CAPACITY REGION OF GAUSSIAN CDMA CHANNELS: THE SYMBOL-SYNCHRONOUS CASE

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Abstract: *The $K-$user capacity region of the white Gaussian Code-Division Multiple-Access channel is found in terms of the crosscorrelations between the assigned signature waveforms and their signal-to-noise ratios under the assumption of power-constrained inputs. Also, the conventional Gaussian multiple-access channel is studied under the assumption of amplitude-constrained inputs and its finite- and infinite-user capacity regions are found under various assumptions on the allowable amplitudes.*

1. Introduction

A major multiaccessing mode in radio networks and other common-channel communication networks is Code-Division Multiple-Access (CDMA). In this technique, each transmitter is assigned a fixed, distinct signature waveform which is used to modulate a stream of digital information as in single-user communication. In this way, the transmitters are not required to cooperate in order to schedule their transmission times, and all of them are allowed to transmit simultaneously and independently. Then, the receiver demodulates the transmitted messages upon observing the sum of the transmitted signals embedded in noise. Ideally, if the assigned waveforms were orthogonal, then (assuming the additive noise is white and Gaussian) the multiuser channel decouples into single-user channels. However, oftentimes in practice nonorthogonal waveforms are assigned because of bandwidth or complexity limitations or because of lack of timing synchronism among the users. In this situation the performance limits of the CDMA system depend on the degree of similarity (i.e., crosscorrelations) between the assigned waveforms. Typically these are pseudo-noise signature sequences operating in high-bandwidth channels. In this paper, the capacity region (i.e., the set of information rates at which reliable communication is possible) of the CDMA channel is found as a function of the crosscorrelations between the assigned waveforms and their signal-to-noise ratios. Therefore, the set of assigned waveforms is fixed and assumed to be part of the channel. But the signalling waveforms are indeed chosen by the designer, so the question arises as to whether such a capacity region really quantifies a fundamental limit. The answer is that analysis is a prerequisite to design; once the maximum throughput is known as a function of the crosscorrelations, the signature waveforms can be optimized under the specific constraints of the problem. Bandwidth is not the only or, often, the major constraint facing the designer of the CDMA system; frequently, the degree of crosscorrelation achievable under the specific limitations of complexity, bandwidth, number of chips-per-symbol, etc. is a harder, more definite constraint.

The chief assumptions under which the capacity region of the CDMA channel is obtained in this paper are a) the receiver observes the sum of transmitted signals imbedded in additive white Gaussian noise, b) the users are symbol-synchronous, i.e., their symbol-epochs coincide at the receiver, and c) the modulation is linear. No assumptions are placed on the assigned waveforms, other than their energy is finite and their duration is the same for all users and finite. Hence, the multiple-access channel under consideration is given by

$$r(t) = \sum_{i} \sum_{k=1}^{K} b_k(i)\, s_k(t - iT) + n(t). \tag{1.1}$$

where K is the number of users, $\{b_k(i)\}_i$ is the sequence of symbols transmitted by the $k-$th user, and $s_k(t)$ ($= 0$, if $t \notin [0,T]$) is the fixed waveform assigned to the $k-th$ user. It should be noted that

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the assumption of symbol-synchronism is crucial for the results of this paper and their proofs. Symbol-asynchronous channels are studied in [1].

The conventional Gaussian multiple-access channel is

$$Y = \sum_{k=1}^{K} X_k + N \tag{1.2}$$

where N is $N(0,\sigma^2)$. Under the assumption that the power of the $k-th$ input is restricted to w_k, Cover [2] and Wyner [3] showed that the capacity region of (1.2) is

$$\bigcap_{J \subset \{1, \cdots K\}} \{(R_1, \cdots R_K) : \; 0 \le \sum_{i \in J} R_i \le \frac{1}{2} \log \left[1 + \sum_{i \in J} \frac{w_i}{\sigma^2} \right] \} \tag{1.3}$$

The conventional Gaussian multiple-access channel does not encompass (1.1) unless all users are assigned the same waveform. As it is shown in Section 2, a multidimensional generalization of (1.2) (or equivalently a scalar channel with memory) is required in order to analyze (1.1). The case where all users are assigned the same waveform is also of interest in connexion with the CDMA channel, because it models the extreme situation where the signature of each user consists exclusively of its assigned codebook. In a Direct-Sequence Spread-Spectrum system, this would correspond to assigning the same chip waveform to each user and to letting each chip be modulated by a different symbol. Although, admittedly, such a system can hardly be categorized as Direct-Sequence Spread-Spectrum since there are no signature sequences and no bandwidth expansion, it is still useful to analyze its capacity and compare it to the assigned waveform channel, especially in the case of amplitude-constrained inputs (Section 3).

Except for a recent paper by Viterbi [4], previous works related to the capacity of CDMA channels (e.g., [5-7]) have all analyzed single-user demodulation approximating the multiple-access interference by a white Gaussian process. Viterbi [4] has examined the infinite-user behavior of the conventional multiple-access channel (1.2) under amplitude constrained inputs, and has shown the achievability (by the "successive cancellation" technique) of the same rate-sum that would be attained by variance constrained users. In Section 3, the capacity region of this infinite-user channel is found.

2. Capacity with Power-Constrained Inputs

In this section, the set of $K-$user rates $(R_1, \cdots, R_K)$ for which reliable communication is possible is found as a function of the crosscorrelations between the assigned signals, the energy of each user and the background white Gaussian noise spectral level. It is assumed that each assigned waveform is linearly modulated by a sequence of real-valued amplitudes that satisfy an average power constraint:

$$\frac{1}{n} \sum_{i=1}^{n} E[b_k^2(i)] \le 1 \qquad k = 1, \cdots, K \tag{2.1}$$

for each block of n transmitted symbols. There is no loss of generality in (2.1) since the energy of the assigned waveforms has not been assumed to be normalized.

Recall that the receiver observes the sum of modulated signals embedded in additive white Gaussian noise with spectral density equal to σ^2, i.e.,

$$r(t) = \sum_{i=1}^{n} \sum_{k=1}^{K} b_k(i)\, s_k(t - iT) + n(t). \tag{2.2}$$

The capacity region of this channel is given by the following result.

Proposition 1: Let $\mathbf{H}$ denote the crosscorrelation matrix of the signal constellation $\{s_1(t), \cdots, s_K(t)\}$, i.e.,

$$H_{ij} = \int_0^T s_i(t)\, s_j(t)\, dt.$$

Then, the capacity region of the Gaussian multiple-access channel (2.2) with power constrained inputs is equal to

$$\bigcap_{J \subset \{1, \cdots K\}} \{(R_1, \cdots R_K) : \; 0 \le \sum_{i \in J} R_i \le \frac{1}{2} \log \left[\det \left(\mathbf{I}_{|J|} + \sigma^{-2}\, \mathbf{H}_J \right) \right] \} \tag{2.3}$$

where $\mathbf{H}_J$ is the $|J| \times |J|$ submatrix of $\mathbf{H}$ obtained by striking out the rows and columns of $\mathbf{H}$ whose indices do not belong to the set J.

Proof: The first step is to derive a discrete-time model for the multiple-access channel (2.2). Since each input is already a discrete-time process (namely, the sequence of symbols that modulate the waveform assigned to the corresponding user), it suffices to find a sequence of observables that are sufficient statistics for the input sequence $\{\mathbf{b}(i) = [b_1(i), \cdots b_K(i)]^T,\ i = 1, \cdots n\}$. To that end, suppose that the finite-energy signals $\{s_1(t), \cdots s_K(t)\}$ span an M-dimensional subspace of $L_2[0,T]$, of which $\{\phi_1(t), \cdots \phi_M(t)\}$ is an orthogonal basis. Then, since the additive noise process in (2.2) is white and Gaussian, it is well-known that $\{\mathbf{y}(i) = [y_1(i), \cdots y_K(i)]^T,\ i = 1, \cdots n\}$ is a sequence of sufficient statistics if

$$y_k(i) = \int_{iT}^{iT+T} r(t)\, \phi_k(t - iT)\, dt. \tag{2.4}$$

Therefore, an equivalent discrete-time model is

$$\mathbf{y}(i) = \mathbf{A}\, \mathbf{b}(i) + \mathbf{n}(i) \tag{2.5}$$

where $\mathbf{A}$ is an $M \times K$ matrix such that

$$s_j(t) = \sum_{i=1}^{M} A_{ij}\, \phi_i(t), \tag{2.6}$$

and $\mathbf{n}(i)$ is a zero-mean M-dimensional Gaussian process with covariance

$$E[\mathbf{n}(i)\, \mathbf{n}(j)^T] = \sigma^2\, \delta_{ij}\, I_M. \tag{2.7}$$

Now, since the discrete-time channel in (2.5) is memoryless its capacity region is equal to the convex hull of the regions (e.g. [8]),

$$\bigcap_{J \subset \{1, \cdots K\}} \{(R_1, \cdots R_K):\ 0 \le \sum_{i \in J} R_i \le I(\{X_i\}_{i \in J}; \mathbf{Y} \mid \{X_i\}_{i \notin J})\} \tag{2.8}$$

over all product distributions of $\mathbf{X} = (X_1, \cdots, X_K)$ on R^K such that $E[X_k^2] \le 1, k = 1, \cdots, K$ and with

$$\mathbf{Y} = \mathbf{A}\,\mathbf{X} + \mathbf{N} \tag{2.9}$$

where $\mathbf{N}$ is a zero mean Gaussian random K-vector with covariance matrix $\sigma^2\, \mathbf{I}_K$.

Analogously to the case of the conventional one-dimensional Gaussian multiple-access channel with power-constrained inputs [3], here we are fortunate that there is a product input distribution whose rate region includes that of any other input distribution. This property is a consequence of the assumption that the users are symbol-synchronous and does not hold otherwise [1].

The proof of (2.3) will follow by showing that for every $J \subset \{1, \cdots K\}$ and every admissible input distribution

$$I(\{X_i\}_{i \in J}; \mathbf{Y} \mid \{X_i\}_{i \notin J}) \le \frac{1}{2} \log\, [\, \det\, (\, I_{|J|} + \sigma^{-2}\, \mathbf{H}_J\,)\,], \tag{2.10}$$

with equality if the inputs $(X_1, \cdots X_K)$ are independent standard Gaussian random variables. Letting $h(F)$ denote the differential entropy of the probability distribution function F, we can write the mutual information in (2.10) as

$$I(\{X_i\}_{i \in J}; \mathbf{Y} \mid \{X_i\}_{i \notin J}) = h(F_{\mathbf{Y}_J}) - h(F_{\mathbf{N}}), \tag{2.11}$$

where $F_{\mathbf{Y}_J}$ is the distribution of $\mathbf{N} + \sum_{i \in J} X_i$ and $F_{\mathbf{N}}$ is the distribution of $\mathbf{N}$.

Letting $\Xi_J = diag\ (\ 1\{1 \in J\}, \cdots 1\{K \in J\}\ )$, the differential entropies in (2.11) satisfy the following relationships (e.g. [9,p.44]),

$$h(F_{\mathbf{N}}) = \frac{M}{2} \log\, [2\pi e] + \log\, [\sigma] \tag{2.12}$$

$$h(F_{\mathbf{Y}_J}) \le \frac{M}{2} \log\, [2\pi e] + \frac{1}{2} \log\, [\, \det\, (\sigma^2 I_M + \mathbf{A}\, \Xi_J\, cov(\mathbf{X})\, \Xi_J\, \mathbf{A}^T)\,] \tag{2.13}$$

with equality in (2.13) if X is a Gaussian K-vector. But since the constraints on the admissible inputs only affect $cov(\mathbf{X})$ it follows that it is enough to restrict attention to Gaussian inputs. Moreover, since the identity is the maximal element of the set of admissible input covariances, it follows that (e.g., [10,p.141])

$$\det\,(\sigma^2 \mathbf{I}_M + \mathbf{A}\,\Xi_J\, cov(\mathbf{X})\,\Xi_J\,\mathbf{A}^T) \le \det\,(\sigma^2 \mathbf{I}_M + \mathbf{A}\,\Xi_J\,\mathbf{A}^T) \tag{2.14}$$

for any subset $J \subset \{1, \cdots K\}$. Consequently, as was to be expected the rate region defined by the mutual informations achieved by independent inputs with maximum allowable variance contains the rate region achieved by any other admissible distribution.

To complete the proof it remains to show that

$$\det\,(\mathbf{I}_M + \sigma^{-2}\mathbf{A}\,\Xi_J\,\mathbf{A}^T) = \det\,(\mathbf{I}_{|J|} + \sigma^{-2}\mathbf{H}_J) \tag{2.15}$$

On the one hand, it is easy to check by row/column permutations of $\Xi_J\,\mathbf{H}\,\Xi_J$ that

$$\det\,(\mathbf{I}_{|J|} + \sigma^{-2}\mathbf{H}_J) = \det\,(\mathbf{I}_K + \sigma^{-2}\Xi_J\,\mathbf{H}\,\Xi_J). \tag{2.16}$$

And on the other hand,

$$\begin{aligned}\det\,(\mathbf{I}_M + \sigma^{-2}\mathbf{A}\,\Xi_J\,\mathbf{A}^T) &= \det\,(\mathbf{I}_M + \sigma^{-2}\mathbf{A}\,\Xi_J\,\Xi_J\,\mathbf{A}^T)\\ &= \det\,(\mathbf{I}_K + \sigma^{-2}\Xi_J\,\mathbf{A}^T\mathbf{A}\,\Xi_J)\end{aligned}$$

where the last equality follows from the fact that if $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$ and $n \times m$ matrices respectively, then $\det\,(\mathbf{I}_m + \mathbf{BC}) = \det\,(\mathbf{I}_n + \mathbf{CB})$ (e.g.,[11,p.651]). Now, the result follows because $\mathbf{A}^T\mathbf{A} = \mathbf{H}$.

□

It is interesting to particularize this result to the two-user case. Letting $w_i = \int_0^T s_i^2(t)\,dt = H_{ii}$ and ρ equal the normalized crosscorrelation between both signals, i.e., $\rho\sqrt{w_1 w_2} = H_{12} = \int_0^T s_1(t)s_2(t)\,dt$, the capacity region in (2.3) becomes

$$0 \le R_1 \le \frac{1}{2}\log\,[1 + \frac{w_1}{\sigma^2}] \tag{2.17a}$$

$$0 \le R_2 \le \frac{1}{2}\log\,[1 + \frac{w_2}{\sigma^2}] \tag{2.17b}$$

$$R_1 + R_2 \le \frac{1}{2}\log\,[1 + \frac{w_1}{\sigma^2} + \frac{w_2}{\sigma^2} + \frac{w_1 w_2}{\sigma^4}\,(1 - \rho^2)] \tag{2.17c}$$

If $|\rho| = 1$, then we obtain the classical Gaussian capacity region [3], in which the sum of the rates depends only on the sum of the signal energies. This succedaneum of Gauss law [2] is no longer valid in our more general multiple-access channel.

A very intuitive and fruitful way to represent the multiple-access capacity region is to consider the ***effective*** signal-to-noise ratio of each user, $\gamma_i = \exp[2R_i] - 1$, defined as the signal-to-noise ratio required to achieve capacity R_i in single-user communication. Note that the effective signal-to-noise ratio and the rate give the same information. Now it is convenient to define the ***efficiency*** of each user as the ratio of effective to actual signal-to-noise ratios, i.e.,

$$\eta_i = \frac{\gamma_i\,\sigma^2}{w_i}. \tag{2.18}$$

The efficiency of each user is a parameter ranging from 0 to 1 that quantifies the degree of performance degradation due to the presence of other active users in the channel. Related measures of degradation due to multiuser interference have proven useful in the analysis of multiuser demodulators for systems with uncoded data [12, 13]. When the efficiency region occupies nearly all of the unit hypercube, then the main mechanism limiting performance is the background Gaussian noise, rather than the multiple-access interference. Conversely, the ***asymptotic efficiency*** region (as $\sigma \to 0$) quantifies the underlying limitation of the multiple-access channel due to the crosscorrelation between the assigned signal waveforms.

The symbol-synchronous K-user effciency region can be readily obtained from Proposition 1. Writing the rates in (2.3) in terms of the effective signal-to-noise ratios we obtain

$$\begin{aligned}\sum_{i \in J} R_i &= \frac{1}{2} \log \left[\prod_{i \in J} (1 + \gamma_i) \right] \\ &= \frac{1}{2} \log \left[\prod_{i \in J} (1 + \sigma^{-2} w_i \eta_i) \right] \\ &= \frac{1}{2} \log \left[\det (\mathbf{I}_{|J|} + \sigma^{-2} \mathbf{W}_J \, \Gamma_J) \right] \end{aligned} \tag{2.18}$$

where $\mathbf{W}_J = diag\{w_i, i \in J\}$ and $\Gamma_J = diag\{\eta_i, i \in J\}$. Uniting (2.3) and (2.18) we obtain the efficiency region:

$$\bigcap_{J \subset \{1, \cdots K\}} \{ (\eta_1, \cdots \eta_K) \in [0,1]^K ; \det (\sigma^2 \mathbf{W}_J^{-1} + \Gamma_J) \leq \det (\sigma^2 \mathbf{W}_J^{-1} + \tilde{\mathbf{H}}_J) \}, \tag{2.19}$$

where $\tilde{\mathbf{H}}_J$ is the normalized crosscorrelation matrix of the users in J, i.e.,

$$\mathbf{H}_J = \mathbf{W}_J^{1/2} \, \tilde{\mathbf{H}}_J \, \mathbf{W}_J^{1/2}.$$

It follows from (2.19) that the asymptotic efficiency region is given by

$$\bigcap_{J \subset \{1, \cdots K\}} \{(\eta_1, \cdots \eta_K) \in [0,1]^K : \prod_{i \in J} \eta_i \leq \det (\tilde{\mathbf{H}}_J) \}, \tag{2.20}$$

which, interestingly, is independent of the relative signal energies. Therefore, in order to obtain the asymptotic efficiency region it suffices to compute the determinants of the matrices of normalized crosscorrelations for each subset of users. The expression in (2.20) is especially useful because it gives the number of dB that each group of users have to add to their powers in order to maintain the same rates as in single-user communication. One of the consequences that can be drawn from the region (2.20) is that the maximum number of users with nonzero asymptotic efficiency is equal to the rank of $\mathbf{H}$.

The asymptotic efficiency region is relevant not only because it quantifies the performance degradation in high signal-to-noise ratio channels, but because it is an inner bound to the efficiency region achievable at any noise level. This is a corollary to the following result.

Proposition 2: The multiuser efficiency region (2.19) is monotonically increasing as a function of the noise level σ^2.

Proof: Suppose that the vector $(\eta_1, \cdots \eta_K)$ belongs to the efficiency region achievable at σ_1^2, i.e., for all $J \subset \{1, \cdots K\}$

$$\det (\sigma_1^2 \mathbf{I}_{|J|} + \Gamma_J \, \mathbf{W}_J) \leq \det (\sigma_1^2 \mathbf{I}_{|J|} + \mathbf{H}_J \, \mathbf{W}_J) \tag{2.21}$$

The goal is to show that (2.21) holds also for $\sigma_2 > \sigma_1$. To that end, recall (e.g., [14,p.70]) that

$$\det (x \mathbf{I}_n + \mathbf{M}) = x^n + \sum_{i=1}^{n} S_i(\mathbf{M}) \, x^{n-i} \tag{2.22}$$

where $S_i(\mathbf{M})$ is the sum of the principal minors of order i of $\mathbf{M}$. Therefore, we can write

$$\det (\sigma_2^2 \mathbf{I}_{|J|} + \Gamma_J \, \mathbf{W}_J) = (\sigma_2^2 - \sigma_1^2)^{|J|} + \sum_{i=1}^{|J|} S_i(\sigma_1^2 \mathbf{I}_{|J|} + \Gamma_J \mathbf{W}_J) \, (\sigma_2^2 - \sigma_1^2)^{|J|-i}$$

$$\det (\sigma_2^2 \mathbf{I}_{|J|} + \mathbf{H}_J \, \mathbf{W}_J) = (\sigma_2^2 - \sigma_1^2)^{|J|} + \sum_{i=1}^{|J|} S_i(\sigma_1^2 \mathbf{I}_{|J|} + \mathbf{H}_J \mathbf{W}_J) \, (\sigma_2^2 - \sigma_1^2)^{|J|-i}.$$

But every principal minor of $\sigma_1^2 \mathbf{I}_{|J|} + \Gamma_J \, \mathbf{W}_J$ [resp. $\sigma_1^2 \mathbf{I}_{|J|} + \mathbf{H}_J \, \mathbf{W}_J$] is equal to $\det (\sigma_1^2 \mathbf{I}_{|Q|} + \Gamma_Q \, \mathbf{W}_Q)$ [resp. $\det (\sigma_1^2 \mathbf{I}_{|Q|} + \mathbf{H}_Q \, \mathbf{W}_Q)$] where $Q \subset J$. Therefore (2.21) implies that

$$S_i(\sigma_1^2 \mathbf{I}_{|J|} + \Gamma_J \mathbf{W}_J) \le S_i(\sigma_1^2 \mathbf{I}_{|J|} + \mathbf{H}_J \mathbf{W}_J), \tag{2.23}$$

for all $i = 1, \cdots |J|$ and the proposition follows.

□

In the two-user case the asymptotic efficiency region converges to the intersection of the unit-square with the hyperbolic region $\eta_1 \eta_2 \le 1-\rho^2$. Therefore one can always achieve efficiency equal to $1-\rho^2$ regardless of how close to the single-user transmission rate is the other user's rate. In the conventional scalar multiple-access channel model, it is easy to see that the efficiency region converges to $\{0 \le \eta_1 \le 1, \eta_2 = 0\} \cup \{\eta_1 = 0, 0 \le \eta_2 \le 1\}$. as $\sigma \to 0$. Hence the best strategy is to let only one of the users talk when $|\rho| = 1$ and the signal-to-noise ratio is very high. Note that this is considerably more efficient than Time-Division-Multiple-Access (TDMA) whose asymptotic efficiency is equal to zero for both users — although if *both rates* are required to be nonzero, then TDMA is indeed almost as good as the best coding for low background noise (cf.[15]). Fortunately, none of these conclusions holds in the case where the assigned waveforms are different ($|\rho| < 1$). For example, suppose that $\rho = 0.1$ and two equal-rate equal-energy users have a signal-to-noise ratio of 20 dB and they transmit at the maximum possible rate given by (2.17). Had the users employed TDMA, each of them would have required approximately 40 dB to attain the same rate. But even in the case where there is heavy crosscorrelation between the signals, TDMA is not near-optimum, e.g., if $\rho = 0.9$, then TDMA would still require 33 dB to attain the same rate.

Figure 1 exhibits the effect of a variation of the noise level on the efficiency region. The crosscorrelation between the signals is $\rho = 0.9$; the ratio between the energies of user 1 and user 2 is 5 dB and the signal-to-noise ratio of user 1 varies from -10 dB to 20 dB. The convergence to the low-noise asymptotic efficiency region can be seen for signal-to-noise ratios higher than approximately 15 dB. On the other hand, when the noise is dominant, the efficiency region nearly fills the entire unit square.

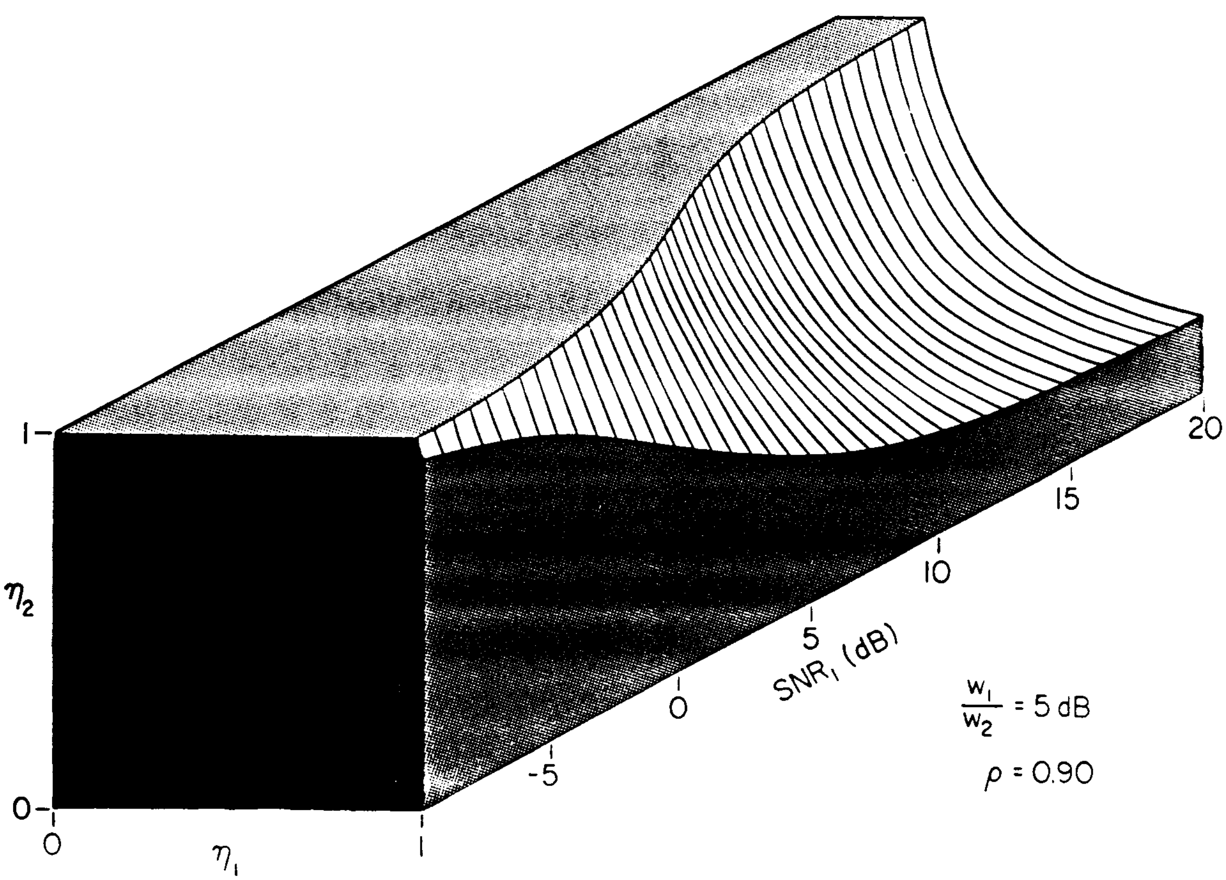

Figure 1. Efficiency region as a function of background Gaussian noise level.

3. Capacity with Amplitude-Constrained Inputs

In this section we investigate the capacity of the amplitude-constrained conventional Gaussian multiple-access channel:

$$Y = \sum_{i=1}^{K} X_i + N \tag{3.1}$$

where the inputs satisfy the constraints $X_i \in [-A_i, A_i]$. This channel has the same capacity as a continuous-time multiple-access white Gaussian channel where all users are assigned the same unit-energy bounded waveform $\{s(t),\ t \in [0,T]\}$:

$$r(t) = \sum_{k=1}^{K} \sum_{i=1}^{n} b_k(i)\, s(t - iT) + n(t) \tag{3.2}$$

and each modulated waveform is restricted to satisfy $|\sum_{i=1}^{n} b_k(i)\, s(t - iT)| \le A_k \max_{0 \le \lambda \le T} s(\lambda)$. Note that unlike the case where the sufficient statistics of the discrete-time channel are obtained via orthogonal signals with infinite support (e.g. via sampling), in this case input amplitude constraints in the discrete-time channel do translate into amplitude constrained inputs to the continuous-time channel. In the single-user case, Smith [16] showed that the capacity of (3.1) is attained by simple random variables and gave an algorithm for finding the optimum input distribution as a function of the noise level. Rather than generalizing these results to the multiuser case, here we will show a) that if the maximum allowed amplitudes are small enough, then the capacity region is attained by binary inputs and b) the infinite-user capacity region under the assumptions that no user is allowed to dominate and the rate-sum is finite.

Proposition 3: Suppose that X_k is a symmetric random variable with points of increase only on $[-A_k, A_k]$, $k = 1, \cdots K$; and suppose that $(A_1, \cdots A_K)$ is such that $\sum_{i \in J} X_i + N$ is unimodal for all $J \subset \{1, \cdots K\}$. Then the capacity region of (3.1) is equal to

$$\bigcap_{J \subset \{1, \cdots K\}} \{(R_1, \cdots R_K) :\ 0 \le \sum_{i \in J} R_i \le h(F^*_{Y_J}) - h(F_N)\} \tag{3.3}$$

where $F^*_{Y_J} = F_N \mathop{*}_{i \in J} F^*_{X_i}$ and $F^*_{X_i}(x) = \frac{1}{2}\ \ -A_i \le x < A_i$.

Proof: The objective is to show that binary equiprobable inputs with maximum allowable amplitude maximize the mutual informations $I(\{X_i\}_{i \in J};\, Y \mid \{X_i\}_{i \notin J})$. Labelling $J = \{a_1, \cdots a_{|J|}\}$ we can write (see [8,p.50])

$$I(\{X_i\}_{i \in J};\, Y \mid \{X_i\}_{i \notin J}) = \sum_{j=1}^{|J|} I(X_{a_j};\, Y \mid \{X_i\}_{i \notin J}, X_{a_1}, \cdots X_{a_{j-1}}). \tag{3.4}$$

Note that the distribution of X_{a_1} only affects the first term in the sum of the right-hand side of (3.4). If we can show that this term is maximized by equiprobable inputs at $\pm A_{a_1}$ regardless of the other input distributions, then the proposition will follow because the ordering of the elements of $J = \{a_1, \cdots a_{|J|}\}$ is arbitrary. In this way, we have converted the problem into one of single-user non-Gaussian capacity. Let F and F_Z be the distribution of X_{a_1} and $Z = \sum_{i=2}^{|J|} X_{a_i} + N$ respectively. Then, the first term in the sum of (3.4) is equal to

$$I(X_{a_1};\, Y \mid \{X_i\}_{i \notin J}) = h(F * F_Z) - h(F_Z). \tag{3.5}$$

Denote the set of symmetric probability distributions with points of increase in $[-A_{a_1}, A_{a_1}]$ by Υ. Then, the problem

$$\max_{F \in \Upsilon} h(F * F_Z) \tag{3.6}$$

is a convex optimization problem for any admissible F_Z (cf. [16]): Υ is a convex set and $h(F * F_Z)$ is concave in $F \in \Upsilon$ (see [17,p.238]).

Therefore, it is enough to show that the directional derivative at $F^*_{X_{a_1}}$:

$$\lim_{\alpha \downarrow 0} \frac{1}{\alpha} \left[h((1-\alpha) F^*_{X_{a_1}} + \alpha F_Z) - h(F^*_{X_{a_1}} * F_Z) \right] \leq 0, \tag{3.7}$$

for all $F \in \Upsilon$. It is easy to check that the left-hand side of (3.7) is equal to

$$\int_{-A_{a_1}}^{A_{a_1}} f(x)\,(dF(x) - dF^*_{X_{a_1}}(x)) \tag{3.8}$$

where

$$f(x) = -\int_{-\infty}^{\infty} p_Z(y-x) \log \left[\frac{1}{2} p_Z(y - A_{a_1}) + \frac{1}{2} p_Z(y + A_{a_1})\right] dy \tag{3.9}$$

Notice that f is an even function since Z is a symmetric random variable. Hence, in order to show (3.7) it will now suffice to show that f is increasing in the positive real line. But since the density p_Z is differentiable we have

$$f'(x) = \int_0^{\infty} p'_Z(\lambda) \left[g(\lambda + x) - g(\lambda - x)\right] d\lambda \tag{3.10}$$

with

$$g(\lambda) = \log \left[\frac{1}{2} p_Z(\lambda - A_{a_1}) + \frac{1}{2} p_Z(\lambda + A_{a_1})\right]. \tag{3.11}$$

If $x \geq 0$ and $\lambda \geq 0$ then $g(\lambda + x) \leq g(\lambda - x)$ because g is even and decreasing for positive arguments (because $N + \sum_{i \in J} X_i$ is unimodal). Moreover, since Z is also unimodal and symmetric it follows that $p'_Z(\lambda) \leq 0$ for $\lambda \geq 0$. So the integrand in (3.10) is nonnegative, and the proof is complete. □

The set of vectors $(A_1, \cdots A_K)$ that result in unimodal outputs as required in Proposition 3 is an open problem. It can checked that $A_1 \in [0,\sigma]$ and $(A_1,A_2) \in [0,\sigma]^2$ guarantee the unimodality of the output for $K = 1$ and $K = 2$, respectively. Moreover, these are tight estimates in the sense that larger sets $[0,x]^i$ do not guarantee unimodality. Lest one gets too optimistic, $(A_1,A_2,A_3) \in [0,\sigma]^3$ does not guarantee unimodality because of a tiny neighborhood of (σ,σ,σ).

Let us now turn our attention to the asymptotic behavior (as $K \to \infty$) of the capacity region of the amplitude-constrained Gaussian multiple-access channel. If no user is allowed a dominant amplitude and in the limit the rate-sum is finite, it is necessary that the allowable amplitude of each user goes to zero. As Proposition 3 indicates, capacity is achieved by binary equiprobable inputs with maximum possible variance, and because of the central-limit theorem the sum of any infinite subset of inputs is asymptotically Gaussian. Therefore, in the nontrivial infinite-user case the amplitude constraints are actually equivalent to variance constraints, and the asymptotic capacity region admits an explicit expression. This reasoning is formalized in the following result.

Proposition 4: Consider the Gaussian multiple-access channel

$$Y = \sum_{i=1}^{K} X_i + N$$

where N is $N(0,\sigma^2)$ and X_i is constrained to the interval $[-A_{Ki}, A_{Ki}]$. Suppose that the allowable amplitudes satisfy:

$$\lim_{K \to \infty} \max_{1 \leq i \leq K} A_{Ki} = 0 \tag{3.12}$$

and

$$\lim_{K \to \infty} \sum_{i=1}^{K} A_{Ki}^2 < \infty \tag{3.13}$$

If R_{Ki} denotes the rate of the i-th user when the number of users is K, then the limit as $K \to \infty$ of the capacity region is equal to

$$\bigcap_{J \subset N} \{ 0 \leq \lim_{K \to \infty} \sum_{i \in J_K} R_{Ki} \leq \frac{1}{2} \log [1 + \lim_{K \to \infty} \sum_{i \in J_K} \frac{A_{Ki}^2}{\sigma^2}] \}$$

where $J_K = J \bigcap \{1, \cdots K\}$.

Proof: Since $var(X_i) \leq A_{Ki}^2$, the $K-$user capacity region is outer-bounded by the conventional Gaussian capacity region [2]

$$\bigcap_{J_K \subset \{1, \cdots K\}} \{ 0 \leq \sum_{i \in J_K} R_{Ki} \leq \frac{1}{2} \log [1 + \sum_{i \in J_K} \frac{A_{Ki}^2}{\sigma^2}] \} \tag{3.14}$$

Also, the $K-$user capacity region is inner-bounded by the region achievable by a particular choice of the input distributions, namely, $X_i \sim ½\, \delta_{-A_{Ki}} + ½\, \delta_{A_{Ki}}$. This region is defined by the mutual informations:

$$I(\{X_i\}_{i \in J_K}; Y \mid \{X_i\}_{i \notin J_K}) = h(F_{J_K} * F_N) - \frac{1}{2} \log [2\pi e \sigma^2] \tag{3.15}$$

where F_{J_K} is the distribution of the binomial random variable $Z_{J_K} = \sum_{i \in J_K} X_i$. The proposition will follow by showing that

$$\lim_{K \to \infty} h(F_{J_K} * F_N) = \frac{1}{2} \log [2\pi e \ (\sigma^2 + \lim_{K \to \infty} \sum_{i \in J_K} A_{Ki}^2)] \tag{3.16}$$

and hence that the outer bound (3.14) is asymptotically tight. But $N + Z_{J_K}$ is a continuous random variable whose density is over-bounded by $1/\sigma \sqrt{2\pi}$ (because $\frac{1}{2} p(y - A) + \frac{1}{2} p(y + A) \leq \max_z p(z)$); then, the product of that density function and its logarithm is uniformly bounded in K, and therefore the left-hand side of (3.16) is equal to the differential entropy of the limit distribution of $N + Z_{J_K}$. Moreover, since N and Z_{J_K} are independent random variables, it suffices to show that Z_{J_K} converges in distribution (as $K \to \infty$) to $N(0, \lim_{K \to \infty} \sum_{i \in J_K} A_{Ki}^2)$. Since $\sum_{i \in J_K} A_{Ki}^2$ is the variance of Z_{J_K}, the result is true (by L_2-convergence) if $\lim_{K \to \infty} \sum_{i \in J_K} A_{Ki}^2 = 0$. Otherwise, $\sum_{i \in J_K} X_i \,/\, (\sum_{j \in J_K} A_{Kj}^2)^{½}$ converges in distribution to a standard Gaussian random variable. In order to show this we verify Lindeberg's condition [18,p. 205]: for each $\delta > 0$,

$$\lim_{K \to \infty} \sum_{i \in J_K} \int_{|x| \leq \delta} x^2 \, dG_{Ki}(x) = 1, \tag{3.17}$$

where G_{ki} is the distribution of $X_i \,/\, (\sum_{j \in J_K} A_{Kj}^2)^{½}$ -a binary symmetric random variable with variance $B_{Ki}^2 = A_{Ki}^2 \,/\, (\sum_{j \in J_K} A_{Kj}^2)$. It follows that

$$\sum_{i \in J_K} \int_{|x| \leq \delta} x^2 \, dG_{Ki}(x) = \sum_{i \in J_K} B_{Ki}^2 \, 1\{B_{Ki}^2 \leq \delta^2\}. \tag{3.18}$$

But since $\sum_{i \in J_K} B_{Ki}^2 = 1$, there are always fewer than δ^{-2} terms in J_K such that $B_{Ki}^2 > \delta^2$. Therefore,

$$1 \geq \sum_{i \in J_K} B_{Ki}^2 \, 1\{B_{Ki}^2 \leq \delta^2\} \geq 1 - \delta^{-2} \max_{i \in J_K} B_{Ki}^2 \tag{3.19}$$

and the right-hand side of (3.18) can be made as close to 1 as desired with large enough K because of assumption (3.12).

□

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