Appendix B

Correlations, power spectra and all that

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Consider a function $x(t)$ that varies in time. We would like to describe a situation in which these variations are random, drawn out of some distribution. But now we need a distribution for a function, rather than for a finite set of variables. This shouldn't bother us, since such constructions are central to much of modern physics, for example in the path integral approach to quantum mechanics. We refer to distributions of functions as "distribution functionals" when we need to be precise.

One strategy for constructing distribution functionals is to start by discretizing time, so that we have at most a countable infinity of variables $x(t_1), x(t_2), x(t_3), \cdots$. Let's assume for simplicity that the mean value of x is zero. Then the first nontrivial characterization of the statistics of x is the covariance matrix,

$$C_{\rm ij} = \langle x(t_{\rm i}) x(t_{\rm j}) \rangle. \tag{B.1}$$

We recall that if a single variable y is drawn from a Gaussian distribution with zero mean, then we have

$$P(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{y^2}{2\sigma^2}\right]. \tag{B.2}$$

The generalization to multiple variables is

$$P(\{x_{\rm i}\}) = \frac{1}{\sqrt{(2\pi)^N \det C}} \exp\left[-\frac{1}{2}\sum_{\rm i,j=1}^{N} x_{\rm i} (C^{-1})_{\rm ij} x_{\rm j}\right], \tag{B.3}$$

where as usual det is the determinant and $(C^{-1})_{\rm ij}$ is the ij element of the matrix inverse to C; if we think of the $\{x_{\rm i}\}$ as a vector $\mathbf{x}$, then we can write, more compactly,

$$P(\{x_{\rm i}\}) = \frac{1}{\sqrt{(2\pi)^N \det C}} \exp\left[-\frac{1}{2}\mathbf{x}^T\cdot C^{-1}\cdot\mathbf{x}\right], \tag{B.4}$$

where $\mathbf{x}^T$ is the transpose of the vector $\mathbf{x}$. Just to be clear, this describes a Gaussian distribution, but we have no guarantee that $\mathbf{x}$ will be Gaussian.

Problem 49: Gaussian integrals. If you haven't done these before, now is a good time to check that the probability distribution in Eq (B.4) is normalized. This requires you to show that

$$\int d^N x \, \exp\left[-\frac{1}{2}\mathbf{x}^T\cdot C^{-1}\cdot\mathbf{x}\right] = \sqrt{(2\pi)^N \det C}. \tag{B.5}$$

While you're at it, you should also show that

$$\ln \det C = {\rm Tr} \ln C. \tag{B.6}$$

This should be straightforward for the case which matters here, where C must have well defined, positive eigenvalues.

In general the covariance matrix $C_{\rm ij}$ can have an arbitrary structure, constrained only by symmetry and positivity of its eigenvalues. But when the index i refers to discrete time points, we have an extra constraint that comes from invariance under translations in time. Because there is no clock, we must have that

$$\langle x(t)x(t')\rangle = C_x(t-t'), \tag{B.7}$$

with no dependence on the absolute time t or t'. As an example, if

$$C_x(t-t') = e^{-|t-t'|/\tau_c}, \tag{B.8}$$

and $t_{\rm n} = n\Delta t$, then

$$C_{\rm ij} = \exp\left[-\left(\frac{\Delta t}{\tau_c}\right)|{\rm i}-{\rm j}|\right]. \tag{B.9}$$

This is shown in Fig B.1 for $\Delta t/\tau_c = 0.1$.

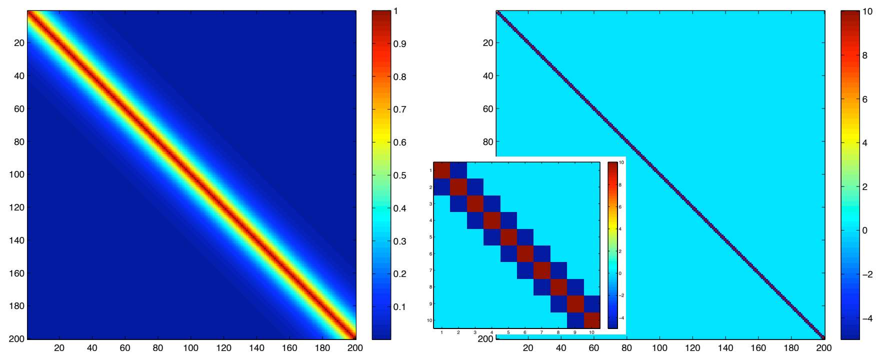

Figure B.1: Covariance matrix and its inverse. At left, the covariance matrix in Eq (B.9), with $\Delta t/\tau_c = 0.1$. At right, the inverse matrix, with inset showing a 10×10 submatrix surrounding the diagonal.

It is useful to look directly at the inverse matrix, also shown in Fig B.1. We see that this inverse matrix consists almost entirely of zeros, except in the immediate neighborhood of the diagonal. This tell us that the inverse matrix actually is the discretizaiton of a differential operator.

Reflexively, seeing that we have to compute inverses and determinants of matrices, we should think about diagonalizing C. We recall from quantum mechanics that the eigenfunctions of an operator have to provide a representation of the underlying symmetries. In this case, the relevant symmetry is time translation, so we know to look at the Fourier functions, $e^{-i\omega t}$. In fact, once we have the hint that we should use a Fourier representation, we don't need the crutch of discrete time points any more. Let's see how this works.

We define the Fourier transform with the conventions

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} dt\, e^{+i\omega t} x(t), \tag{B.10}$$

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-\omega t} \tilde{x}(\omega). \tag{B.11}$$

Now if we compute the covariance of two frequency components, we have

$$\begin{aligned}
\langle \tilde{x}(\omega)\tilde{x}(\omega')\rangle &= \left\langle \int_{-\infty}^{\infty} dt\, e^{+i\omega t} x(t) \int_{-\infty}^{\infty} dt'\, e^{+i\omega' t'} x(t') \right\rangle && \text{(B.12)}\\
&= \int_{-\infty}^{\infty} dt\, e^{+i\omega t} \int_{-\infty}^{\infty} dt'\, e^{+i\omega' t'} \langle x(t)x(t')\rangle && \text{(B.13)}\\
&= \int_{-\infty}^{\infty} dt\, e^{+i\omega t} \int_{-\infty}^{\infty} dt'\, e^{+i\omega' t'} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} e^{-i\Omega(t-t')} S_x(\Omega), && \text{(B.14)}
\end{aligned}$$

where we introduce the Fourier transform of the correlation function,

$$\begin{aligned}
S_x(\Omega) &= \int_{-\infty}^{\infty} d\tau\, e^{+i\Omega\tau} C_x(\tau) && \text{(B.15)}\\
C_x(t-t') &= \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} e^{-i\Omega(t-t')} S_x(\Omega). && \text{(B.16)}
\end{aligned}$$

Now we can rearrange the integrals in Eq (B.14):

$$\begin{aligned}
\langle \tilde{x}(\omega)\tilde{x}(\omega')\rangle &= \int_{-\infty}^{\infty} dt\, e^{+i\omega t} \int_{-\infty}^{\infty} dt'\, e^{+i\omega' t'} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} e^{-i\Omega(t-t')} S_x(\Omega),\\
&= \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} S_x(\Omega) \left[\int_{-\infty}^{\infty} dt\, e^{i(\omega-\Omega)t}\right] \left[\int_{-\infty}^{\infty} dt'\, e^{i(\omega'+\Omega)t'}\right]. && \text{(B.17)}
\end{aligned}$$

This is moment to recall the Fourier representation of the Dirac delta function. The delta function has the property that

$$\begin{aligned}
\delta(z) &= 0 \quad z \neq 0, && \text{(B.18)}\\
\int dz\, \delta(z) &= 1, && \text{(B.19)}
\end{aligned}$$

if the domain of the integral includes $z = 0$. Then

$$\delta(z) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-iqz}. \qquad \text{(B.20)}$$

Thus we recognize, in Eq (B.17),

$$\begin{aligned}
\int_{-\infty}^{\infty} dt\, e^{i(\omega-\Omega)t} &= 2\pi\delta(\omega-\Omega), && \text{(B.21)}\\
\int_{-\infty}^{\infty} dt'\, e^{i(\omega'+\Omega)t'} &= 2\pi\delta(\omega'+\Omega). && \text{(B.22)}
\end{aligned}$$

Substituting back into Eq (B.17), we have

$$\begin{aligned}
\langle \tilde{x}(\omega)\tilde{x}(\omega')\rangle &= \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}S_x(\Omega)\left[\int_{-\infty}^{\infty}dt\, e^{i(\omega-\Omega)t}\right]\left[\int_{-\infty}^{\infty}dt'\, e^{i(\omega'+\Omega)t'}\right]. \\
&= \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}S_x(\Omega)2\pi\delta(\omega-\Omega)2\pi\delta(\omega'+\Omega) && \text{(B.23)} \\
&= S_x(\omega)2\pi\delta(\omega'+\omega). && \text{(B.24)}
\end{aligned}$$

We see that, while different time points can be correlated with one another in complicated ways, the covariance of frequency components has a much simpler structure: $\tilde{x}(\omega)$ is correlated only with $\tilde{x}(-\omega)$.

This covariance structure, which couples positive and negative frequency components, makes sense when we realize that we are using a complex representation for real variables. To make a real variable $x(t)$, the Fourier transform must obey

$$\tilde{x}(-\omega) = \tilde{x}^*(\omega), \qquad \text{(B.25)}$$

so positive and negative frequency components are not independent—in fact they are redundant. We could instead think about the real and imaginary parts of the positive frequency components, which can be written as

$$\begin{aligned}
\tilde{x}_{\mathrm{Re}}(\omega) &= \frac{1}{2}\left[\tilde{x}(\omega)+\tilde{x}(-\omega)\right] && \text{(B.26)} \\
\tilde{x}_{\mathrm{Im}}(\omega) &= \frac{1}{2i}\left[\tilde{x}(\omega)-\tilde{x}(-\omega)\right]. && \text{(B.27)}
\end{aligned}$$

With this representation, we can use the result in Eq(B.24):

$$\begin{aligned}
\langle \tilde{x}_{\mathrm{Re}}(\omega)\tilde{x}_{\mathrm{Re}}(\omega')\rangle &= \left\langle \frac{1}{2}\left[\tilde{x}(\omega)+\tilde{x}(-\omega)\right]\frac{1}{2}\left[\tilde{x}(\omega')+\tilde{x}(-\omega')\right]\right\rangle && \text{(B.28)} \\
&= \frac{1}{4}[\langle\tilde{x}(\omega)\tilde{x}(\omega')\rangle + \langle\tilde{x}(\omega)\tilde{x}(-\omega')\rangle \\
&\qquad + \langle\tilde{x}(-\omega)\tilde{x}(\omega')\rangle + \langle\tilde{x}(-\omega)\tilde{x}(-\omega')\rangle] && \text{(B.29)} \\
&= \frac{S_x(\omega)}{4}2\pi[\delta(\omega+\omega')+\delta(\omega-\omega') \\
&\qquad +\delta(-\omega+\omega')+\delta(-\omega-\omega')]. && \text{(B.30)}
\end{aligned}$$

Because we are looking only at positive frequencies, $\omega+\omega'$ can never be zero, and hence the first and last delta functions can be dropped. The remaining two are actually the same, so we have

$$\langle \tilde{x}_{\mathrm{Re}}(\omega)\tilde{x}_{\mathrm{Re}}(\omega')\rangle = \frac{1}{2}S_x(\omega)2\pi\delta(\omega-\omega'). \qquad \text{(B.31)}$$

Similar calculations show that the imaginary parts of $\tilde{x}(\omega)$ have the same variance,

$$\langle \tilde{x}_{\rm Im}(\omega)\tilde{x}_{\rm Im}(\omega')\rangle = \langle \tilde{x}_{\rm Re}(\omega)\tilde{x}_{\rm Re}(\omega')\rangle = \frac{1}{2}S_x(\omega)2\pi\delta(\omega-\omega'), \tag{B.32}$$

while real and imaginary parts are uncorrelated,

$$\langle \tilde{x}_{\rm Re}(\omega)\tilde{x}_{\rm Im}(\omega')\rangle = 0. \tag{B.33}$$

Problem 50: The other phase. Derive Eq's (B.32) and (B.33).

What does all this mean? We think of the random function of time $x(t)$ as being built out of frequency components, and each component has a real and imaginary part. The structure of the covariance matrix is such that different frequency components do not covary, and this makes sense—if we have covariation of different frequency components then we can beat them against each other to make a clock running at the difference frequency, and this would violate time translation invariance. Similarly, the fact that real and imaginary components do not covary means that there is no preferred phase, which again is consistent with (indeed, required by) time translation invariance.

We should be able to put these results on the covariance matrix together to describe the distribution functional for a Gaussian function of time. Since the real and imaginary parts are independent, let's start with just the real parts. We should have

$$P[\{\tilde{x}_{\rm Re}(\omega)\}] \;\;\propto\;\; \exp\left[-\frac{1}{2}\int_0^\infty \frac{d\omega}{2\pi}\int_0^\infty \frac{d\omega'}{2\pi}\tilde{x}_{\rm Re}(\omega)\mathcal{A}(\omega,\omega')\tilde{x}_{\rm Re}(\omega')\right], \tag{B.34}$$

where $\mathcal{A}$ is the inverse of the covariance,

$$\int \frac{d\omega'}{2\pi}\mathcal{A}(\omega,\omega')\langle \tilde{x}_{\rm Re}(\omega')\tilde{x}_{\rm Re}(\omega'')\rangle = 2\pi\delta(\omega-\omega''). \tag{B.35}$$

We can find $\mathcal{A}$ by substituting the explicit expression for the covariance and doing the integrals:

$$\begin{aligned}
2\pi\delta(\omega-\omega'') &= \int \frac{d\omega'}{2\pi}\mathcal{A}(\omega,\omega')\langle \tilde{x}_{\rm Re}(\omega')\tilde{x}_{\rm Re}(\omega'')\rangle \\
&= \int \frac{d\omega'}{2\pi}\mathcal{A}(\omega,\omega')\frac{1}{2}S_x(\omega')2\pi\delta(\omega'-\omega'') &\text{(B.36)}\\
&= \frac{1}{2}\mathcal{A}(\omega,\omega'')S_x(\omega'') &\text{(B.37)}\\
\Rightarrow \mathcal{A}(\omega,\omega'') &= \frac{1}{S_x(\omega'')}4\pi\delta(\omega-\omega''). &\text{(B.38)}
\end{aligned}$$

Substituting back into Eq (B.34) for the probability distribution, we have

$$P[\{\tilde{x}_{\rm Re}(\omega)\}] \propto \exp\left[-\frac{1}{2}\int_0^\infty \frac{d\omega}{2\pi}\int_0^\infty \frac{d\omega'}{2\pi}\tilde{x}_{\rm Re}(\omega)\mathcal{A}(\omega,\omega')\tilde{x}_{\rm Re}(\omega')\right],$$

$$= \exp\left[-\frac{1}{2}\int_0^\infty \frac{d\omega}{2\pi}\int_0^\infty \frac{d\omega'}{2\pi}\tilde{x}_{\rm Re}(\omega)\frac{4\pi\delta(\omega-\omega'')}{S_x(\omega'')}\tilde{x}_{\rm Re}(\omega')\right] \tag{B.39}$$

$$= \exp\left[-\int_0^\infty \frac{d\omega}{2\pi}\frac{\tilde{x}^2_{\rm Re}(\omega)}{S_x(\omega)}\right]. \tag{B.40}$$

Exactly the same argument applies to the imaginary parts of the Fourier components, and these are independent of the real parts, so we have

$$\begin{aligned}
P[x(t)] &= P[\{\tilde{x}_{\rm Re}(\omega),\tilde{x}_{\rm Im}(\omega)\}] &\text{(B.41)}\\
&\propto \exp\left[-\int_0^\infty \frac{d\omega}{2\pi}\frac{\tilde{x}^2_{\rm Re}(\omega)+\tilde{x}^2_{\rm Im}(\omega)}{S_x(\omega)}\right] &\text{(B.42)}\\
&= \frac{1}{Z}\exp\left[-\int_0^\infty \frac{d\omega}{2\pi}\frac{|\tilde{x}(\omega)|^2}{S_x(\omega)}\right] &\text{(B.43)}\\
&= \frac{1}{Z}\exp\left[-\frac{1}{2}\int_{-\infty}^\infty \frac{d\omega}{2\pi}\frac{|\tilde{x}(\omega)|^2}{S_x(\omega)}\right], &\text{(B.44)}
\end{aligned}$$

where we have introduced the normalization constant Z.

It's useful to look at the example illustrated in Fig B.1. Here we have $C_x(\tau) = \exp(-|\tau|/\tau_c)$, so the power spectrum is

$$\begin{aligned}
S_x(\omega) &= \int_{-\infty}^\infty d\tau\, e^{+i\omega\tau}e^{-|\tau|/\tau_c} &\text{(B.45)}\\
&= \int_{-\infty}^0 d\tau\, e^{(+i\omega+1/\tau_c)\tau} + \int_0^\infty d\tau\, e^{(+i\omega-1/\tau_c)\tau} &\text{(B.46)}
\end{aligned}$$

$$= \frac{1}{(+i\omega + 1/\tau_c)} + \frac{1}{-(+i\omega - 1/\tau_c)} \quad \text{(B.47)}$$

$$= \frac{\tau_c}{1 + i\omega\tau_c} + \frac{\tau_c}{1 - i\omega\tau_c} \quad \text{(B.48)}$$

$$= \frac{2\tau_c}{1 + (\omega\tau_c)^2}. \quad \text{(B.49)}$$

This means that the probability distribution functional has the form

$$\begin{aligned} P[x(t)] &= \frac{1}{Z} \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{|\tilde{x}(\omega)|^2}{S_x(\omega)}\right] \\ &= \frac{1}{Z} \exp\left[-\frac{1}{4\tau_c}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[1 + (\omega\tau_c)^2]|\tilde{x}(\omega)|^2\right]. \end{aligned} \quad \text{(B.50)}$$

We recall that

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}|\tilde{x}(\omega)|^2 = \int dt\, x^2(t). \quad \text{(B.51)}$$

More subtly,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}(\omega\tau_c)^2|\tilde{x}(\omega)|^2 = \tau_c^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}| - i\omega\tilde{x}(\omega)|^2 \quad \text{(B.52)}$$

$$= \tau_c^2 \int dt \left[\frac{dx(t)}{dt}\right]^2, \quad \text{(B.53)}$$

where we recognize $-i\omega\tilde{x}(\omega)$ as the Fourier transform of $dx(t)/dt$. Thus we can write

$$\begin{aligned} P[x(t)] &= \frac{1}{Z} \exp\left[-\frac{1}{4\tau_c}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[1 + (\omega\tau_c)^2]|\tilde{x}(\omega)|^2\right] \\ &= \frac{1}{Z} \exp\left[-\frac{1}{4\tau_c}\int dt \left(\tau_c^2\dot{x}^2(t) + x^2(t)\right)\right]. \end{aligned} \quad \text{(B.54)}$$

This shows explicitly, as promised above, that inverting the covariance matrix gives rise to differential operators. This example also is nice because it produces a probability distribution functional for trajectories $x(t)$ that reminds us of a (Euclidean) path integral in quantum mechanics, in this case for the harmonic oscillator.

Let's push a little further and see if we can evaluate the normalization constant Z. By definition, we have

$$Z = \int \mathcal{D}x \exp\left[-\frac{1}{4\tau_c}\int dt \left(\tau_c^2\dot{x}^2(t) + x^2(t)\right)\right], \quad \text{(B.55)}$$

where $\int \mathcal{D}x$ denotes an integral over all the functions $x(t)$. We have the general result for an N dimensional Gaussian integral,

$$\int d^N x \exp\left[-\frac{1}{2}\mathbf{x}^T\cdot\hat{A}\cdot\mathbf{x}\right] = \sqrt{\frac{(2\pi)^N}{\det \hat{A}}} \tag{B.56}$$

$$= \sqrt{(2\pi)^N} \exp\left[-\frac{1}{2}\mathrm{Tr}\ln\hat{A}\right], \tag{B.57}$$

where $\hat{A}$ is a matrix. Here we need to let the number of dimensions become infinite, since we are integrating over functions. As you may recall from discussions of the path integral in quantum mechanics, there is some arbitrariness about how we do this, or, more formally, in how we define the measure $\mathcal{D}x$. A fairly standard choice is to absorb the $\sqrt{2\pi}$, so that, in the time window $0 < t < T$,

$$\mathcal{D}x = \lim_{dt\to 0} \prod_{\mathrm{n}=0}^{T/dt} \frac{dx(t_\mathrm{n})}{\sqrt{2\pi}}, \qquad t_\mathrm{n} = \mathrm{n}\cdot dt. \tag{B.58}$$

Notice that before we send $dt \to 0$, we have an integral over a finite number of points, so we should be able to carry over the results we know, and just interpret the limits correctly.

The Gaussian functional integrals that we want to do have the general form

$$\int \mathcal{D}x \exp\left[-\frac{1}{2}\int dt \int dt' x(t)\hat{K}(t,t')x(t')\right],$$

where $\hat{K}$ is an operator. Carrying over what we know from the case of finite matrices [Eq (B.57)], we have

$$\int \mathcal{D}x \exp\left[-\frac{1}{2}\int dt \int dt' x(t)\hat{K}(t,t')x(t')\right] = \exp\left[-\frac{1}{2}\mathrm{Tr}\ln\hat{K}\right]. \tag{B.59}$$

Our only problem is to say what we mean by $\mathrm{Tr}\ln\hat{K}$. Since $\hat{K}$ is an operator, we can ask for its spectrum, that is the eigenvalues and eigenfunctions. This means that we need to solve the equations

$$\int_0^T dt' \hat{K}(t,t')u_\mu(t') = \Lambda_\mu u_\mu(t), \tag{B.60}$$

where we are careful here to note that we are working in window $0 < t < T$. In the basis formed by the eigenfucntions, of course $\hat{K}$ is diagonal. As with matrices, when an operator is diagonal we can take the log element by element, and then computing the trace requires us to sum over these diagonal elements; recall that traces and determinants are invariance, se we can use this convenient basis and not worry abotu generality. Thus,

$$\mathrm{Tr}\ln\hat{K} = \sum_\mu \ln\Lambda_\mu. \tag{B.61}$$

How does this work for our case? First, we need to identify the operator $\hat{K}$. In the exponential of $P[x(t)]$ we have

$$\int dt \left[\frac{\tau_c}{2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2\tau_c}x^2(t)\right].$$

To get this into a more standard form we need to integrate by parts,

$$\int dt \left[\frac{\tau_c}{2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2\tau_c}x^2(t)\right] \quad = \quad \int dt\, x(t)\left[-\frac{\tau_c}{2}\frac{d^2}{dt^2} + \frac{1}{2\tau_c}\right]x(t). \tag{B.62}$$

We now see that our integral for Z in Eq (B.55) can we written

$$\begin{aligned} Z &= \int \mathcal{D}x \, \exp\left[-\frac{1}{4\tau_c}\int dt\,\left(\tau_c^2\dot{x}^2(t) + x^2(t)\right)\right], \\ &= \int \mathcal{D}x \, \exp\left[-\frac{1}{2}\int dt' \int dt x(t')\hat{K}(t',t)x(t)\right] \quad \text{(B.63)} \\ \hat{K}(t',t) &= \delta(t'-t)\left[-\frac{\tau_c}{2}\frac{d^2}{dt^2} + \frac{1}{2\tau_c}\right]. \quad \text{(B.64)} \end{aligned}$$

This is a linear operator, and also time translation invariant (again). So we know that the eigenfunctions are $e^{-i\omega t}$, and since we are in a finite window of duration T we should use only those frequency components that 'fit' into the window, $\omega_\mathrm{n} = 2\pi\mathrm{n}/T$ for integer n. We have

$$\int_0^T dt\,\delta(t'-t)\left[-\frac{\tau_c}{2}\frac{d^2}{dt^2} + \frac{1}{2\tau_c}\right]e^{-i\omega_\mathrm{n}t} = \left(\frac{\tau_c\omega_\mathrm{n}^2}{2} + \frac{1}{2\tau_c}\right)e^{-i\omega_\mathrm{n}t'}, \tag{B.65}$$

so that the eigenvalues are

$$\Lambda(\omega_\mathrm{n}) = \left(\frac{\tau_c\omega_\mathrm{n}^2}{2} + \frac{1}{2\tau_c}\right) = \frac{1+(\omega_\mathrm{n}\tau_c)^2}{2\tau_c}. \tag{B.66}$$

Notice that these are just the inverses of the power spectrum,

$$\Lambda(\omega_\mathrm{n}) = \frac{1}{S_x(\omega_\mathrm{n})}. \tag{B.67}$$

This makes sense, of course, when we look back at Eq (B.44).

To finish the calculation, we have

$$Z = \exp\left[-\frac{1}{2}\sum_{\mu}\Lambda_\mu\right] \quad \text{(B.68)}$$

$$= \exp\left[-\frac{1}{2}\sum_{\mathrm{n}}\ln\left(\frac{1}{S_x(\omega_{\mathrm{n}})}\right)\right] \quad \text{(B.69)}$$

$$= \exp\left[\frac{1}{2}\sum_{\mathrm{n}}\ln S_x(\omega_{\mathrm{n}})\right]. \quad \text{(B.70)}$$

Finally, we need to do the sum. As the time window T becomes large, the spacing between frequency components, $\Delta\omega = 2\pi/T$, become small, and we expect that the sum approaches an integral.[1] Thus, for any function of ω_{n},

$$\sum_{\mathrm{n}} f(\omega_{\mathrm{n}}) = \frac{1}{\Delta\omega}\sum_{\mathrm{n}}\Delta\omega f(\omega_{\mathrm{n}}) \quad \text{(B.72)}$$

$$\to \frac{1}{\Delta\omega}\int d\omega\, f(\omega) \quad \text{(B.73)}$$

$$= T\int\frac{d\omega}{2\pi}f(\omega). \quad \text{(B.74)}$$

At last, this gives us

$$Z = \exp\left[\frac{T}{2}\int\frac{d\omega}{2\pi}\ln S_x(\omega)\right]. \quad \text{(B.75)}$$

Putting the pieces together, we have the probability distribution functional for a Gaussian $x(t)$,

$$P[x(t)] = \exp\left[+\frac{T}{2}\int_{-\infty}^{\infty}\ln S_x(\omega) - \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{|\tilde{x}(\omega)|^2}{S_x(\omega)}\right]. \quad \text{(B.76)}$$

Not every case we look at will be Gaussian, but this helps to get us started. In the interest of posting these notes sooner rather than later, I will stop

[1] There is an analogous result for summing over the states of particles in a box in quantum systems; recall that the states are labelled by their wavevector $\mathbf{k}$, and in three dimensions we have

$$\sum_{\mathbf{k}} \to V\int\frac{d^3k}{(2\pi)^3}, \quad \text{(B.71)}$$

where V is the volume of the box.

here. We should go back and see if there are other things we need as the course progresses.

Problem 51: Generality. We made an effort to evaluate Z in the specific case where $C_x(\tau) = e^{-|\tau|/\tau_c}$, but we wrote the final result in a very general form, Eq (B.76). Show that this slide into generality was justified.

Problem 52: Nonzero means and signal to noise ratios. We should be able to carry everything through in the case where the mean $x(t)$ is not zero. For example, if we just have background noise described by some spectrum $\mathcal{N}(\omega)$, then

$$P_{\rm noise}[x(t)] = \exp\left[+\frac{T}{2}\int_{-\infty}^{\infty} \ln\mathcal{N}(\omega) - \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{|\tilde{x}(\omega)|^2}{\mathcal{N}(\omega)}\right]. \quad (B.77)$$

If there is an added signal $x_0(t)$, the distribution functional becomes

$$P_{\rm signal}[x(t)] = \exp\left[+\frac{T}{2}\int_{-\infty}^{\infty} \ln\mathcal{N}(\omega) - \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{|\tilde{x}(\omega)-\tilde{x}_0(\omega)|^2}{\mathcal{N}(\omega)}\right]. \quad (B.78)$$

Suppose that you observe some particular $x(t)$, and you have to decide whether this came from the signal or noise distribution, that is, you have to decide whether the signal was present; for simplicity assume that the two possibilities are equally likely a priori. As discussed in Chapter 1, to make such decisions optimally you should use the relative probabilities that the signal or noise could give rise to your data. In particular, consider computing the "log likelihood ratio,"

$$\lambda[x(t)] \equiv \ln\left(\frac{P_{\rm signal}[x(t)]}{P_{\rm noise}[x(t)]}\right) \quad (B.79)$$

(a.) Give a simple expression for $\lambda[x(t)]$. Show that it is a linear functional of $x(t)$.

(b.) Show that, when the $x(t)$ are drawn at random out of either $P_{\rm signal}$ or $P_{\rm noise}$, $\lambda[x(t)]$ is a Gaussian random variable. Find the means, $\langle\lambda\rangle_{\rm noise}$ and $\langle\lambda\rangle_{\rm signal}$, and the variances $\langle(\delta\lambda)^2\rangle_{\rm noise}$ and $\langle(\delta\lambda)^2\rangle_{\rm signal}$, in the two distributions. Hint: you should see that $\langle(\delta\lambda)^2\rangle_{\rm noise} = \langle(\delta\lambda)^2\rangle_{\rm signal}$.

(c.) Sketch the distributions $P_{\rm noise}(\lambda)$ and $P_{\rm signal}(\lambda)$. Show that your ability to make reliable discriminations is determined only by the signal to noise ratio,

$$SNR = \frac{(\langle\lambda\rangle_{\rm signal} - \langle\lambda\rangle_{\rm noise})^2}{\langle(\delta\lambda)^2\rangle}, \quad (B.80)$$

and that we can write

$$SNR = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{|\tilde{x}_0(\omega)|^2}{\mathcal{N}(\omega)}. \quad (B.81)$$

(d.) In rod cells, a single photon produces a current pulse with the approximate form $x_0(t) = I_1(t/\tau)^3 e^{-t/\tau}$. The power spectrum of continuous background noise is approximately $\mathcal{N}(\omega) = A/[1+(\omega\tau)^2]^2$, with the same value of τ. Evaluate the peak current, $I_{\rm peak}$, and total variance of the background noise, σ_I^2. A naive estimate of the signal to noise ratio is just $SNR_{\rm naive} = (I_{\rm peak}/\sigma_I)^2$. Show that the optimal signal to noise ration, computed from Eq (B.81), is larger. Why?
