Chapter 2

Resonance and response

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In this section of the course we begin with a very simple system—a mass hanging from a spring—and see how some remarkable ideas emerge. We will see, for example, that it is useful to use imaginary numbers to describe real things. Most importantly, we will understand how to describe the way systems respond to small perturbations, and this turns out to be very general. Following this path, our intuitive notions that something is stable or unstable can be given precise mathematical formulations. We will take all of this far enough to see how the ideas can be used in describing complex biological phenomena, from the switches that control the expression of genes to the electrical impulses that carry information throughout the brain.

2.1 The simple harmonic oscillator

We have been talking about mechanics problems in which there is (a) no force, (b) a constant force, or (c) a force proportional to the velocity. The other "simple" case is when the force is proportional to the position, as is the case when we stretch a spring. Notice that we do these simple cases not because we want to torture you with simplified problems that are irrelevant in nature, but rather because, from our discussion of Taylor series and "laws" like Hooke's law or Ohm's law, we know that these simple cases are the leading approximations to more complex situations. Hopefully this will be clear before too long.

So, let us consider, as in Fig 2.1, a mass M hanging at the end of a spring with stiffness κ; you can imagine either that the system lies on its

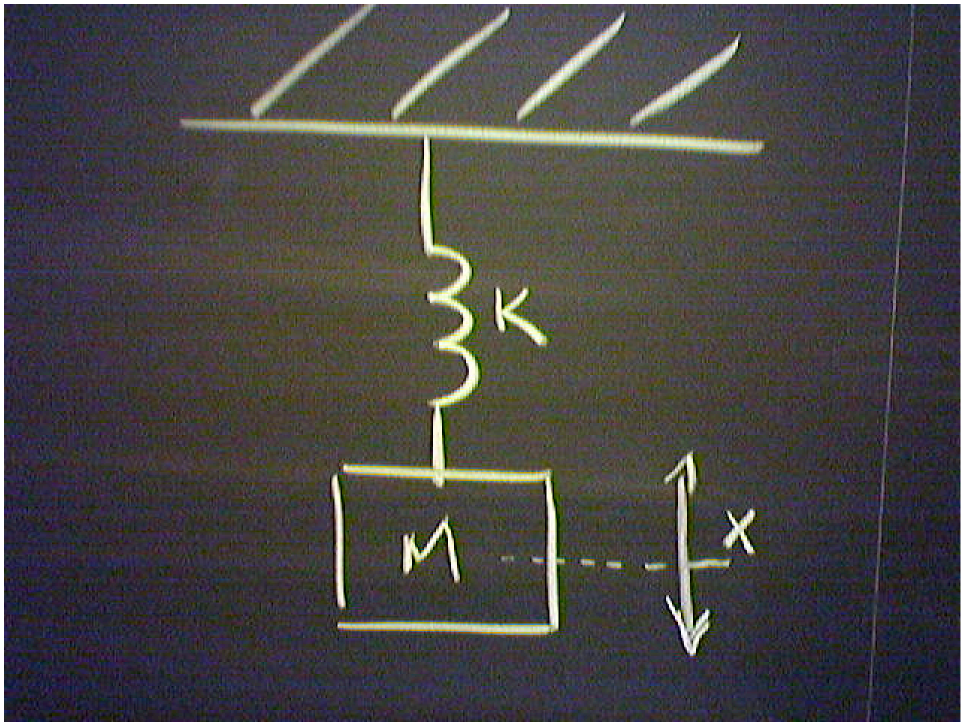

Figure 2.1: A mass M bound by a spring of stiffness κ, as in Eq (2.2).

side, or that we ignore the force of gravity. If we measure the position x of the spring in coordinates such that the equilibrium position is $x = 0$,[1] then the force on the mass is

$$F = -\kappa x. \tag{2.1}$$

In this problem $F = ma$ therefore corresponds to the differential equation

$$M\frac{d^2x(t)}{dt^2} = -\kappa x(t). \tag{2.2}$$

This is an example of a system usually called a *simple harmonic oscillator*, for reasons that I hope will become clear as we go along.

We will see, remarkably, that to give a full solution it is natural to write the real displacement of the mass in terms of complex numbers. Once we understand how to do this we will see that we can generate a rather complete

[1]You should convince yourself that we can include gravity just by redefining the zero point on the x axis.

view of the problem. It is important that the seemingly special problem of the harmonic oscillator comes up in many different guises.

Take a moment to think about what Eq (2.2) means. We can draw the function $x(t)$. At every point on the graph we can compute the local slope (the derivative or velocity) and then from this new graph we can compute the local slope again (the acceleration). Up to constants, Eq. (2.2) is telling us that this function we obtain by differentiating twice is just the negative of the original function $x(t)$—if we graph the second derivative and flip it upside down it should overly the original graph. Obviously not all functions have this property, and indeed you will learn in your math courses the very important theorem that (once we specify the initial conditions) the functions which satisfy differential equations are unique. This is crucial because it means that if we find *a* solution of a differential equation that satisfies all the initial conditions, even if we have to guess the form of the solution, then we're done, because there can't be any other solutions.

Equation (2.2) has a very simple form. Notice that there are two derivatives, so we say it is a second order equation. Further, the equation is linear, which means all the terms are proportional to x. The fact that equation is linear implies that the sum of two solutions is also a solution. This is a subtle idea, and we will come back to it. Finally, all the coefficients which appear in the equation are constants, with no explicit dependence on time. While we may not know how to solve all differential equations, we'll make a lot of progress on this important special class.

As noted previously, the best way to solve a differential equation is to ask someone who knows the answer. In this case, *you* know the answer from your calculus course. Recall that

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t) \tag{2.3}$$

$$\frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t). \tag{2.4}$$

Then if we take two derivatives we have

$$\frac{d^2}{dt}\sin(\omega t) = \frac{d}{dt}[\omega\cos(\omega t)] = -\omega^2\sin(\omega t) \tag{2.5}$$

$$\frac{d^2}{dt}\cos(\omega t) = \frac{d}{dt}[-\omega\sin(\omega t)] = -\omega^2\cos(\omega t). \tag{2.6}$$

Thus, sine and cosine have the properties of the function that we are looking for: when you differentiate twice, you get back something proportional to the function itself, with a minus sign.

To be more explicit, let's rewrite Eq (2.2), dividing through by the mass:

$$\frac{d^2x(t)}{dt^2} = -\left(\frac{\kappa}{M}\right)x(t). \tag{2.7}$$

Then we also have

$$\frac{d^2\sin(\omega t)}{dt^2} = -\omega^2\sin(\omega t), \tag{2.8}$$

which means that $x(t) = \sin(\omega t)$ is a solution to the equation, provided that we identify

$$\omega^2 = \kappa/M. \tag{2.9}$$

For the same reasons, $x(t) = \cos(\omega t)$ is also a solution, with the same value of ω.

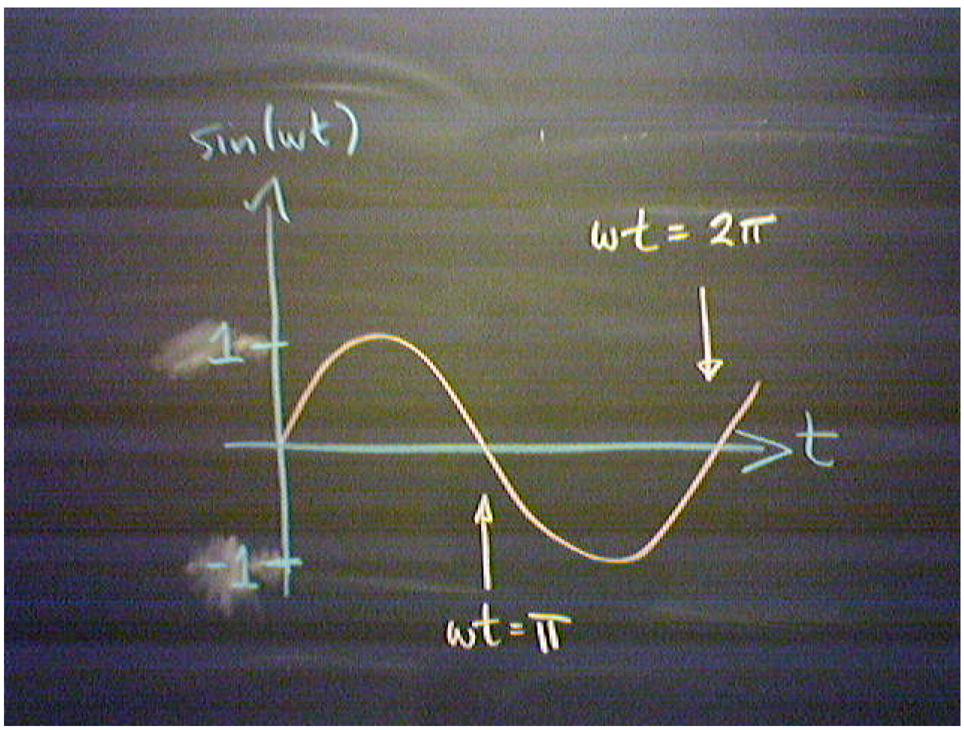

Figure 2.2: The function $\sin(\omega t)$.

Before proceeding it is worth remembering a few facts about sines and cosines. As shown in Fig 2.2, the sine function oscillates between +1 and

-1; when the time t shifts by an amount T such that $\omega T = 2\pi$, the function has the same value. We say that $T = 2\pi/\omega$ is the period of the oscillation and $f = 1/T = \omega/2\pi$ is the frequency. Sometimes we are sloppy and refer to ω as the frequency.

Problem 27: When a 50 kg person sits on top of a car, the car body moves down toward the ground by 3 inches. The car body itself weighs one ton.

(a.) What is the effective stiffness of the spring which supports the weight of the car? Use some useful set of units!

(b.) The stiffness comes from the shock absorbers. Suppose that there is no viscosity or damping in the shocks. Then if you are sitting on top of the car and suddenly jump off, the height of the car body should oscillate. What is the oscillation frequency?

What we have found is that $x(t) = \sin(\omega t)$ is "a solution" of the differential equation that encapsulates $F = ma$ for this system, and we have also found that $x(t) = \cos(\omega t)$ is "a solution." But have we found "the solution"? What do we mean by this? Why is there more than one solution?

Let's step back from the particular equations and think more generally. As emphasized by Laplace, Newtonian mechanics presents us with a dramatic view of the world in which, given the initial conditions, we can solve the differential equations describing the motion of all the relevant objects and hence predict the future with, it would seem, complete certainty. Obviously this depends on the differential equations having unique solutions—if we claim to be able to predict the trajectory of a falling object starting with $F = ma$, then this equation had better have a unique solution once we specify all the initial conditions. If the solutions weren't unique, then maybe each time we drop a ball something different would happen (!).

So the correct statement is that solutions of the differential equations are unique *once we specify all the initial conditions.* How many initial conditions are there? We saw in simple cases that we needed to specify the initial position and the initial velocity in order to integrate the equations of motion, and this is quite general. So when we talk about "a solution" we mean a solution that is consistent with *some* set of initial conditions; "the solution" means the solution consistent with the initial conditions in our particular physical setting.

What we might be able to do, and indeed what we might hope to do, is to write down a solution that has parameters, and show that by setting these parameters we can be consistent with any set of initial conditions. Then what we really have a is a family of solutions, and at this point we really are done with our problem—we have a whole family of functions $x(t)$, each of which solves the differential equation, and by picking the right member of this family we can agree with the initial position and velocity of the particle.

You have already seen this idea of *families of solutions* in the simplest case of zero force. Recall that in this case the position as a function of time is given by $x(t) = x(0) + v(0)t$. This function describes a straight line when we plot x vs t, but this straight line can have any slope [depending on $v(0)$] and any intercept [depending on $x(0)$)]. So really we have a family of lines, all of which solve the differential equation $F = ma$, but to pick one of them we need to know the initial position and initial velocity of the particle.

Our example of the mass on a spring is a little more complicated than the case of zero force, but again simpler than it could be. The simplicity here is that the equation is *linear*. That is, if we look at the terms in

$$\frac{d^2x(t)}{dt^2} = -\left(\frac{\kappa}{M}\right) x(t), \tag{2.10}$$

we see that both terms are proportional to x, and this is what we mean by linearity. There is a special consequence of linearity, and this is the *superposition* of solutions. Suppose that we have found one solution of our equation, a solution consistent with one set of initial conditions, and let's call this solution $x_1(t)$. Suppose also that we have found another solution, consistent with a different set of initial conditions, and let's call this $x_2(t)$. What is remarkable about linear equations is that now we can construct another function, $x(t) = Ax_1(t) + Bx_2(t)$, and this is also a solution, one which matches yet a third set of initial conditions. To see this, we can just check by substitution:

$$\frac{d^2x(t)}{dt^2} = \frac{d^2[Ax_1(t) + Bx_2(t)]}{dt^2} \tag{2.11}$$

$$= A\frac{d^2x_1(t)}{dt^2} + B\frac{d^2x_2(t)}{dt^2} \tag{2.12}$$

$$= -\left(\frac{\kappa}{M}\right) Ax_1(t) - \left(\frac{\kappa}{M}\right) Bx_2(t) \tag{2.13}$$

$$= -\left(\frac{\kappa}{M}\right) [Ax_1(t) + Bx_2(t)] \tag{2.14}$$

$$= -\left(\frac{\kappa}{M}\right) x(t), \tag{2.15}$$

where in going from Eq (2.12) to (2.13) we use the fact that $x_1(t)$ and $x_2(t)$ each are solutions, which means that $d^2x_1(t)/dt^2 = -(\kappa/M)x_1(t)$, and similarly for $x_2(t)$. Thus we see that we can use these two solutions to construct a whole family of new solutions just by linear combination. Notice that this argument doesn't depend on knowing the exact form of the solutions, since all we use is the linearity of the equation.

Why is this so important? By combining two solutions we generate a whole family of solutions, but these have two parameters, A and B. But we know that once we match the initial position and initial velocity, the solution is unique. Thus if we can adjust A and B to match the initial position and velocity, we are done: We have constructed the whole family of solutions that we need.

Let's see how this plays out with our particular example. We have seen that one possible solution to our problem is $x_1(t) = \sin(\omega t)$, and another is $x_2(t) = \cos(\omega t)$. Thus we can construct the linear combination

$$x(t) = A\sin(\omega t) + B\cos(\omega t), \tag{2.16}$$

and this should also be a solution. Now because $\sin(0) = 0$ and $\cos(0) = 1$, we can see that

$$x(0) = B. \tag{2.17}$$

If we differentiate to find the velocity,

$$v(t) \equiv \frac{dx(t)}{dt} = \frac{d[A\sin(\omega t) + B\cos(\omega t)]}{dt} \tag{2.18}$$

$$= A\omega\cos(\omega t) - B\omega\sin(\omega t), \tag{2.19}$$

so that

$$v(0) = A\omega. \tag{2.20}$$

So in this case the relationship between the coefficients A, B and the initial conditions is quite simple:

$$A = x(0), \tag{2.21}$$

$$B = \frac{v(0)}{\omega}. \tag{2.22}$$

Let's summarize what we have done:

- The differential equation $F = ma$ that describes a mass M hanging from a spring of stiffness κ is

$$M\frac{d^2x(t)}{dt^2} = -\kappa x(t). \tag{2.23}$$

- Solutions to this equation include

$$x_1(t) = \sin(\omega t), \text{ and} \tag{2.24}$$
$$x_2(t) = \cos(\omega t), \tag{2.25}$$

  where we have to choose

$$\omega = \sqrt{\frac{\kappa}{M}}. \tag{2.26}$$

- Because Eq (2.23) is linear, we can combine these solutions to form a family of solutions

$$x(t) = A\sin(\omega t) + B\cos(\omega t). \tag{2.27}$$

- Finally, we can adjust the constants A and B to match the initial position and initial velocity:

$$x(t) = \frac{v(0)}{\omega}\sin(\omega t) + x(0)\cos(\omega t). \tag{2.28}$$

You might want to play with this solution a little bit, plotting it and seeing what it looks like. We will come back and do this, but first let's look at a very different way of finding these solutions, one which is much more general.

Problem 28: Show that you can rewrite Eq (2.28) in the form

$$x(t) = A\cos(\omega t + \phi), \tag{2.29}$$

where A is called the amplitude and ϕ is called the phase of the oscillation. Draw (by hand, not with the computer!) the function $x(t)$, being careful to show units on both axes and marking the point where $t = 0$. Indicate the features of the graph that correspond to the amplitude, phase and frequency.

Recall our (only partly) joking idea that there are three ways to solve a differential equation. What we did last time was to ask someone who knew the answer—you knew the properties of sine and cosine from your calculus course, and you could see that this is what you needed to solve the equation. But you remember that when we looked at a mass moving through a fluid, or first order chemical kinetics, we also encountered linear differential equations with constant coefficients. The simplest equation in this class of interest is of the form

$$\frac{dx(t)}{dt} = ax(t), \tag{2.30}$$

where a is a constant. We found that we could solve this by guessing a solution of the form $x(t) = Ae^{\lambda t}$, and then everything works if we set $\lambda = a$ and $A = x(0)$. Can we use this same "guess and check" method (the second of the three methods) to solve the mass–spring problem?

To get started, suppose that we have a second order differential equation which I'll write in the suggestive form

$$\frac{d^2x}{dt^2} = a^2x. \tag{2.31}$$

You can see that $x(t) = Ae^{at}$ still is a solution. In fact, the exponential function has the property that differentiating is just multiplication by a constant:

$$\begin{aligned}
\frac{d}{dt}\exp(\lambda t) &= \lambda\exp(\lambda t) && (2.32)\\
\frac{d^2}{dt^2}\exp(\lambda t) &= \lambda^2\exp(\lambda t) && (2.33)\\
&\cdots && (2.34)\\
\frac{d^n}{dt^n}\exp(\lambda t) &= \lambda^n\exp(\lambda t). && (2.35)
\end{aligned}$$

Thus if we try to solve Eq. (2.31) by *guessing* a solution of the form $x(t) \propto \exp(\lambda t)$, we see that this will work if (and only if)

$$\lambda^2 = a^2, \tag{2.36}$$

which means that $\lambda = \pm a$. Now we use the idea of combining solutions to generate a whole family:

$$x(t) = A\exp(at) + B\exp(-at), \tag{2.37}$$

and we have to set the two constants A and B by fixing the initial position and initial velocity as in the discussion above. Again there are two arbitrary constants because we are looking at a second order equation.

All this is fine, but how do we use it to solve the equation we really are interested in, for the mass on a spring? As before we can write this as

$$\frac{d^2x(t)}{dt^2} = \left[-\frac{\kappa}{M}\right] x(t). \tag{2.38}$$

This is just like Eq. (2.31) if we identify $a^2 = -\kappa/M$. To put it another way, we can try to solve Eq. (2.38) by guessing that $x(t) = A\exp(\lambda t)$, and if we substitute we find

$$\begin{aligned}
\frac{d^2}{dt^2} A\exp(\lambda t) &= \left[-\frac{\kappa}{M}\right] A\exp(\lambda t) &(2.39)\\
A\lambda^2 \exp(\lambda t) &= \left[-\frac{\kappa}{M}\right] A\exp(\lambda t) &(2.40)\\
\lambda^2 &= \left[-\frac{\kappa}{M}\right] &(2.41)\\
\lambda^2 + \frac{\kappa}{M} &= 0. &(2.42)
\end{aligned}$$

Thus we see that λ obeys a quadratic equation, although a very simple one in this case. There are two solutions, and this is related to the fact that this is a second order equation:

$$\lambda = \pm\sqrt{-\frac{\kappa}{M}}. \tag{2.43}$$

Now we see something strange, namely that our solution is the square root of a negative number.

For most of you, at some early point in your education you were taught about square roots, and your teachers explained that you can't take the square root of a negative number. Then at some point in high school, perhaps, they told you that it's OK, but it gets a special name: $i = \sqrt{-1}$ is the unit imaginary number. So, we have the result that the solution of $F = ma$ for this problem must be of the form

$$x(t) = A\exp(+i\omega t) + B\exp(-i\omega t). \tag{2.44}$$

It's absolutely fantastic that imaginary numbers appear in the solution to a physics problem.