information theory and applications

ucsd

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shannon's inequality

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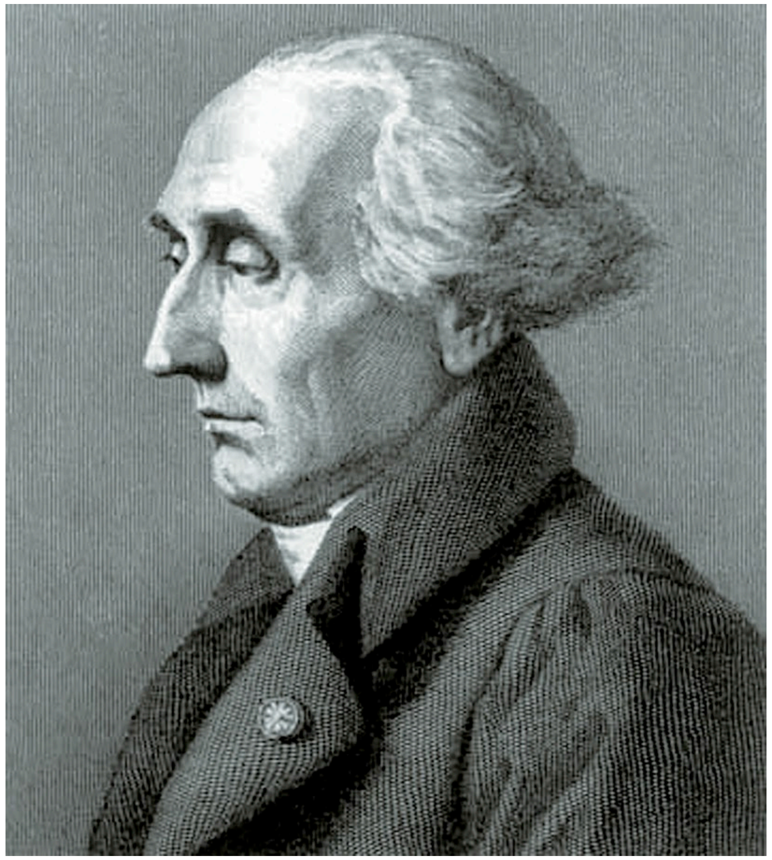
C. E. Shannon,
"Channels with Side Information at the Transmitter,"
IBM Journal of Research and Development,
vol.2, no.4, pp.289-293, Oct. 1958

a mathematical theory of communication, 1948

- channel capacity achievability in the sense of vanishing error probability
- channel capacity converse in the sense of vanishing conditional entropy rate



R. Fano

$$H(X|Y) \leq \mathbb{P}[X \neq Y] \log(M-1) + h(\mathbb{P}[X \neq Y])$$

a mathematical theory of communication, 1948

- channel capacity achievability in the sense of vanishing error probability
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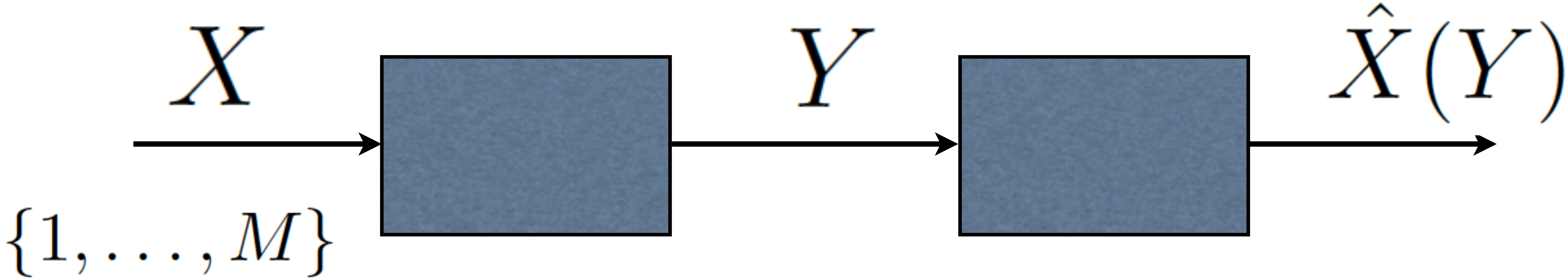
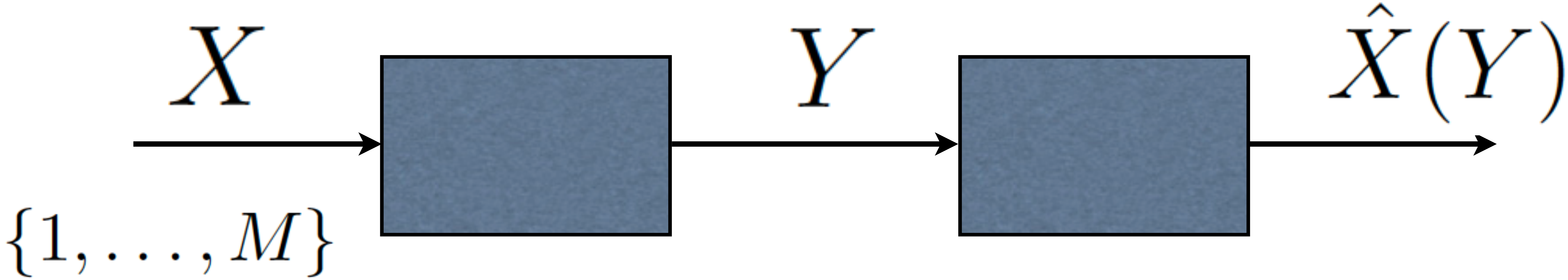


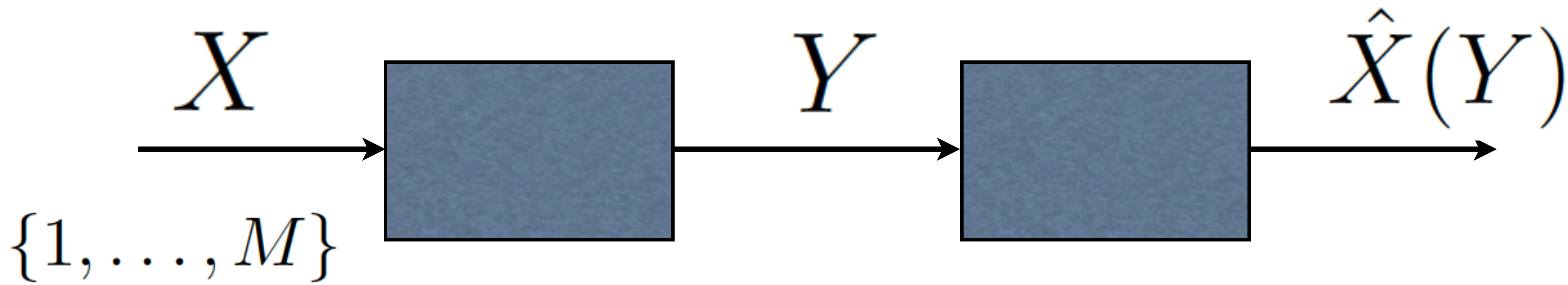
$$P_e = \min \mathbb{P}[X \neq \hat{X}(Y)]$$

$$H(X|Y) \leq P_e \log(M-1) + h(P_e)$$



$$P_e \geq \frac{H(X|Y) - 1}{\log(M-1)}$$

shannon's inequality



$$P_e \geq \frac{1}{6} \frac{H(X|Y)}{\log M + \log\log M - \log H(X|Y)}$$

proof: 1. blind is enough

$$P_e = 1 - p_{\max}$$

$$P_e \geq \frac{1}{6} \frac{H(X)}{\log M + \log\log M - \log H(X)}$$

convex in $H(X)$

proof: 2

$$H(X) \le P_e \log(M-1) + h(P_e)$$

Proof: $P_e \log(M-1) + h(P_e) - H(X) = D(P_X \,||\, \quad)$

proof: 3

$$H(X) \leq P_e \log(M-1) + h(P_e)$$

$$\leq \quad P_{\mathrm{e}} \log \frac{(M-1)e}{P_{\mathrm{e}}}$$

$$(1-x)\log \tfrac{1}{1-x} \leq x \log e \text{ on } x \in (0,1)$$

shannon's inequality

$$P_{\mathsf{e}} \quad \geq \quad \frac{H(X)}{6\log(M\theta)}$$

$$\theta = \frac{\log M}{H(X)} \geq 1$$

proof: 4: by contradiction

$$H(X) \leq P_{\mathrm{e}} \log \frac{(M-1)e}{P_{\mathrm{e}}}$$

$$P_{\mathrm{e}} < \frac{H(X)}{6 \log(M\theta)}$$

proof: 5

$$6 < \frac{\log((M-1)\theta) + \log(6e) + \log\frac{\log(M\theta)}{\log M}}{\log(M\theta)}$$

proof: 6

$$\frac{\log((M-1)\theta)}{\log(M\theta)} \leq 1$$

proof: 7

$$\frac{\log(6e)}{\log(M\theta)} \leq \log_2(6e)$$

proof: 8

$$\frac{\log_2 \frac{\log_2(M\theta)}{\log_2 M}}{\log_2(M\theta)} \leq \frac{\log_2 \log_2(M\theta)}{\log_2(M\theta)}$$

$$\leq \frac{\log_2 e}{e}$$

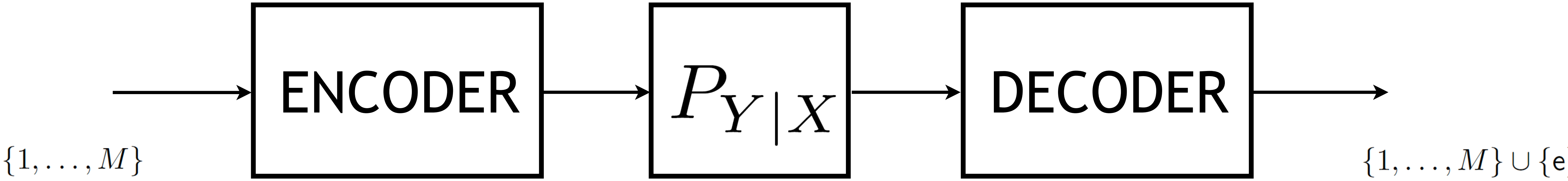
$x^{-1} \log_2 x$ is maximized at $x = e$.

proof: 9

$$6 < \frac{\log((M-1)\theta) + \log(6e) + \log\frac{\log(M\theta)}{\log M}}{\log(M\theta)}$$

proof: end

$$6 < 1 + \log_2(6e) + \frac{\log_2 e}{e} < 5.56$$



$$\epsilon^*(M) \geq \frac{\log M - \sup_X I(X;Y)}{\log M - \log\left(1 - \frac{\sup_X I(X;Y)}{\log M}\right)}$$

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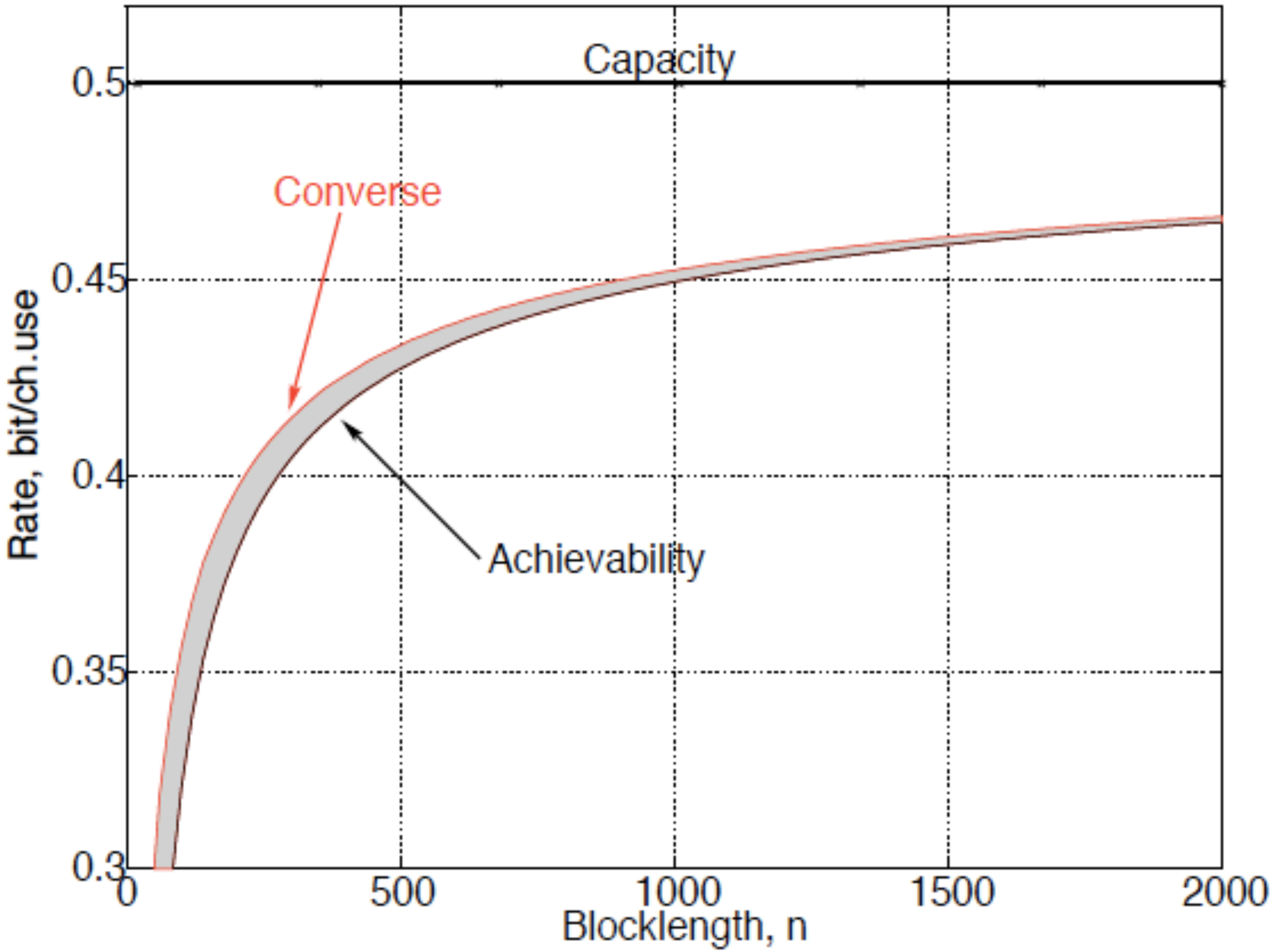
$$P_e \geqslant \frac{R-C}{6\left(R+\frac{1}{n}\ln\frac{R}{(R-C)}\right)}$$


Capacity
Converse
Achievability
Rate, bit/ch.use
Blocklength, n
0.5
0.45
0.4
0.35
0.3
0
500
1000
1500
2000

addendum: tightening

M	$\to 6$
2	2.495
3	1.963
4	1.771
5	1.667
6	1.601
1001	1.049
10^{20}	1.022