_{n=l+1}^{K-1} |R_{n\kappa(t-l)}| \right) \right] \\ &\quad + \sum_{i=q+1}^{t-1} |u_i| z_{\kappa(i)} \geq w \quad [\text{condition iii}]. \end{aligned} \quad (\text{A.18})$$

Proof of Proposition 5: It is straightforward (cf. [23]) to show that for every walk of weight w , starting from ground, there exists a sequence whose energy is equal to w , and that for any indecomposable sequence ϵ (i.e., ϵ is such that there exists no subsequence ϵ^1 such that $\|S(\epsilon)\|^2 \geq \|S(\epsilon^1)\|^2 + \|S(\epsilon - \epsilon^1)\|^2$) there exists a walk along the digraph whose weight is equal to $\|S(\epsilon)\|^2$. ∇

Proof of Proposition 6: First, we show by induction that $J_n^i(z)$, i.e., the value of $J_n(z)$ at the beginning of the i th iteration of the algorithm, is equal to the minimum cost from ground to $z \in L_n$ of any path with no more than i arcs. After the initialization of the algorithm, $i = 1$ and the claim is true because of the definition of $h_n(z)$. Assuming that the claim holds for the i th iteration, it follows that the minimum distance to $z \in L_n$ with no more than $i + 1$ arcs is equal to

$$\min \{J_n^i(z),$$

$$\min \{J_{\kappa(n-1)}^i(z^1) + g_{\kappa(n-1)}(z^1, z); z^1 \in p_n(z)\}.$$

In order to show that this expression is indeed equal to $J_n^{i+1}(z)$, it is enough to show that if $a_{\kappa(n-1)}(z^1) = N$ (see Fig. 3), then $J_{\kappa(n-1)}^i(z^1) + g_{\kappa(n-1)}(z^1, z) \geq J_n^i(z)$. To see this, notice that if $z^1 \in p_n(z) \subset L_{\kappa(n-1)}$ is not available for the i th iteration, i.e., $a_{\kappa(n-1)}(z^1) = N$, then $J_{\kappa(n-1)}^i(z^1)$ is the minimum distance of a path from ground to z^1 with $i - 1$ arcs or fewer since it was not updated during the $(i - 1)$ th iteration. Therefore, the left-hand side of (4.1) is equal to the cost of a path from ground to $z \in L_n$ with i arcs or fewer, and (4.1) follows. Hence, after the $(l_n(z) - 1)$ th iteration the value of $J_n(z)$ remains unchanged, so at the D th iteration no update takes place, and the algorithm terminates. The running time readily follows from the fact that at each iteration the algorithm carries out at most I sums and I comparisons for each node in each layer. ∇

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