

Chapter 1

Newton's laws, chemical kinetics, ...

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When you look around you, you see many things changing in time. Our most powerful tools for describing such dynamics are based on differential equations. This mathematical approach to the description of nature started with mechanics, and grew to encompass other phenomena. In this section of the course, we'll introduce you to these ideas using what we think are the simplest examples. Following the historical path, we'll begin with mechanics, but we'll quickly see how similar equations arise in chemical kinetics, electric circuits and population growth. Sometimes the simple equations have simple solutions, but even these have profound consequences, such as understanding that most of the chemical elements in our solar system were created at some definite moment several billion years ago. In other cases simple equations have strikingly complex solutions, even generating seemingly random patterns. This is just a first look at this whole range of phenomena.

1.1 Starting with $F = ma$

By the time you arrive at the University, you have heard many things about elementary mechanics. In fact, much of what we cover in these first lectures are things you already know. We hope to emphasize several points: (1) Many of the things which you have may have remembered as isolated facts about the trajectories of objects really all follow from Newton's laws by direct calculation. (2) You need to take seriously the fact that Newton's

$F = ma$ is a differential equation. (3) Hidden inside some elementary facts that you learned in high school are some remarkably profound truths about the natural world. We won't have a chance to discuss their consequences, but we'd like to give you some flavor for these advanced but fundamental ideas.

Let us begin with Newton's famous equation,

$$F = ma. \tag{1.1}$$

At the risk of being pedantic, let's be sure we know what all the symbols mean. We all have an intuitive feeling for the mass m , although again we'll see that there is something underneath your intuition that you might not have appreciated. Acceleration is the clearest one: We describe the position of a particle as a function of time as $x(t)$, and then the velocity

$$v(t) = \frac{dx(t)}{dt} \tag{1.2}$$

and the acceleration

$$a(t) = \frac{d^2x(t)}{dt^2}. \tag{1.3}$$

As a warning, we'll sometimes write dx/dt and sometimes $dx(t)/dt$. These two ways of writing things mean the same thing; the second version reminds us that we are talking not about variables but about *functions*—algebra is about equations for variables, but now we have equations for functions. Alternatively we can say that equations like $F = ma$ are statements that are true at every instant of time, so really when we write $F = ma$ we are writing an infinite number of equations (!). This may not make you feel better.

We have defined all the terms in Newton's famous Eq. (1.1)—all except for the force F . The definition of force is a minor scandal.¹ As far as I know, there is no independent definition of force other than through $F = ma$. If you want to go out and measure a force you might arrange for that force to stretch a spring, then look how far it was stretched, and if you know the spring constant you can determine the force. But how did you measure the spring constant? You see the problem.

In effect what Newton did was to say that when we observe accelerations we should look for explanations in terms of forces. This embodies the Galilean notion of inertia, that objects in motion *tend to keep moving* and

¹See, for example, F Wilczek, Whence the force of $F = ma$? I: Culture shock, *Physics Today* **57**, 11–12 (2004); <http://www.physicstoday.org/vol-57/iss-10/p11.html>.

hence if they change their velocity there should be a reason. If it turns out that forces are arbitrarily complicated, then we're in deep trouble. In this sense, $F = ma$ is a framework for thinking about motion, and its success depends on whether the rules that determine the forces in different situations are simple and powerful.

Leaving aside these difficulties with the definition of force, Newton's law becomes a differential equation

$$m \frac{d^2 x(t)}{dt^2} = F. \quad (1.4)$$

To build up some intuition, and some practice with the mathematics, we will start with three simple cases: zero force, a constant force, and a force that is proportional to velocity. Of course these are not just simple examples, they actually correspond to situations that are fairly common in the real world and that you will study in the laboratory. Again you probably know much of what will be said here, but it's worth going through carefully and being sure you understand how it emerges from the differential equation.

These problems are designed to make you comfortable, once again, with the ideas from calculus that we will need in the next sections.

Problem 4: In Fig 1.1 we plot the velocity vs time $v(t)$ for an object moving in one dimension. Sketch the corresponding plots of position $x(t)$ and acceleration $a(t)$ vs time. If you need additional assumptions, please state them clearly. Be careful about units.

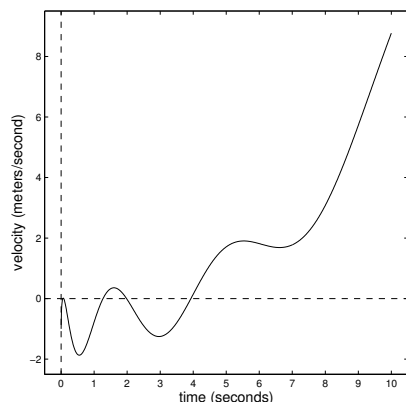


Figure 1.1: Velocity vs time for some hypothetical particle.

We are going to use MATLAB repeatedly in the course. Princeton students can go to <http://www.princeton.edu/licenses/software/matlab.xml> to find out about how to get started with their own computers; we'll also make sure that you get access to local computers that have MATLAB running on them. Hopefully, this problem is a good introduction. Note that you can type `help command` to get MATLAB to tell you how things work; for example, `help plot` will tell you something about those mysterious symbols such as `'k--'` below.

Problem 5: In fact the funny looking plot in Fig 1.1 corresponds to

$$v(t) = \sin(2\pi\sqrt{t}) + \left(\frac{t}{5}\right)^3 - \exp(-t/4). \quad (1.5)$$

(a.) Find analytic expressions for the position and acceleration as functions of time. You may refer to a table of integrals (or to its electronic equivalent), but you must give references in your written solutions.

(b.) Use MATLAB to plot your results in [a]. To get you started, here's a small bit of MATLAB code that should produce something like Fig 1.1:

```
t = [0:0.01:10];
v = sin(2*pi*sqrt(t)) + (t/5).^3 - exp(-t/4);
figure(1)
plot(t,v); hold on
plot([-1 11],[0 0],'k--',[0 0], [-3 10],'k--');
hold off
axis([-0.5 10.5 -2.5 9.5])
```

There are just two lines of math, and the rest is to make the graph and have it look nice. How do these plots compare with your sketches in the problem above?

Zero force

When there are no forces, $F = 0$, Eq (1.4) becomes

$$m \frac{d^2 x(t)}{dt^2} = 0. \quad (1.6)$$

Notice that this equation, as always with differential equations, is telling us about how things change from moment to moment. If we imagine knowing where things start, we should be able add up all the changes from this starting point (which we can call $t = 0$) until now (t). In this simplest of cases, “adding up all the changes” really is a matter of doing integrals.

Although professors sometimes forget this, it's important to be careful about limits when you do integrals. In this case, we want to know how things evolve from a starting moment until now, so all integrals should be

definite integrals from some initial time $t = 0$ up to now (t). Going carefully through the steps

$$\begin{aligned} m \frac{d^2 x(t)}{dt^2} &= 0 \\ \int_0^t dt m \frac{d^2 x(t)}{dt^2} &= \int_0^t dt [0] \end{aligned} \quad (1.7)$$

$$m \int_0^t dt \frac{d^2 x(t)}{dt^2} = \int_0^t dt [0] \quad (1.8)$$

$$m \left[\frac{dx(t)}{dt} \Big|_t - \frac{dx(t)}{dt} \Big|_{t=0} \right] = 0 \quad (1.9)$$

$$\frac{dx(t)}{dt} = \frac{dx(t)}{dt} \Big|_{t=0} \quad (1.10)$$

$$\frac{dx(t)}{dt} = v(0). \quad (1.11)$$

You should get in the habit of following these derivations with a pen in your hand, not just reading. Whenever we go through a long series of steps, you have to ask yourself both (a) if you understand where we are going and why, and (b) if you understand how we take each step. Near the start of the course, it seems best to lead you in this process, but by the end you should be doing it yourself. So, in this case, let's see how each step worked:

Eq (1.7) $\rightarrow$ (1.8) Since the mass m doesn't change with time (in this problem!) you can take it outside the integral.

Eq (1.8) $\rightarrow$ (1.9) Taking the integral of zero gives zero, while taking the integral of a derivative gives back the function itself.

Eq (1.9) $\rightarrow$ (1.10) Since the mass isn't zero, we can divide it through, and then rearrange.

Eq (1.10) $\rightarrow$ (1.11) Finally, since dx/dt is the velocity, we call $dx/dt|_{t=0} = v(0)$, the initial velocity.

What we have shown so far is that the velocity at time t is the same as at time $t = 0$: Objects in motion stay in motion, as promised.

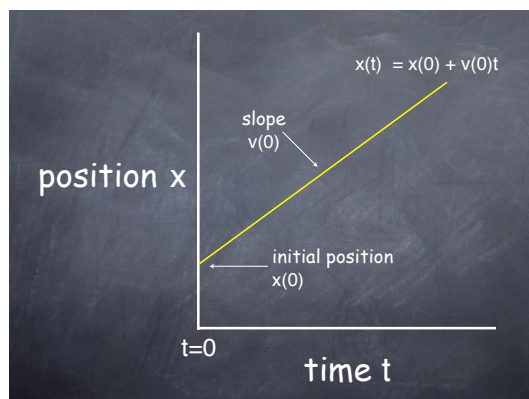


Figure 1.2: Trajectory of an object moving with zero force, from Eq. (1.14). Position vs. time is a straight line, with a slope equal to the initial velocity and an intercept equal to the initial position.

Now we go further, integrating once more:

$$\begin{aligned}\frac{dx(t)}{dt} &= v(0) \\ \int_0^t dt \frac{dx(t)}{dt} &= \int_0^t dt v(0)\end{aligned}\tag{1.12}$$

$$x(t) - x(0) = v(0)t\tag{1.13}$$

$$x(t) = x(0) + v(0)t.\tag{1.14}$$

What this shows is that if we plot position vs. time, we should find a straight line, as shown in Fig 1.2.

An important thing to remember is that position and force really are vectors. Thus if the (vector) force is equal to zero, then there is an equation like Eq (1.14) along each direction. As an example, in two dimensions we might write

$$x(t) = x(0) + v_x(0)t\tag{1.15}$$

$$y(t) = y(0) + v_y(0)t.\tag{1.16}$$

This is important, because the plot of x vs. t (which is what we solve for most directly!) is not what you see when you watch things move. What you actually see is something more like y vs. x as the object moves through space. In this case, if you plot $y(t)$ vs. $x(t)$, you get a straight line. You can see this by a little bit of algebra:

$$\begin{aligned}x(t) &= x(0) + v_x(0)t \\ x(t) - x(0) &= v_x(0)t\end{aligned}\tag{1.17}$$

$$\frac{x(t) - x(0)}{v_x(0)} = t \quad (1.18)$$

$$\Rightarrow y(t) = y(0) + v_y(0)t = y(0) + v_y(0) \cdot \frac{x(t) - x(0)}{v_x(0)} \quad (1.19)$$

$$y(t) = \frac{v_y(0)}{v_x(0)}x(t) + \left[y(0) - \frac{v_y(0)}{v_x(0)}x(0) \right], \quad (1.20)$$

and we recognize Eq (1.20) as the equation for a line with slope $v_y(0)/v_x(0)$. So motion without forces is motion at constant velocity, but also motion in a straight line.

Constant force

The standard example of motion with a constant force is the effect of gravity here on earth. This is a slight cheat, since of course the gravitational pull should depend on how far we are from the center of the earth. But if we do our experiments in a room (even a large room) it's hard to change this distance by more than a few meters, while the radius of the earth is measured in thousands of kilometers, so the changes in distance are only one part in a million. One can measure forces with enough accuracy to see such effects, but for now let's neglect them.

So, in the approximation that we don't move too far, and hence the pull of the earth's gravity is constant, we write

$$F = -mg, \quad (1.21)$$

with the convention that x is measured upward so that the downward force

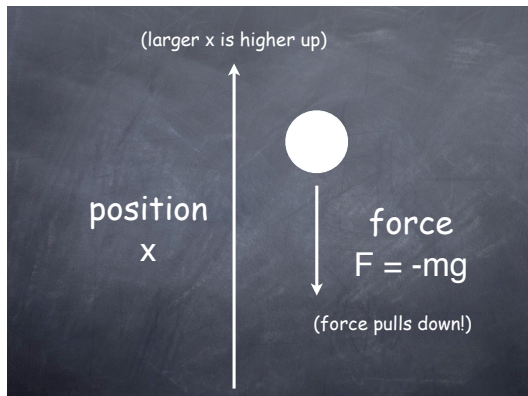


Figure 1.3: Setting up our coordinates for a particle moving under the influence of gravity, as in Eq (1.22).

of gravity is negative (Fig 1.3). Putting this together with $F = ma$, we have

$$m \frac{d^2 x(t)}{dt^2} = -mg. \quad (1.22)$$

The extraordinary thing is that the mass m appears on both sides of the equation, so we can cancel it, leaving

$$\frac{d^2 x(t)}{dt^2} = -g. \quad (1.23)$$

Now in this equation, $x(t)$ denotes the position of the object, and g is a property of the earth—none of the properties of the object appear in the equation! Even without solving the equation we thus make the prediction that all objects should fall toward the earth in exactly the same way, and this is what Galileo famously is supposed to have tested by dropping different objects from the Tower of Pisa and finding that they hit the ground at the same time.

The statement that every object falls in the same way obviously is wrong, as you know by watching leaves float and flutter to the ground. The idea is that all these differences arise from forces exerted by the air, and so if we could take these away and “purify” the effects of gravity we would really would see *everything* fall in the same way.² A number of science museums have beautiful demonstrations of this, with long tubes out of which they can pump all the air and then drop either a rock or a feather. Even if you know the principles it is pretty compelling to see a feather drop like a rock!

One might be tempted to think that our ability to cancel the masses in Eq (1.22) is an approximation. Perhaps. But in the 1950s here at Princeton, Robert Dicke and his colleagues did an amazing experiment to show that this approximation is accurate to about 11 decimal places. This certainly makes us think that what we have here is not an approximation but really something that one can call a law of nature.

Just so that you know all the words, the mass which appears in $F = ma$ is called the *inertial mass*, since this is what determines the inertia of an object. Inertia expresses the tendency of objects to keep moving in the absence of forces, and corresponds intuitively to the effort that we have to expend in stopping or deflecting the object. We also use inertia in everyday English to mean something quite similar, although not only in reference to mechanics. In contrast, the mass in $F = -mg$ is called the *gravitational mass*, for more obvious reasons. The statement that the masses cancel

²One should take a moment to appreciate Galileo's insight, separating these effects in his mind in advance of methods for doing the experiments.

thus is the “equivalence of gravitational and inertial masses,” or simply the “principle of equivalence.”

The essential content of the principle of equivalence is clear from Eq (1.23): You actually can’t tell the difference between a little extra acceleration (on the left hand side of the equation) and slightly stronger gravity (on the right). Einstein made the point in a thought experiment, imagining himself trapped in an elevator. Unable to see outside, he argued that he couldn’t tell the difference between falling freely in a gravitational field and being accelerated (e.g. by rocket jets attached to the elevator). From the Newtonian point of view, this equivalence is a coincidence. After all, there are other forces such as electricity and magnetism which aren’t proportional to mass, and thus one could have imagined that the gravitational force wasn’t proportional to mass either. Indeed, you may remember that when we go beyond the approximation of gravity as a constant force, if two objects with masses m_1 and m_2 are a distance r apart, then the force that one objects exerts on the other is given by

$$F = -\frac{Gm_1m_2}{r^2}, \quad (1.24)$$

where the minus sign indicates that the force is attractive, and G is a constant (called Newton’s constant). This is very much like Coulomb’s law for the force between two particles with charges q_1 and q_2 , again separated by a distance r ,

$$F = \frac{q_1q_2}{r^2}. \quad (1.25)$$

Thus, except for the constant, the masses act like “gravitational charges,” and it’s a mystery why the gravitational charge should be the same as the mass in $F = ma$.

In 1905, Einstein wrote a series of papers that shook the world—on what we now call the special theory of relativity, on the idea that energy carried by light is quantized into photons, and on Brownian motion and the size of atoms. Fresh from these triumphs, he decided that the mysterious coincidence between inertial and gravitational masses was a central fact about nature, indeed *the* central fact that needed his attention, and he set out to construct a theory of gravity in which the principle of equivalence is fundamental. It took him a decade, but the result was the general theory of relativity, arguably the greatest among his many great achievements. As you may have heard, general relativity involves a radical rethinking of our ideas about space and time and predicts the existence of black holes, the expansion of the universe, and other astonishing (but true!) things. We aren’t ready

for all this ... so reluctantly we will go back to the more mundane falling of things to the ground. But for now we'd like you to remember that when you read about the black hole in the center of our galaxy, the theory which predicts the existence of these exotic objects grew out of Einstein's taking *very* seriously a seemingly simple and obvious coincidence in the physics of everyday objects.

So, back to Eq (1.23). By now it should be clear what to do—integrate twice, as in the case of zero force:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -g \\ \int_0^t dt \frac{d^2x(t)}{dt^2} &= \int_0^t dt [-g]\end{aligned}\tag{1.26}$$

$$\left. \frac{dx(t)}{dt} - \frac{dx(t)}{dt} \right|_{t=0} = -gt\tag{1.27}$$

$$\frac{dx(t)}{dt} = \left. \frac{dx(t)}{dt} \right|_{t=0} - gt\tag{1.28}$$

$$\frac{dx(t)}{dt} = v(0) - gt\tag{1.29}$$

$$\int_0^t dt \frac{dx(t)}{dt} = \int_0^t dt [v(0) - gt]\tag{1.30}$$

$$x(t) - x(0) = v(0)t - \frac{1}{2}gt^2\tag{1.31}$$

$$x(t) = x(0) + v(0)t - \frac{1}{2}gt^2.\tag{1.32}$$

Thus we recover the $\frac{1}{2}gt^2$ that you all remember from high school.

Once again, $x(t)$ is not something you literally “see,” since it is what you get by plotting position vs. time. On the other hand, position and force are both vectors, as noted above, but gravity only acts along one dimension (up/down). So if x is the up/down direction and y is measured parallel to the surface of the earth—opposite the usual convention!—then x obeys Eq (1.32) while y obeys Eq (1.14):

$$x(t) = x(0) + v_x(0)t - \frac{1}{2}gt^2\tag{1.33}$$

$$y(t) = y(0) + v_y(0)t.\tag{1.34}$$

But nobody told you where you should put $y = 0$, so you might as well choose this point so that $y(0) = 0$. Then the position y is proportional to t , and hence plotting x vs. y is just like plotting x vs. t except for the units

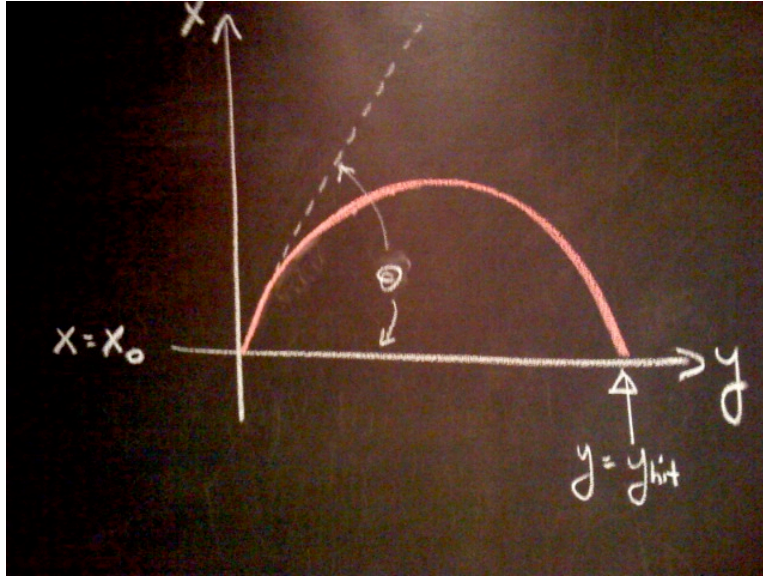


Figure 1.4: Launching an object from the ground. Initial position is $[x(0), y(0)]$, chosen for convenience as $(0, 0)$. Initial velocity launches the object in a direction θ , and the object returns to $x = 0$ at some point y as in Eq. (1.44).

on the horizontal axis. Thus one of the nice things about the trajectories of objects in our immediate environment is that distance parallel to the earth provides a surrogate for time, and we can literally see the trajectories played out in front of us. In particular, this means that when you throw something it follows a parabolic trajectory.

It's worth going through the algebra of the parabolic trajectory, choosing $y(0) = 0$ as suggested:

$$y(t) = v_y(0)t \quad (1.35)$$

$$t = \frac{y(t)}{v_y(0)} \quad (1.36)$$

$$x(t) = x(0) + v_x(0)t - \frac{1}{2}gt^2 = x(0) + v_x(0)\frac{y(t)}{v_y(0)} - \frac{1}{2}g\left[\frac{y(t)}{v_y(0)}\right]^2 \quad (1.37)$$

$$x = x(0) + \left[\frac{v_x(0)}{v_y(0)}\right] \cdot y - \left[\frac{g}{2v_y^2(0)}\right] \cdot y^2. \quad (1.38)$$

I hope it's clear that this is a parabola.

Standard questions at this point are of the following sort: How far along the y axis does the object go before hitting the ground? To answer this question you choose the ground to be at $x = 0$ and solve for the value of $y = y_{\text{hit}}$ that results in $x = 0$. This is especially simple if the object *starts* at $x = 0$, which kind of makes sense if you fire a rocket off the ground (see Fig 1.4). Then $x(0) = 0$, and the condition $x = 0$ is equivalent to

$$0 = \left[\frac{v_x(0)}{v_y(0)} \right] \cdot y_{\text{hit}} - \left[\frac{g}{2v_y^2(0)} \right] \cdot y_{\text{hit}}^2 \quad (1.39)$$

$$= y_{\text{hit}} \left[\frac{v_x(0)}{v_y(0)} - \left(\frac{g}{2v_y^2(0)} \right) y_{\text{hit}} \right]. \quad (1.40)$$

So one solution is that the object is on the ground at $y = 0$, but this is where we start (remember that we chose $y(0) = 0$). So the interesting solution is found by dividing through by y_{hit} ,

$$\begin{aligned} 0 &= y_{\text{hit}} \left[\frac{v_x(0)}{v_y(0)} - \left(\frac{g}{2v_y^2(0)} \right) y_{\text{hit}} \right] \\ &= \frac{v_x(0)}{v_y(0)} - \left(\frac{g}{2v_y^2(0)} \right) y_{\text{hit}} \end{aligned} \quad (1.41)$$

$$y_{\text{hit}} = \frac{2v_x(0)v_y(0)}{g}. \quad (1.42)$$

This is the answer, but it's a little messy, so we'll see if we can simplify.

We see that that, from Fig 1.4, $v_x(0) = v(0) \sin \theta$, where $v(0)$ is the initial speed of the object and θ is the angle that its initial velocity makes with the ground; $\theta = \pi/2$ corresponds to shooting the object straight up and $\theta = 0$ corresponds to skimming along the ground. Similarly $v_y(0) = v(0) \cos \theta$, so that the particle hits the ground at

$$y = \frac{2v_x(0)v_y(0)}{g} = \frac{2v^2(0) \sin \theta \cos \theta}{g}. \quad (1.43)$$

But you may recall that $\sin(2\theta) = 2 \sin \theta \cos \theta$, so we have

$$y = \frac{v^2(0)}{g} \sin(2\theta), \quad (1.44)$$

which is a nice, compact result.

Problem 6: Use Eq (1.38) to find the maximum height that the object reaches along its trajectory. Recall that to find the maximum of a function you find the place where the

derivative is zero. Notice that in this case you are looking for the maximum value of x viewed as function of y , opposite the usual conventions in textbooks. You should be able to do the same calculation directly from Eq (1.32). Show that you get the same answer.

Perhaps you have seen Eq (1.44) before, in your high school course. What is important here is to emphasize that this, like all the other formulae of mechanics, are *derivable* from Newton's laws. If we had to remember a different formula for each different situation, it wouldn't really be much of a science. The great achievement of our scientific culture is to have a small set of principles from which everything can be worked out.

Drag forces

When you move your arm through the water you feel a force opposing the motion. Part of this force is the inertia of the water that you are moving, but if you go very slowly then the dominant component is the *drag* generated by the viscosity of the water, and this force is proportional to the velocity v . The sign of the force is to oppose motion, so we write $F_{\text{drag}} = -\gamma v$, where γ is called the drag coefficient.

Problem 7: Imagine that we have two flat parallel plates, each of area A , separated by a distance L , and that this space is filled with fluid. If we slide the plates relative to each other slowly, at velocity v (parallel to plates), then we will find that there is a drag force $F_{\text{drag}} = -\gamma v$ which acts to resist the motion. Intuitively, the bigger the plates (larger A) and the closer they are together (smaller L) the larger the drag, and in fact over a range of interesting scales one finds experimentally that $\gamma = \eta A/L$, where the proportionality constant η is called the viscosity of the fluid.

(a.) What are the units of viscosity? Instead of expressing your answer in terms of force, length and time, try to express the viscosity as a combination of energy, length and time.

(b.) Viscosity is something we can measure (and “feel”) on a macroscopic scale. But the properties of a fluid depend on the properties of the molecules out of which it is made. So if we want to understand why the viscosity of water is $\eta = 0.01$ in the cgs (centimeter–gram–second) system of units, we need to think about the scales of energy, length and time that are relevant for the water molecules. Plausibly relevant energy scales are the energies of the hydrogen bonds between the water molecules (which you can look up), and the thermal energy $k_B T \sim 4 \times 10^{-21}$ J at room temperature, which is the average kinetic energy of molecules as they jiggle around in the fluid (more about this later in the semester). The characteristic length is the size of an individual water molecule, or the distance between molecules. What is the range of time scales that combines with these

energies and volume to give the observed viscosity? What do you think this time scale means—i. e., what event actually happens on this time scale?

Newton's basic equation

$$m \frac{d^2 x(t)}{dt^2} = F \quad (1.45)$$

can also be written as

$$m \frac{dv(t)}{dt} = F, \quad (1.46)$$

which in this case becomes

$$m \frac{dv(t)}{dt} = -\gamma v(t). \quad (1.47)$$

Here I am being careful to show you that v is a function that depends on time.

It is often said that there are three good ways to solve a differential equation. Best is to ask someone who knows the answer. Next one guesses the form of the solution and checks that it is correct. Finally, there are some more systematic approaches. Let's try one of these, largely so we can build up our intuition and make better guesses next time we need them!

We'd like to solve Eq (1.47) the same way that we did in previous cases, by integrating, but this doesn't work directly—on the right hand side we'd have to integrate $v(t)$ itself, and clearly we don't know how to do this. So we play a little with the equation, doing something which would make a real mathematician cringe:

$$\begin{aligned} m \frac{dv}{dt} &= -\gamma v \\ \frac{dv}{dt} &= -\frac{\gamma}{m} v \end{aligned} \quad (1.48)$$

$$\frac{dv}{v} = -\frac{\gamma}{m} dt. \quad (1.49)$$

Now we can integrate, since on the left we have v and on the right we have dt , with no mixing. Again we should be careful to do definite integrals from

some initial time $t = 0$ up until now (t), during which time the velocity runs from its initial value $v(0)$ to its current value $v(t)$:³

$$\begin{aligned}\frac{dv}{v} &= -\frac{\gamma}{m}dt \\ \int_{v(0)}^{v(t)} \frac{dv}{v} &= -\frac{\gamma}{m} \int_0^t dt\end{aligned}\tag{1.50}$$

$$[\ln v] \Big|_{v(0)}^{v(t)} = -\frac{\gamma}{m}t\tag{1.51}$$

$$\ln v(t) - \ln v(0) = -\frac{\gamma}{m}t\tag{1.52}$$

$$\ln \left[\frac{v(t)}{v(0)} \right] = -\frac{\gamma}{m}t\tag{1.53}$$

$$v(t) = v(0)e^{-\gamma t/m}.\tag{1.54}$$

Thus the solution is an *exponential decay*.

Let's be sure we understand the steps leading to Eq (1.54):

Eq (1.50) $\rightarrow$ (1.51) On the right hand side we just use $\int dt = t$, and on the left we use $\int \frac{dv}{v} = \ln v$, where $\ln$ denotes the natural logarithm. Note that this is why natural logarithms are natural!

Eq (1.51) $\rightarrow$ (1.52) This is just evaluating the indefinite integral at it's endpoints.

Eq (1.52) $\rightarrow$ (1.53) Now we use $\ln a - \ln b = \ln(a/b)$.

Eq (1.53) $\rightarrow$ (1.54) Finally, to get rid of the logarithm we exponentiate both sides of the equation. We are using $\ln(e^x) = x$, or equivalently $e^{\ln x} = x$.

Another way of writing our result in Eq (1.54) is

$$v(t) = v(0)e^{-t/\tau},\tag{1.55}$$

³It's interesting that notice that we don't actually know the value of $v(t)$; indeed this is what we are trying to find. Nonetheless we can put this value as the endpoint of our integral, and solve at the end.

where the time constant $\tau = m/\gamma$. We can see that this is the characteristic time scale in the problem by going back to the original equation:

$$\begin{aligned} m \frac{dv(t)}{dt} &= -\gamma v(t) \\ \frac{m}{\gamma} \frac{dv(t)}{dt} &= -v(t). \end{aligned} \tag{1.56}$$

The combination $\tau = m/\gamma$ must be a time scale in order to balance the units on either side of the equation. This “characteristic time scale” is the only term in the equation that has the units of time, and thus we expect that when we plot the solution we will see all the important variations occurring on this time scale. This is an important idea—we can say on what scale we expect to see things happen even before we solve the equation—and we will come back to it several times in the course.

This is a good place to remind ourselves of a special feature of the exponential function. With $v(t) = v(0) \exp(-t/\tau)$, there is a unique time $t_{1/2}$ such that v is reduced by a factor of two:

$$v(t_{1/2}) \equiv \frac{1}{2}v(0) \tag{1.57}$$

$$v(0) \exp(-t_{1/2}/\tau) = \tag{1.58}$$

$$\exp(-t_{1/2}/\tau) = 1/2 \tag{1.59}$$

$$-t_{1/2}/\tau = \ln(1/2) \tag{1.60}$$

$$t_{1/2}/\tau = \ln(2) \tag{1.61}$$

$$t_{1/2} = \tau \ln 2. \tag{1.62}$$

So as t runs from 0 up to $t_{1/2}$, the velocity goes down by a factor of two. The special feature of the exponential function is that when t advances further, from $t_{1/2}$ to $2 \times t_{1/2}$, the velocity goes down by another factor of two. Thus whenever a time $t_{1/2}$ elapses, the velocity falls to half its value. For this reason we can call $t_{1/2}$ the *half life*: this is the time for the velocity to fall by half, no matter what velocity we start with. More generally, if we look at the evolution from time t to $t + T$, it “looks the same” no matter what point in time t we start with, as long as we rescale the initial value of the function—the change over a window of time T depends on duration of the window (T), not on *when* we look (t). This is illustrated in Fig 1.5.

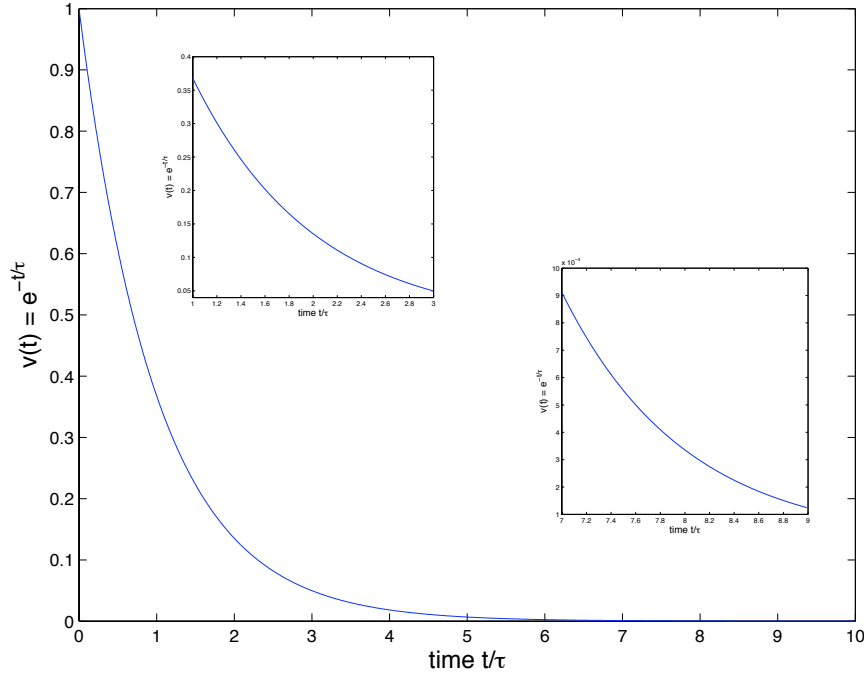


Figure 1.5: Exponential decay, as in Eq (1.55) with $v(0) = 1$. In the insets we focus on two windows of time that have a duration of $T = 2\tau$, starting at different moments. You see that, once we set the scale on the y -axis correctly, the plots look the same.

Problem 8: Consider the motion of a particle subject to a drag force, as in the experiments you are doing in the lab. In the absence of any other forces (including, for the moment, gravity), Newton's equation $F = ma$ can be written as

$$M \frac{dv}{dt} = -\gamma v, \quad (1.63)$$

where M is the mass of the particle and γ is the drag coefficient; we assume that the velocities are small, so the drag force is proportional to the velocity. For a spherical particle of radius r in a fluid of viscosity η , we have the Stokes' formula, $\gamma = 6\pi\eta r$. Assume that the particle also has a mass density of ρ . As shown above, the solution to Eq (3.3) is an exponential decay: $v(t) = v(0) \exp(-t/\tau)$, where the time constant τ determined by all the other parameters in the problem. Be sure that you understand this before doing the rest of this problem!

(a.) Write the time constant τ in terms of M and γ . How does τ scale with the radius of the particle?

(b.) Suppose that the density ρ is close to that of water, and that the relevant viscosity is also that of water. What value (in seconds) do you predict for the time constant τ when the particle has a radius $r \sim 1$ cm? What about $r \sim 1$ mm or $r \sim 10 \mu\text{m}$? Be careful about units!

(c.) A bacterium like *E coli* is approximately a sphere with radius $r = 1 \mu\text{m}$. Will you ever see the bacterium moving in a straight line because of its inertia?

(d.) What is the relationship between the position $x(t)$ and the velocity $v(t)$? Given that $v(t) = v(0)e^{-t/\tau}$, find a formula for $x(t)$ and sketch the result. Label clearly the major features of your sketch. What happens at long times, $t \gg \tau$?

(e.) *E coli* can swim at a speed of $\sim 20 \mu\text{m/s}$. Imagine that the motors which drive the swimming suddenly stop at time $t = 0$. Now there are no forces other than drag, but the bacterium is still moving at velocity $v(0) = 20 \mu\text{m/s}$. How far will the bacterium move before it finally comes to rest?

Problem 9: Let's try to use these same ideas to describe the motion of a person through a swimming pool. Once again the fluid is water, and the density of the "object" is also close to that of water. When a person curls up into a ball, they have a radius of about 50 cm (a meter in diameter). If a person starts moving at speed v_0 through a swimming pool while in this position, then by analogy with the previous problem, what is your prediction about how long it will take for their velocity to fall from v_0 down to $v_0/2$? Does this make sense given your own experience in the water? If not, what do you think has gone wrong? We know that none of you are spherical. You'll have to decide if this is a key issue, or if these calculations are wrong even for the case of the spherical student.

A very different sort of drag arises when objects move more rapidly. Although this isn't the same sort of rigorously justifiable approximation as $F_{\text{drag}} = -\gamma v$, one often finds that drag forces are roughly proportional to the square of the velocity at higher velocities. One then has to be careful about the sign of the force; if the velocity is positive then the force is negative, opposing the motion, so we'll write $F_{\text{drag}} = -cv^2$. Then $F = ma$ becomes

$$m \frac{dv(t)}{dt} = -cv^2(t). \quad (1.64)$$

We proceed as before to integrate the equation:

$$m \frac{dv}{dt} = -cv^2$$

$$\frac{dv}{dt} = -\left(\frac{c}{m}\right)v^2 \quad (1.65)$$

$$\frac{dv}{v^2} = -\left(\frac{c}{m}\right)dt \quad (1.66)$$

$$\int_{v(0)}^{v(t)} \frac{dv}{v^2} = -\left(\frac{c}{m}\right) \int_0^t dt \quad (1.67)$$

$$\left[-\frac{1}{v} \right] \Big|_{v(0)}^{v(t)} = -\left(\frac{c}{m}\right)t \quad (1.68)$$

$$-\frac{1}{v(t)} + \frac{1}{v(0)} = -\left(\frac{c}{m}\right)t \quad (1.69)$$

$$\frac{1}{v(0)} + \left(\frac{c}{m}\right)t = \frac{1}{v(t)} \quad (1.70)$$

$$v(t) = \frac{1}{\frac{1}{v(0)} + \left(\frac{c}{m}\right)t} \quad (1.71)$$

$$= \frac{v(0)}{1 + [cv(0)/m]t}. \quad (1.72)$$

It is convenient to write this as

$$v(t) = \frac{v(0)}{1 + t/t_c}, \quad (1.73)$$

where $t_c = m/[cv(0)]$ is the time at which the velocity has fallen to half of its initial value. Notice that we don't really have a half life in the way that we do for the exponential decay, because this time for falling by half depends on where we start.

Problem 10: Go through the derivation from Eq (1.64) to (1.73) and explain what happens at each step. The strategy for solving the equation is the same as before, but the details are different.

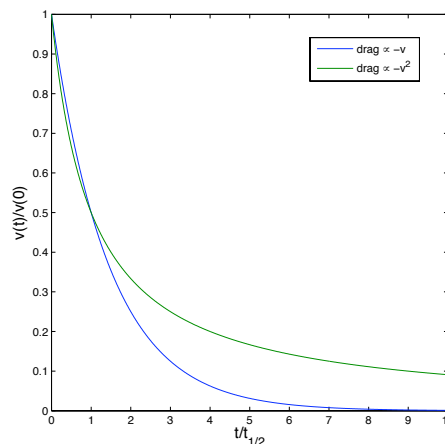
Figure 1.6 shows the solutions for both $F_{\text{drag}} = \gamma v$ and $F_{\text{drag}} = -cv^2$, with parameters chosen so that the time to reach half of the initial velocity is the same in both cases. Notice that the behavior at small times is quite similar, but that real differences appear at long times.

It's worth playing with these results, and seeing how the two cases differ, because the same equations arise in thinking about different chemical kinetic schemes, as we'll see in the next section. One interesting point to think about: If we look at the case where $F_{\text{drag}} = -cv^2$, then at long times

$$v(t) = \frac{v(0)}{1 + [cv(0)/m]t} \rightarrow \frac{v(0)}{[cv(0)/m]t} = \frac{m}{ct}. \quad (1.74)$$

Thus, after a while ($t \gg t_{1/2}$), the velocity still is decaying with time but the actual value doesn't depend any more on the velocity that we started with!

Figure 1.6: Time dependence of velocity for particles experiencing fluid drag. When the drag force is proportional to velocity, the decay is exponential, $v(t) = v(0) \exp(-t/\tau)$, as in Eq (1.55), where $t_{1/2} = \tau \ln 2$. When the drag force is proportional to velocity squared, the decay is asymptotically $\propto 1/t$, as in Eq (1.73).



One last point: when do we expect to see the drag be linear, and when do we expect it will go as the square of the velocity? This is a great question, and you'll be addressing it in the lab, so we'll leave it for now.

This problem is about an object falling under the influence of gravity, and hence fits with the text a few paragraphs back. It is, however, a bit more open ended than the previous problems, so we place it here at the end of our introduction to $F = ma$.

Problem 11: A simple model of shooting a basketball is that the ball moves through the air influenced only by gravity, so we neglect air resistance. Let's also simplify and not worry about the rotation of the ball, so the dynamics is described just by its position as a function of time. Choose coordinates so the basket is at position $x = 0$ and at a height $y = h$ above the floor (in fact $h = 10$ ft, but it's best in these problems not to plug in numbers until the end). When a player located at $x = L$ shoots the ball, it leaves his or her hand at a speed v and at an angle θ measured from the floor (i.e., $\theta = \pi/2$ would be shooting straight up, $\theta = 0$ would correspond to throwing the ball horizontally, parallel to the floor). Assume that the shooter is standing still, and the release of the ball happens at some initial height $y = h_0$ above the floor (in practice h_0 is somewhere between 5 and 7 ft, depending on who's playing).

(a.) Draw a diagram that represents everything you know about the problem, labeling things with all the right symbols. Notice that we are treating this as a problem in two dimensions, whereas of course the real problem is three dimensional.

(b.) What is the equation for the trajectory of the ball with as a function of time after the player releases it? Write your answer as $x(t)$ and $y(t)$, with $t = 0$ the moment of release.

(c.) A perfect shot must arrive at the point $x = 0$, $y = h$ at some time. Presumably the ball also has to be traveling downward at this time. Express these conditions as equations

that constrain the trajectory $\{x(t), y(t)\}$, and solve to find allowed values of the speed v and angle θ .

(d.) Saying that the ball must be traveling downward might not be enough. In fact the ball has radius $r = 4.5''$ and the basket has radius $R = 9''$. Continuing with the assumption that we want the ball to pass perfectly through the center of the basket (that is, $x = 0, y = h$), what is the real condition on the trajectory?

(e.) The fact that the basket is bigger than the ball means that you don't have to have x exactly equal to zero when $y = h$. To keep things simple let's assume that the shot still will go so long as we get within some critical distance $|x| < x_c$ at the moment when $y = h$. Given what you know so far, what is a plausible value of x_c ? Turn this condition on the end of the trajectory into a range of allowed values for v and θ . With typical values for L (think about what these are, or go out to a basketball court and measure!), how accurately does someone need to control v and θ in order to make the shot?

(f.) What we have done here is oversimplified. You are invited to see how far you can go in making a more realistic calculation.⁴ Some things to think about are the third dimension (e.g., how accurately does the trajectory need to be "pointed" toward the basket?), and a more careful treatment of the ball going through the hoop so that you can state more precisely the condition for making the shot. If you were really ambitious you could think about shots that bounce off the backboard, but that's probably too much for now!

⁴You might reasonably ask why we care. The fact that people (well, some people, at least) can make these shots with high probability from many different distances is telling us something about ability of the brain to deliver precise motor commands to our muscles, since it is the action of our muscles that determine the initial conditions of the ball leaving the hand of the shooter. Although the mechanisms are biological, the constraints are physical. Exploring the constraints makes precise what the system must do in order to achieve the observed level of performance.

