“Foundations of Wireless Networks and Beyond”
A symposium in honor of Tony Ephremides on his 65th birthday
Friday, October 17, 2008
Inn and Conference Center, University of Maryland, College Park, Md.

Shannon and Poisson

sergio verdú
• deaths from horse kicks in the Prussian cavalry.
• photons arriving at photodetector
• packets arriving at a router
• DNA mutations

\[ P_\lambda(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad k = 0, 1, 2, \ldots \]
Poisson entropy

\[ H(P_\lambda) = \lambda \log \frac{e}{\lambda} + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k \log k!}{k!} \]
**law of small numbers**

\[ D([X_1]_{1/n} + \ldots + [X_n]_{1/n} \| P_\lambda) \to 0 \]

\[ X_1, X_2, \ldots \text{ iid} \]

\[ \mathbb{E}[X_1] = \lambda \]

\[ [X]_a = \sum_{i=1}^{X} B_i \quad B_i \text{ iid Bernoulli } \mathbb{P}[B_i = 1] = a. \]

Harremoes, Johnson and Kontoyiannis (2007)  
“Thinning and the law of small numbers”
Poisson as maximum entropy distribution

$X \in \{0, 1, 2, \ldots\}$

If $X$ is ultra-log-concave, $H(X) \leq H(P_{E[X]}(x))$

ultra-log-concave: $kP_X^2(k) \geq (k + 1)P_X(k + 1)P_X(k - 1)$

Johnson (2008) “Log-concavity and the maximum entropy property of the Poisson distribution”
Poisson entropy

\[ H(P_\lambda) = \lambda \log \frac{e}{\lambda} + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k \log k!}{k!} \]
If $X$ is Poisson with mean $\lambda$, the number of bits, $N(X)$, required for almost-lossless compression of $X$ satisfies

$$
\lim_{\lambda \to \infty} \frac{N(X)}{\frac{1}{2} \log_2 \lambda} = 1
$$

Verdú and Han (1997) “The role of the asymptotic equipartition property in noiseless source coding”
lossy compression of Poisson processes
lossy compression of Poisson processes


Gallager (1976) “Basic limits on protocol information in data communication networks”

lossy compression of Poisson processes

lossy compression of Poisson processes

\[ R(d) = \lambda \left[ \log_2 \frac{1}{\lambda d} \right]^+ \text{ bits/arrival} \]

lossy compression of Poisson processes


Coleman, Kiyavash, Subramanian (2008) “Rate distortion function of a Poisson process with a queueing distortion measure”
lossy compression of Poisson processes

\[ R(d) = \lambda \left[ \log_2 \frac{1}{\lambda d} \right]^+ \text{ bits/arrival} \]


Coleman, Kiyavash, Subramanian (2008) “Rate distortion function of a Poisson process with a queueing distortion measure”
Shannon capacity of a queue

ENCODER

QUEUE

DECODER

01111001010000101110001110100

01111001010000101110001110100
Shannon capacity of a queue

- Exponential server capacity: $e^{-1}$ nats per average service time
- Non-exponential server capacity: $\geq e^{-1}$ nats per average service time

Anantharam and Verdú (1996) “Bits through queues”
Shannon capacity of a queue

- if the arrival rate is $\lambda$ and the service rate is $\mu$, the capacity $= \lambda \log \frac{\mu}{\lambda}$
- optimum arrival rate $\lambda = e^{-1} \mu$

Anantharam and Verdú (1996) “Bits through queues”
Shannon capacity of a queue

- capacity does not increase with feedback

Anantharam and Verdú (1996) “Bits through queues”
Shannon capacity of a queue

Anantharam and Verdú (1996) “Bits through queues”
error exponent of the exponential server

Arikan (2002) “On the reliability function of the exponential timing channel”
Shannon capacity of a discrete-time geometric-server queue

Wagner and Anantharam (2005) “Zero-rate reliability of the exponential timing channel”
Shannon capacity of a discrete-time geometric-server queue

\[
\log \left( 1 + \mu (1 - \mu)^{\frac{1 - \mu}{\mu}} \right)
\]


Prabhakar and Gallager (2005) “Entropy and the timing capacity of discrete queues”
Shannon capacity of the direct-detection Poisson photon-counting channel

\[ 0 \leq \lambda(t) \leq A \quad \frac{1}{T} \int_0^T \lambda(t) \, dt \leq \sigma A \]

\[ C = A \left[ q^*(1 + s) \log(1 + s) + (1 - q^*)s \log s - (q^* + s) \log(q^* + s) \right] \]

\[ s = \frac{\lambda_0}{A} \quad q^* = \min\{\sigma, \frac{(1 + s)^{1+s}}{s^s e} - s\} \]

Kabanov (1978) “Capacity of a channel of a Poisson type”

Davis (1980) “Capacity and cutoff rate for Poisson-type channels”

Wyner (1988) “Capacity and error exponent for the direct detection photon channel”
error exponent of the direct-detection Poisson photon-counting channel

\[ E(R) = \max \left[ AE_1(\rho, q) - \rho R \right], \quad (1.9a) \]

where the maximization is over \( \rho \in [0, 1] \), and \( q \in [0, \sigma] \). \( E_1(\cdot) \) is defined by

\[ E_1(\rho, q) = (q + s) - s [1 + \tau q]^{1+\rho}, \quad (1.9b) \]

where \( s = \lambda_0/A \) and

\[ \tau = \left(1 + \frac{1}{s}\right)^{1/(1+\rho)} - 1. \quad (1.9c) \]

Wyner (1988) “Capacity and error exponent for the direct detection photon channel”
error exponent of the direct-detection Poisson photon-counting channel with feedback

given for $\sigma \leq 1/e$ and $0 < R \leq C$ by

$$E_f(R) = A \left( \sigma + \frac{R}{A \ln \sigma} \right) + R \left( 1 - \frac{1}{\ln \sigma} \ln \left( -\frac{R}{A \ln \sigma} \right) \right)$$

Lapidoth (1993) “On the reliability function of the ideal Poisson channel with noiseless feedback”
capacity of the direct-detection Poisson photon-counting channel with fading

\[ C_N = \max_{0 \leq \mu \leq \sigma} \mathbb{E} [\mu \zeta(S\alpha, \lambda_0) - \zeta(\mu S\alpha, \lambda_0)] \]

\[ \zeta(x, y) \triangleq (x + y) \ln(x + y) - y \ln y, \]

pulse-amplitude modulation Poisson photon-counting channel

\[
\Pr[Y=y \mid X=x] = e^{-(x+\lambda_0)} \frac{(x + \lambda_0)^y}{y!}, \quad y \in \mathbb{Z}^+, \ x \geq 0.
\]

\[
C(A, \mathcal{P}) = \frac{1}{2} \log A - \frac{1}{2} \log \frac{\pi e}{2} + o(1), \quad \frac{1}{3} \leq \alpha \leq 1,
\]

where the \( o(1) \) term tends to zero as \( \mathcal{P} \) and \( A \) tend to infinity with their ratio \( \mathcal{P}/A \) held fixed at a value between 1/3 and 1.

When \( 0 < \alpha < 1/3 \) we have

\[
C(A, \mathcal{P}) = \frac{1}{2} \log A + \alpha u - u - \log \left( \frac{1}{2} - \alpha u \right) - \frac{1}{2} \log 2\pi e + o(1),
\]

where \( u \) is the non-zero solution to

\[
\sqrt{\pi} \text{erf} \left( \sqrt{u} \right) \left( \frac{1}{2} - \alpha u \right) - \sqrt{u} e^{-u} = 0,
\]

mutual information and estimation

\[ X \xrightarrow{\alpha} \mathcal{P}(\cdot) \xrightarrow{\mathcal{P}(\lambda)} \mathcal{P}(\alpha X + \lambda) \]

\[
\frac{d}{d\lambda} I (X; \mathcal{P}(X + \lambda)) = \mathbb{E} \left\{ \log \frac{X + \lambda}{\langle X + \lambda \rangle} \right\}
\]

Guo, Shamai, Verdú (2008) “Mutual information and conditional mean estimation in Poisson Channels”
mutual information and estimation

\[ \frac{\partial}{\partial \alpha} I(X; \mathcal{P}(\alpha X + \lambda)) = E \left\{ X \log \frac{\alpha X + \lambda}{\langle \alpha X + \lambda \rangle} \right\} \]

Guo, Shamai, Verdú (2008) “Mutual information and conditional mean estimation in Poisson Channels”
filtering with Poisson observations
mutual information and causal filtering

\[ I(\nu_0^T; \lambda_0^T) = \int_0^T \left\{ E\phi(\lambda_t) - E\phi(\hat{\lambda}_t) \right\} dt, \] (3)

where

\[ \phi(\alpha) = (\lambda_0 + x) \log(\lambda_0 + x) - \lambda_0 \log \lambda_0 \] (4)

(with natural logarithms used hereafter) and \( \hat{\lambda}_t \) denotes the causal conditional mean estimator of \( \lambda_t \), i.e.,

\[ \hat{\lambda}_t = E(\lambda_t | \nu_0^t). \] (5)

Kabanov (1978) "Capacity of a channel of a Poisson type"
Liptser and Shiryaevo (1978) "Statistics of Random Processes"
causal and noncausal filtering

\[
E \left\{ X_t \log X_t - \langle X_t \rangle_t \log \langle X_t \rangle_t \right\} \\
= \int_0^\infty E \left\{ \log \frac{\langle X_t + \lambda \rangle_\infty}{X_t + \lambda} \right\} d\lambda \\
= \int_0^1 E \left\{ X_t \log \frac{\alpha X_t}{\langle \alpha X_t \rangle_\infty} \right\} d\alpha
\]

Guo, Shamai, Verdú (2008) “Mutual information and conditional mean estimation in Poisson Channels”
“Life is good for only two things: discovering mathematics and teaching mathematics”

Simeon Denis Poisson
“Life is good for only two things: discovering mathematics and teaching mathematics”

Simeon Denis Poisson

“I can think of other things”

Anthony Ephremides