When does the Source–Channel Separation Theorem Hold?

Sridhar Vembu, Sergio Verdú and Yossef Steinberg

Department of Electrical Engineering, Princeton University, Princeton, NJ 08544

The meeting point of the two main branches of the Shannon theory is the joint source-channel coding theorem. This theorem has two parts: a direct part that states that if the minimum achievable source coding rate of a given source is strictly below the capacity of a channel, then the source can be reliably transmitted through the channel by appropriate encoding and decoding operations; and a converse part stating that if the source coding rate is strictly greater than capacity, then reliable transmission is impossible. Implicit in the direct source-channel coding theorem is the fact that reliable transmission can be accomplished by separate source and channel coding, where the source [resp. channel] encoder and decoder need not take into account the channel [resp. source] statistics. Because of the converse part (and except for the residual uncertainty in the case when the minimum source coding rate is equal to the channel capacity) it follows that either reliable transmission is possible by separate source-channel coding or it is not possible at all. This is the reason why the joint source-channel coding theorem is commonly referred to as the separation theorem.

Ever since Shannon's 1948 paper [1], where the result was stated for stationary memoryless sources and channels, the separation theorem has received considerable attention, with a number of researchers proving versions that apply to more and more general classes of sources and channels. Even though most analytically tractable channels and sources are encompassed by previous versions of the separation theorem, it is of considerable theoretical interest to study the validity of this theorem in the context of very general sources and channels. In particular, we do not impose restrictions such as memorylessness, stationarity, ergodicity, causality, information stability, etc. This is motivated by the recent papers [2] and [3], which find general expressions for the minimal source coding rate and channel capacity that apply without those restrictions. A result (Theorem 4 of [3]) which leads to a new general converse to the channel coding theorem in [3], proves to be a key tool in our investigation of the source-channel coding theorem. Despite the generality of [2, 3] and the present paper, the proofs are, in fact, conceptually simple.

Let us first show an example of a memoryless information stable source-channel pair for which the separation theorem fails to hold. The source switches between a binary Bernoulli($\frac{1}{2}$) source and a deterministic binary sequence; the channel switches between a noiseless binary channel and a BSC with crossover probability equal to $\frac{1}{2}$. Switching takes place deterministically at times 2^i. The capacity of the channel is $\frac{1}{3}$ bit/use and the minimum achievable fixed-length source coding rate is $\frac{2}{3}$ bit/symbol. Yet, the source is transmissible through the channel (with zero error). Note that all previous instances where the separation theorem was known to fail were always within the context of multiterminal sources and channels.

This example reveals that, in general, the channel capacity and the minimum source coding rate do not provide sufficient knowledge in order to determine whether the source can be reliably transmitted through the channel. A finer look at the statistical structure of the channel and source is necessary. We prove that two similar conditions which we call *domination* and *strict domination* are necessary and sufficient, respectively, for reliable transmissibility which is defined as

Definition 1 *A source Z is said to be reliably transmissible over W if there exists a sequence of encoders and decoders such that the $\lim_{n\to\infty} P(Z^n \neq \tilde{Z}^n) = 0$, where $\tilde{Z}^n$ is the response of the cascade encoder-channel-decoder to Z^n.*

The domination conditions involve the overlap between the information spectra of the source and channel. Let

$$h_{Z^n}(a^n) = \log \frac{1}{P_{Z^n}(a^n)}$$

and

$$i_{X^nW^n}(a^n; b^n) = \log \frac{P_{X^nW^n}(a^n; b^n)}{P_{X^n}(a^n)P_{Y^n}(b^n)}.$$

Then the definitions of strict domination and domination are as follows:

Definition 2 A channel W is said to *strictly dominate* a source Z if there exists a $\delta > 0$ and a channel input process X such that

$$\lim_{n\to\infty} \inf_{c_n \in R} \left\{ P\left[\frac{1}{n} h_{Z^n}(Z^n) \geq c_n\right] + P\left[\frac{1}{n} i_{X^nW^n}(X^n; Y^n) \leq c_n + \delta\right]\right\} = 0.$$

Definition 3 A channel W is said to *dominate* a source Z if for any $\delta > 0$ and any sequence of nonnegative numbers $\{c_n\}_{n=1}^{\infty}$, there exists X such that

$$\lim_{n\to\infty} P\left[\frac{1}{n} h_{Z^n}(Z^n) \geq c_n\right] P\left[\frac{1}{n} i_{X^nW^n}(X^n; Y^n) \leq c_n - \delta\right] = 0$$

Using these results we characterize those channels for which the classical statement of the separation theorem holds for every source. It turns out that those are the channels whose definition of capacity is insensitive to whether good codes are required for all sufficiently long blocklengths or for only infinitely many blocklengths. We also characterize those sources for which the separation theorem holds for every channel. This class of sources includes but is not restricted to stationary sources. A conclusion to be drawn from our results is that when dealing with nonstationary probabilistic models, care should be exercised before applying the separation theorem.

References

[1] C. E. Shannon, "A Mathematical Theory of Communication," *Bell System Technical Journal*, Vol. 27, pp. 379-423, July 1948.

[2] T. S. Han, S. Verdú, "Approximation Theory of Output Statistics," *IEEE Trans. Inf. Theory*, pp. 752-772, Vol. 39, No. 3, May 1993.

[3] S. Verdú, T. S. Han, "A General Formula For Channel Capacity," *IEEE Trans. Inf. Theory*, to appear, 1994.

0 - 7803-2015-8/94/$4.00