Poisson Communication Theory

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Abstract— **This paper gives a literature review of detection theory and information theory in the Poisson regime, where the channel model is a Poisson point process whose instantaneous rate is controlled by the input.**

I. Communication under the Poisson Regime

In the semi-classical model of photodetection, the output of the photodetector (e.g. a p-i-n diode) is the superposition of two independent Poisson point processes: one with constant rate λ_0 ("dark current noise") and another whose instantaneous rate $\lambda(t)$ is proportional to the instantaneous squared magnitude of the impinging optical field integrated over the receiving aperture. This simple model is representative of the *direct-detection photon-limited* optical communication channel.

In 1969, Israel Bar-David discovered the "Poisson matched filter" in his paper [1] entitled *Communication under the Poisson Regime.* The Poisson matched filter (matched to rate $\lambda_1(\cdot)$) processes a point process realization $t_1, \ldots, t_n$ by

$$\alpha_1 = \sum_{i=1}^{n} \log\left(1 + \frac{\lambda_1(t_i)}{\lambda_0}\right). \tag{1}$$

The scalar in (1) plays a role analogous to the conventional matched filter output $\langle y(t), s_1(t)\rangle$ for detection of signals embedded in white Gaussian noise. In particular, in order to decide whether an observed point process is produced by a Poisson law with rate $\lambda_1(t)+\lambda_0$ or by a Poisson law with rate $\lambda_2(t)+\lambda_0$, the scalar $\alpha_1 - \alpha_2$ is a sufficient statistic.

Throughout the sixties, a number of works [2], [3], [4], [5] had failed to reach beyond the special case of piecewise constant rate functions. It was not until Bar-David's seminal paper [1] that Poisson communication theory acquired a full-fledged identity as an elegant conceptual analog of the venerable additive white Gaussian noise channel. Contemporary to [1] is Helstrom's treatise [6], [7] on quantum communication theory, which while occupying a seminal, and yet largely untapped, role in optical communication theory is devoted to a model with different physical underpinnings.

II. Detection

Despite the conceptual parallels with classical additive Gaussian channels, Bar-David [1] noticed important differences such as the fact that performance was no longer determined by the L_2 distance (or any other distance) between the signal shapes.

Pulse-energy modulation (a possibly different energy level is sent in each signaling interval) is a generalization of Pulse-position modulation (PPM) whose bit error rate was studied by R. Gagliardi and S. Karp [5].

In the early seventies, references [9], [10], [11], [12], [13] undertook the generalization of Bar-David's Poisson matched filter to the case when the rate is random, accounting, for example, for random turbulence and other types of fading in a free-space optical channel. D. Snyder [10], [14] showed a counterpart of the estimator-correlator (cf. [15]), where the rate in the correlator is replaced by its minimum mean-square error causal estimate. Another generalization of Bar-David's likelihood ratio formula was accomplished by T. Kadota [16] in a model where the point process is not observable directly, but its points determine the location of realizations of nonstationary Gaussian processes.

The counterpart in the Poisson setting of Forney's maximum likelihood sequence detector [17] for channels subject to intersymbol interference was obtained by R. Morley and D. Snyder [18]. Robustified versions of the Poisson matched filter were obtained by E. Geraniotis and V. Poor [19], [20].

New impetus for optical detection theory was seen in the early eighties with the successful experimental implementation of coherent-detection optical photodetection. In this technique [21], a strong nonmodulated optical source in phase with the incoming signal is superposed at the photodetector. The analyses of the error probability of coherent-detection optical photodetection [23], [24], [25] reverted to additive-noise Gaussian models with the exception of the Poisson analysis in [26], where Bar-David's optical matched filter is shown to achieve asymptotically (as the strength of the local oscillator goes to infinity) a bit error rate which depends on the rates only through their Hellinger distance, i.e. the distance between their square roots.

Sequential detection problems with Poisson proint processes have been studied in [27], [28].

III. Estimation and Filtering

In [1] Bar-David was motivated, not only by detection, but by delay estimation, and proposed the use of the Poisson matched filter for distance measurements. In fact, a general setup for parameter estimation is given in [1]. Recent advances in this area can be found in [29].

Stochastic integration with Poisson point-process observations has been developed for various filtering problems by, among others, R. Boel, V. Benes, P. Bremaud, M. Davis, S. Marcus, A. Segall, D. Snyder, and Y. Yavin [10], [30], [31], [32], [33], [34].

IV. Channel Capacity

The first study of the Shannon capacity of photon counting channels dates back to the late sixties [35]. In 1973, Bar-David [36] analyzed the capacity of the PPM photon counting optical channel, which would later be popularized in the information theory and coding theory communities by J. Pierce's 1978 paper [37]. Pierce argued that the capacity of the PPM channel is equal to hf/kT nats per photon, where h and k are Planck's and Boltzmann constants, resp., f is the center frequency and T is the noise temperature. Further work on the capacity of the PPM channel and its variants was published in the eighties by Bar-David, Pierce, coworkers and others [38], [39], [40], [41], [42]. Recently, C. Georghiades [43] has compared the capacity, cutoff rate and error probability of on-off keying, PPM, overlapping PPM, and "combinatorial" PPM.

The martingale techniques used in filtering problems proved successful in the derivation of the Shannon capacity of the Poisson channel without the PPM constraint. This result is due to Y. Kabanov [44] and M. Davis [45]: If the codewords (i.e information bearing rate functions) are constrained to satisfy

$$0 \leq \lambda(t) \leq A \tag{2}$$

$$\frac{1}{T}\int_0^T \lambda(t)\, dt \leq \sigma A, \tag{3}$$

then, the Shannon capacity is given by the formula

$$\begin{aligned} C &= A\left[q^*(1+s)\log(1+s) + (1-q^*)s\log s\right. \\ &\quad \left. - (q^*+s)\log(q^*+s)\right] \end{aligned} \tag{4}$$

with

$$s = \frac{\lambda_0}{A} \tag{5}$$

$$q^* = \min\{\sigma, \frac{(1+s)^{1+s}}{s^s e} - s\}. \tag{6}$$

An alternative derivation of (4) which was more palatable to information theorists was given by A. Wyner [46]. Wyner [46], [47] also found a rare exact expression for the reliability function of this channel for all rates below capacity. The increase in the reliability function afforded by feedback was studied by A. Lapidoth [48], where an exact expression is found in the absence of dark current noise.

To achieve (4), the bandwidth of the rate functions grows without bound. A more realistic model (cf. [49], [50]) has been studied by S. Shamai (Shitz) and A. Lapidoth [51], [52]. The capacity of the Poisson channel has been obtained under various other constraints in [53], [54], [55].

In the early eighties, cutoff rate was more in vogue than nowadays as a measure of channel transmission capabilities. J. Massey and D. Snyder [56], [57] (see also [58]) showed that the cutoff-rate of a q-ary Poisson direct-detection channel is attained by a simplex q-set whose signals correspond to the square roots of the rates. If the common distance between those signals is allowed to be d, the cutoff rate is given by

$$R_0 = \log q - \log\left(1 + (q-1)e^{-d^2/2}\right). \tag{7}$$

Although outside the scope of this paper, the information theoretic problems introduced in [59] also deal with point process observations.

V. Multiple-Access Channels

The study of Poisson channels where the rate is the superposition of the rates transmitted by several users was apparently introduced in [60], [61] which found a maximum-likelihood multiuser detector for asynchronous users and an upper bound on minimum bit error rate. Other multiuser detectors have been proposed in the last five years in [62], [63], [64], [65], [66], [67].

The analysis of the bit error rate of Bar-David's Poisson matched filter in the presence of multiaccess interference has been undertaken from several directions in [68], [69], [70], [71], [72].

The capacity of the Poisson multiple-access channel was first addressed in 1994 by I. Bar-David, E. Plotnik and R. Rom [73]. A full generalization of the Kabanov-Davis formula to the multiuser case was not attempted until the thorough study [74] by A. Lapidoth and S. Shamai (Shitz) which puts particular emphasis in the assessment of the loss in capacity if signaling is restricted to TDMA.

A few works have also considered a different multiaccess-channel model where electric fields rather than energies superpose at the receiver [60], [61], [75]. P. Narayan and D. Snyder [75] showed the optimality of TDMA with respect to cutoff rate. With this model and in the hypothetical scenario that full phase synchronism were feasible, an interfering user can actually lower the bit error rate of the desired user [60].

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