

2.3 Damping, phases and all that

If we imagine taking our idealized mass on a spring and dunking it in water (or, more dramatically, in molasses), then there will be a viscous friction or drag force which opposes the motion and is proportional to the velocity:²

$$M \frac{d^2 x(t)}{dt^2} = -\kappa x(t) - \gamma \frac{dx(t)}{dt}, \quad (2.129)$$

where κ is the spring constant as before and γ is the damping constant. The convention is to put all of these terms on one side of the equation,

$$M \frac{d^2 x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + \kappa x(t) = 0. \quad (2.130)$$

We're going to solve this using our trick of guessing a solution with the form $x(t) = A \exp(\lambda t)$. Recall that

$$x(t) = A e^{\lambda t} \quad (2.131)$$

$$\Rightarrow \frac{dx(t)}{dt} = A \lambda e^{\lambda t} \quad (2.132)$$

$$\text{and } \frac{d^2 x(t)}{dt^2} = A \lambda^2 e^{\lambda t}. \quad (2.133)$$

Then can substitute into Eq (2.130):

$$M A \lambda^2 \exp(\lambda t) + \gamma A \lambda \exp(\lambda t) + \kappa A \exp(\lambda t) = 0 \quad (2.134)$$

$$M \lambda^2 + \gamma \lambda + \kappa = 0, \quad (2.135)$$

where in the last step we divide through by $A \exp(\lambda t)$ since this can't be zero unless we are in the uninteresting case $A = 0$.

What we have shown is that there are solutions of the form $x \propto \exp(\lambda t)$ provided that λ obeys a quadratic equation,

$$M \lambda^2 + \gamma \lambda + \kappa = 0. \quad (2.136)$$

It's convenient to divide through by the mass M , which gives us

$$\lambda^2 + \left(\frac{\gamma}{M}\right) \lambda + \left(\frac{\kappa}{M}\right) = 0. \quad (2.137)$$

This is a quadratic equation, which means that λ can take on two values, which we will call λ_{\pm} ,

$$\lambda_{\pm} = \frac{1}{2} \left[-\frac{\gamma}{M} \pm \sqrt{\left(\frac{\gamma}{M}\right)^2 - 4 \left(\frac{\kappa}{M}\right)} \right]. \quad (2.138)$$

²Let's assume that things move slowly enough to make this approximation. You'll look at the case of $F_{\text{drag}} \propto v^2$ in one of the problems.

We will see that these roots of the quadratic are all we need to construct the trajectory $x(t)$.

Before calculating any further it is useful to recall that we have already solved two limits of this problem. When $\gamma = 0$ it is just the harmonic oscillator without damping. Then we see that

$$\lambda_{\pm}(\gamma = 0) = \frac{1}{2} \left[\pm \sqrt{-4 \left(\frac{\kappa}{M} \right)} \right] \quad (2.139)$$

$$= \frac{1}{2} \left[\pm i 2 \sqrt{\frac{\kappa}{M}} \right] \quad (2.140)$$

$$= \pm i\omega, \quad (2.141)$$

where as before we write $\omega = \sqrt{\kappa/M}$. So we recover what we had in the absence of damping, as we should.

Actually we have also solved already another limit, which is $\kappa = 0$, because this is just a particle subject to damping with no other forces. We know that in this case the velocity decays exponentially, $v(t) = v(0) \exp(-\gamma t/M)$, so we should recover $\lambda = -\gamma/M$. But why are there two values of λ ? Let's calculate:

$$\lambda_{\pm}(\kappa = 0) = \frac{1}{2} \left[-\frac{\gamma}{M} \pm \sqrt{\left(\frac{\gamma}{M} \right)^2} \right] \quad (2.142)$$

$$= \frac{1}{2} \left[-\frac{\gamma}{M} \pm \frac{\gamma}{M} \right]. \quad (2.143)$$

Thus we see that

$$\lambda_{-}(\kappa = 0) = \frac{1}{2} \left[-\frac{\gamma}{M} - \frac{\gamma}{M} \right] = -\frac{\gamma}{M}, \quad (2.144)$$

which is what we expected. On the other hand,

$$\lambda_{+}(\kappa = 0) = \frac{1}{2} \left[-\frac{\gamma}{M} + \frac{\gamma}{M} \right] = 0. \quad (2.145)$$

What does this mean? If say that $x(t) \propto e^{\lambda t}$, and $\lambda = 0$, then we are really saying that $x(t) \propto 1$ is a solution— $x(t)$ is constant. This is right, because in the absence of the spring there is nothing to say that the particle should come to rest at $x = 0$; indeed, no particular position is special, and where you stop just depends on where you start. Thus, the solution has a piece that corresponds to adding a constant to the position.

You should see that something interesting has happened: In one limit ($\gamma \rightarrow 0$) we have imaginary values of λ and we know that this describes

sinusoidal oscillations. In the other limit ($\kappa \rightarrow 0$) we have purely real values of λ , and this describes exponential decays, plus constants. Obviously it's interesting to ask how we pass from one limit to the other ...

One of the important ideas here is that looking at λ itself tells us a great deal about the nature of the dynamics that we will see in the function $x(t)$, even before we finish solving the whole problem. We already know that when λ is imaginary, we will see a sine or cosine oscillation. On the other hand, if λ is a real number and negative, then we will see an exponential decay, as in the case of a mass moving through a viscous fluid. Finally, when λ is real and positive, we see exponential growth, as in the case of a bacterial population (cf Section 1.5). In the present case of the damped harmonic oscillator, we will see cases where λ is real and where it is complex, and we will have to understand what this combination of real and imaginary parts implies about $x(t)$.

Before going any further, let's understand how to use these roots in constructing the full solution $x(t)$. The general principle again is to make a solution by linear combination,

$$x(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}. \quad (2.146)$$

Then we have to match the initial conditions:

$$x(0) = A + B, \quad (2.147)$$

$$v(0) \equiv \left. \frac{dx(t)}{dt} \right|_{t=0} \quad (2.148)$$

$$= \left[A\lambda_+ e^{\lambda_+ t} + B\lambda_- e^{\lambda_- t} \right] \Big|_{t=0} \quad (2.149)$$

$$= A\lambda_+ + B\lambda_-. \quad (2.150)$$

After a little algebra you can solve these equations to find

$$A = \frac{\lambda_- x(0) - v(0)}{\lambda_- - \lambda_+} \quad (2.151)$$

$$B = \frac{-\lambda_+ x(0) + v(0)}{\lambda_- - \lambda_+}. \quad (2.152)$$

Problem 33: Derive Eq's (2.151) and (2.152).

Thus the general solution to our problem is

$$x(t) = \frac{[\lambda_- x(0) - v(0)] \exp(\lambda_+ t) + [-\lambda_+ x(0) + v(0)] \exp(\lambda_- t)}{\lambda_- - \lambda_+}. \quad (2.153)$$

Admittedly this is a somewhat complicated looking expression. Let's focus on the case where $v(0) = 0$, so that things get simpler:

$$x(t) = \frac{[\lambda_- x(0)] \exp(\lambda_+ t) + [-\lambda_+ x(0)] \exp(\lambda_- t)}{\lambda_- - \lambda_+} \quad (2.154)$$

$$= x(0) \frac{\lambda_- \exp(\lambda_+ t) - \lambda_+ \exp(\lambda_- t)}{\lambda_- - \lambda_+}. \quad (2.155)$$

Notice that it doesn't matter if we exchange λ_+ and λ_- , which makes sense since our choice of the signs \pm in the roots of a quadratic equation is just a convention. Checking for this sort of invariance under different choices of convention is a good way to be sure you haven't made any mistakes!

When we construct $x(t)$, three rather different things can happen, depending on the term under the square root in Eq. (2.138). To see this it is useful to define $\omega_0 = \kappa/M$ as the “natural frequency” of the oscillator, that is the frequency at which we'd see oscillations if there were no damping. Then

$$\lambda_{\pm} = \frac{1}{2} \left[-\frac{\gamma}{M} \pm \sqrt{\left(\frac{\gamma}{M}\right)^2 - 4\omega_0^2} \right]. \quad (2.156)$$

Problem 34: Go back to the differential equation that we started with,

$$M \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \kappa x = 0, \quad (2.157)$$

and gives the units for all the parameters M , γ and κ . Then show that what the natural frequency ω_0 really does have the units of frequency or 1/time. What are the units of $\Gamma = \gamma/2M$? Explain why your answer makes sense given the formula for λ_{\pm} .

The key point is to look at the square root in the formula for λ_{\pm} , Eq (2.156). If $\gamma/M > 2\omega_0$, then the term under the square root is positive, so that its square root is a real number, and hence λ_{\pm} itself is real. Presumably this describes exponential decays—the damping or viscous drag is so large

that it destroys the oscillation completely. On the other hand, if $\gamma/M < 2\omega_0$, then the term under the square root is negative and its square root is imaginary. So now λ_{\pm} will be complex numbers. Clearly this is different, and the two cases are called *overdamped* ($\gamma/M > 2\omega_0$) and *underdamped* ($\gamma/M < 2\omega_0$), respectively.

Underdamping. This is the case where the combination under the square root is negative, that is

$$\frac{\gamma}{2M} < \omega_0. \quad (2.158)$$

In this case, λ_{\pm} has an imaginary part,

$$\lambda_{\pm} = -\frac{\gamma}{2M} \pm i\omega \quad (2.159)$$

$$\omega = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2M}\right)^2}. \quad (2.160)$$

This means that the time dependence of x is given by

$$x(t) = A \exp(\lambda_+ t) + B \exp(\lambda_- t) \quad (2.161)$$

$$= A \exp\left(-\frac{\gamma}{2M}t + i\omega t\right) + B \exp\left(-\frac{\gamma}{2M}t - i\omega t\right) \quad (2.162)$$

Notice that for x to be real A and B must once again be complex conjugates, so that if $A = |A| \exp(i\phi_A)$ as before, we can write

$$x(t) = A \exp\left(-\frac{\gamma}{2M}t + i\omega t\right) + \left[A \exp\left(-\frac{\gamma}{2M}t + i\omega t\right)\right]^* \quad (2.163)$$

$$= |A| \exp\left(+i\phi_A - \frac{\gamma}{2M}t + i\omega t\right) + \left[|A| \exp\left(+i\phi_A - \frac{\gamma}{2M}t + i\omega t\right)\right]^* \quad (2.164)$$

$$= 2\operatorname{Re} \left[|A| \exp\left(+i\phi_A - \frac{\gamma}{2M}t + i\omega t\right) \right] \quad (2.165)$$

$$= 2|A| \exp\left(-\frac{\gamma}{2M}t\right) \cos(\omega t + \phi_A) \quad (2.166)$$

We see that the introduction of damping causes oscillations to occur at a lower frequency, since $\omega < \omega_0$, and causes these oscillations to decay according to the exponential ‘envelope’ outside the cosine. See Fig. 2.5.

Overdamping. This is when

$$\frac{\gamma}{2M} > \omega_0, \quad (2.167)$$

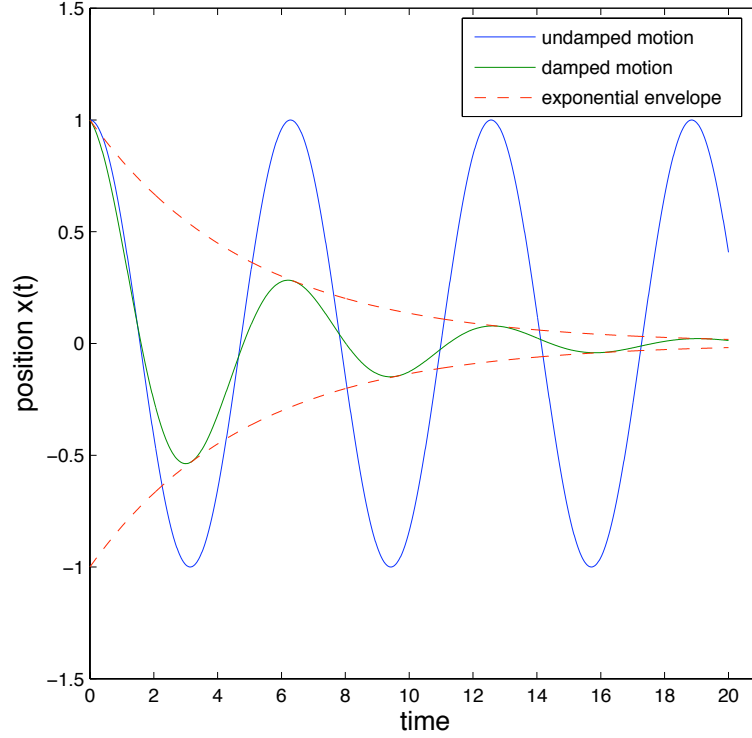


Figure 2.5: Comparing the motion of a damped oscillator, following the predictions from Eq (2.166), with that of an undamped oscillator (same equation but with $\gamma = 0$). We choose units such that $\omega_0 = 1$, and for the damped oscillator we take $\gamma/2M = 0.2$; in both cases the initial phase $\phi_A = 0$.

and now the term inside the square root in Eq (2.156) is positive. This means that both λ_+ and λ_- are real, and in fact both are negative. Thus we can write $\lambda_+ = -|\lambda_+|$ and $\lambda_- = -|\lambda_-|$, so that

$$x(t) = A \exp(-|\lambda_+|t) + B \exp(-|\lambda_-|t). \quad (2.168)$$

Thus the displacement consists just of decaying exponentials, and hence there is **no** oscillation.

To understand what happens it is convenient to go into the *strongly* overdamped limit, where $\gamma/(2M) \gg \omega_0$. Then we can do some algebra to work out the values of λ_{\pm} . The easy case is λ_- :

$$\lambda_-(\gamma/(2M) \gg \omega_0) = \frac{1}{2} \left[-\frac{\gamma}{M} - \sqrt{\left(\frac{\gamma}{M}\right)^2 - 4\omega_0^2} \right]$$

$$\approx \frac{1}{2} \left[-\frac{\gamma}{M} - \sqrt{\left(\frac{\gamma}{M}\right)^2} \right] \quad (2.169)$$

$$= -\frac{\gamma}{M}. \quad (2.170)$$

To estimate λ_+ in this limit we have to be a bit more careful:

$$\lambda_- = \frac{1}{2} \left[-\frac{\gamma}{M} + \sqrt{\left(\frac{\gamma}{M}\right)^2 - 4\omega_0^2} \right] \quad (2.171)$$

$$= \frac{1}{2} \left[-\frac{\gamma}{M} + \sqrt{\left(\frac{\gamma}{M}\right)^2 \left(1 - 4\left(\frac{\omega_0}{\gamma/M}\right)^2\right)} \right] \quad (2.172)$$

$$= \frac{1}{2} \left[-\frac{\gamma}{M} + \frac{\gamma}{M} \sqrt{1 - 4\left(\frac{\omega_0}{\gamma/M}\right)^2} \right] \quad (2.173)$$

$$= \frac{\gamma}{2M} \left[-1 + \sqrt{1 - 4\left(\frac{\omega_0}{\gamma/M}\right)^2} \right] \quad (2.174)$$

$$\approx \frac{\gamma}{2M} \left[-1 + 1 - \frac{1}{2} 4\left(\frac{\omega_0}{\gamma/M}\right)^2 + \dots \right] \quad (2.175)$$

$$= -\frac{\gamma}{2M} \frac{1}{2} 4\left(\frac{\omega_0}{\gamma/M}\right)^2 = \frac{M\omega_0^2}{\gamma} \quad (2.176)$$

$$= \frac{\kappa}{\gamma}. \quad (2.177)$$

The key steps were to use the Taylor expansion of the square root,

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x + \dots, \quad (2.178)$$

and to notice that since the natural frequency is given by $\omega_0^2 = \kappa/M$, we have $M\omega_0^2 = \kappa$. To summarize, in the extreme overdamped limit, we have

$$x(t) = A \exp\left(-\frac{\gamma}{M}t\right) + B \exp\left(-\frac{\kappa}{\gamma}t\right). \quad (2.179)$$

Notice that both are exponential decays, one gets faster at large γ and one gets slower at large γ . Intuitively, the fast decay is the loss of inertia (forgetting the initial velocity) and the slow decay is the relaxation of the spring back to its equilibrium position. Roughly speaking, γ/M is the rate at which the initial velocity is forgotten, while κ/γ describes the slow relaxation of the position back to equilibrium at $x = 0$.

Problem 35: If the interpretation we have just given for the behavior of λ_{\pm} in the overdamped limit is correct, then the decay of the velocity gets faster as γ gets larger, while the decay of the position gets slower. Explain, intuitively, why this makes sense.

To be sure that this interpretation is correct, consider the case where the initial velocity is zero, which case we should see relatively little contribution from the term $\sim \exp(-\gamma t/M)$. To satisfy the initial conditions we must have

$$0 = \left. \frac{dx(t)}{dt} \right|_{t=0} \quad (2.180)$$

$$= \left. \frac{d}{dt} \left[A \exp\left(-\frac{\gamma}{M}t\right) + B \exp\left(-\frac{\kappa}{\gamma}t\right) \right] \right|_{t=0} \quad (2.181)$$

$$= \left[A \left(-\frac{\gamma}{M}\right) \exp\left(-\frac{\gamma}{M}t\right) + B \left(-\frac{\kappa}{\gamma}\right) \exp\left(-\frac{\kappa}{\gamma}t\right) \right] \Big|_{t=0} \quad (2.182)$$

$$= -A \frac{\gamma}{M} - B \frac{\kappa}{\gamma} \quad (2.183)$$

$$\Rightarrow A \frac{\gamma}{M} = -B \frac{\kappa}{\gamma} \quad (2.184)$$

$$A = -B \frac{M\kappa}{\gamma^2} = -B \frac{M \cdot M\omega_0^2}{\gamma^2} = -B \left(\frac{M\omega_0}{\gamma} \right)^2. \quad (2.185)$$

Thus we see that in the strongly overdamped limit, where $\gamma \gg M\omega_0$, we have $A \ll B$ if the initial velocity is zero, as promised.

Problem 36: Many proteins consist of separate “domains,” often with flexible connections between the domains. Imagine a protein that is sitting still, with one extra domain that can move. Assume that this mobile domain is roughly spherical with a radius of $r \sim 1$ nm, and that it has a molecular weight $m \sim 30,000$ a.m.u.³ The small piece of the molecule which connects this domain to the rest of the protein acts like a spring with stiffness $\kappa \sim 1$ N/m.

(a.) If there were no drag, what differential equation would describe the motion of the domain? Would the domains oscillate? At what frequency?

³Reminder: one mole of atomic mass units (a.m.u.) has a total mass of one gram.

(b.) Estimate the drag coefficient for motion of the domain through water. You should use Stokes' formula, which you explored in the lab.

(c.) Write out the differential equation that describes motion in the presence of damping. Given the parameters above, is the resulting motion of the domain underdamped or overdamped?

(d.) If the spring that attaches the mobile domain to the rest of the protein is stretched and released with zero velocity, estimate how much time it takes before this displacement decays to half its initial value.

Problem 37: In your ear, as in the ears of other animals, the “hair cells” which are sensitive to sound have a bundle of small finger-like structures protruding from their surface. These hairs, or stereocilia, bend in response to motion of the surrounding fluid. Directly pushing on the hairs one measures a stiffness of $\kappa \sim 10^{-3}$ N/m. In this problem you'll examine the possibility that the stereocilia form a mass-spring system that resonates in the $\omega_0 \sim 2\pi \times 10^3$ Hz frequency range that corresponds to the most sensitive range of human hearing.

(a.) What mass would the stereocilia have to have in order that their natural frequency would come out to be $\omega_0 \sim 2\pi \times 10^3$ Hz?

(b.) The entire bundle of stereocilia in human hair cells ranges from 1 to $5\mu\text{m}$ in height, and the cross-sectional area of the bundle typically is less than $1\mu\text{m}^2$. Is it plausible that this bundle has the mass that you derived in [a]? You'll need to make some assumptions about the density of the hairs, and you should state your assumptions clearly.

(c.) Independent of your answer to [b], it still is possible that the stiffness of the stereocilia is the spring in mass-spring resonance, perhaps with the mass provided by some other nearby structure. But when the stereocilia move through the fluid, this will generate a damping or drag coefficient γ . How small would γ have to be in order that this system exhibit a real resonance?

(d.) The geometry of the stereocilia is complicated, so actually calculating γ is difficult. You know that for spherical objects $\gamma = 6\pi\eta R$, with R the radius and η the viscosity of the surrounding fluid (water, in this case). You also experimented with objects of different shape and learned something about how damping coefficients depend on size and shape. Using what you know, decide whether γ for the stereocilia can be small enough to satisfy the conditions for underdamping that you derived in [c].

Problem 38: For the damped harmonic oscillator,

$$m \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \kappa x = 0, \quad (2.186)$$

we found that the solution can be written as $x(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$, where λ_{\pm} are the roots of a quadratic equation, $m\lambda^2 + \gamma\lambda + \kappa = 0$.

(a.) Find the constants A and B in the case where the initial conditions are $x(t) = 1$ and $(dx/dt)|_{t=0} = 0$. Write the function $x(t)$ only in terms of λ_{\pm} .

(b.) To be sure you understand what underdamping and overdamping really mean, we'd like you to plot the function $x(t)$ in different cases. To make things clear, choose units of time so that $\kappa/m = 1$. Then if $\gamma = 0$ you should just have $x(t) = \cos(t)$. Now consider values of $\gamma/2m = 0.1, 0.9, 1.1, 10$. In each case, use MATLAB to plot $x(t)$ over some interesting range of times; part of the problem here is for you to decide what is interesting.

(c.) Write a brief description of your results in [b]. Can you show the slowing of oscillations in the presence of a small amount of drag? The exponential envelope for the decay? The disappearance of oscillations in the overdamped regime?

(d.) Use your mathematical expression for $x(t)$ to explore what happens right at the transition between overdamped and underdamped behavior (“critical damping”), where $\lambda_+ = \lambda_-$. Hint: Recall l’Hopital’s rule, which states that if two functions $f(y)$ and $g(y)$ are both zero at $y = y_0$, then

$$\lim_{y \rightarrow y_0} \frac{f(y)}{g(y)} = \frac{f'(y_0)}{g'(y_0)}, \quad (2.187)$$

where $f'(y_0)$ is another way of writing $(df/dy)|_{y=y_0}$. Hopefully you will discover that, in addition to sines, cosines and exponentials, this gives yet another functional form. It seems remarkable that by varying parameters in one equation we can get such different predictions. Maybe even more remarkable is that we can capture this wide range of behaviors by using the complex exponentials, although we do have to use them carefully.

Problem 39: So far we have discussed damping of the harmonic oscillator in the case where the damping is linear, $F_{\text{drag}} = -\gamma v$. What happens if $F_{\text{drag}} = -cv^2$? In an oscillator, the velocity changes sign during each period of oscillation, so we should be careful and write $F_{\text{drag}} = -c|v|v$ so that the drag force always opposes the motion. Then the relevant differential equation is

$$M \frac{d^2 x(t)}{dt^2} + c \left| \frac{dx(t)}{dt} \right| \frac{dx(t)}{dt} + \kappa x(t) = 0, \quad (2.188)$$

where as usual M is the mass and κ is the stiffness of the spring to which the mass is attached. Consider an experiment in which we stretch the spring to an initial displacement $x(0) = x_0$ and release it with zero velocity.

(a.) This problem seems to have lots of parameters: M , c , κ and x_0 . It’s very useful to simplify the problem by changing units, so you can see that there really aren’t so many parameters. Consider measuring position in units of the initial displacement, $X = x/x_0$ and measuring time in units related to the period of the oscillations, $T = \omega t$, where as usual $\omega = \sqrt{\kappa/M}$. Show that Eq (2.188) is equivalent to

$$\frac{d^2 X}{dT^2} + B \left| \frac{dX}{dT} \right| \frac{dX}{dT} + X = 0, \quad (2.189)$$

where B is a dimensionless combination of parameters. What is the formula for B in relation to all the original parameters?

(b.) Write a simple program in MATLAB to solve Eq (2.189). Clearly you will need to choose time steps $\Delta T \ll 1$, but you don’t know in advance what to choose so leave this as a parameter.

(c.) Small values of B should generate relatively small amounts of damping. Try $B = 0.1$, and run your program for a time long enough to see 20 oscillations; a reasonable value for the time step is $\Delta T = 0.01$. Can you see the effects of the damping? Does it look different from the case of linear damping?

(d.) Do some numerical experiments to see if the choice $\Delta T = 0.01$ really gives a reliable solution.

(e.) With linear damping, there is a critical value that destroys the oscillation and leads to “overdamping.” Try increasing the value of B and running your program to see if there is a similar transition in the case of nonlinear damping.

(f.) Is $\Delta T = 0.01$ still sufficiently small as you explore larger values of B ?