2.6 The driven oscillator

We would like to understand what happens when we apply forces to the harmonic oscillator. That is, we want to solve the equation

$$M\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \kappa x(t) = F(t). \tag{2.239}$$

The problem is that, of course, the solution depends on what we choose for the force. It seems natural to ask what happens, but we don't want to have to answer with a long list—if the force looks like this, then the displacement looks like that; if the force is different in this way, then the displacement is different in that way ... and so on. Is there any way to give **one** answer to the question of what happens in response to applied forces?

One idea is to think of an arbitrary function $F(t)$ as a sequence of short pules, occurring at the right times with the right amplitudes. This is useful because if we can solve for the response to one pulse, then the response to many pulses is just the sum of the individual responses. To see this, imagine that $x_1(t)$ is the time dependent displacement that is generated by the force $F_1(t)$, and similarly $x_2(t)$ is generated by $F_2(t)$. This means that

$$\begin{aligned} M\frac{d^2x_1(t)}{dt^2} + \gamma\frac{dx_1(t)}{dt} + \kappa x_1(t) &= F_1(t) && (2.240)\\ M\frac{d^2x_2(t)}{dt^2} + \gamma\frac{dx_2(t)}{dt} + \kappa x_2(t) &= F_2(t). && (2.241) \end{aligned}$$

Now we add these two equations together and notice that adding and diferentiating commute:

$$\begin{aligned} \left[M\frac{d^2x_1(t)}{dt^2} + \gamma\frac{dx_1(t)}{dt} + \kappa x_1(t)\right] &+ \left[M\frac{d^2x_2(t)}{dt^2}\gamma\frac{dx_2(t)}{dt} + \kappa x_2(t)\right] \\ &= F_1(t) + F_2(t) \qquad (2.242) \end{aligned}$$

$$\begin{aligned} M\left[\frac{d^2x_1(t)}{dt^2} + \frac{d^2x_2(t)}{dt^2}\right] + \gamma\left[\frac{dx_1(t)}{dt} + \frac{dx_2(t)}{dt}\right] &+ \kappa\left[x_1(t) + x_2(t)\right] \\ &= F_1(t) + F_2(t) \end{aligned} \tag{2.243}$$

$$\begin{aligned} M\frac{d^2[x_1(t)+x_2(t)]}{dt^2} + \gamma\frac{d[x_1(t)+x_2(t)]}{dt} &+ \kappa\left[x_1(t) + x_2(t)\right] \\ &= F_1(t) + F_2(t). \end{aligned} \tag{2.244}$$

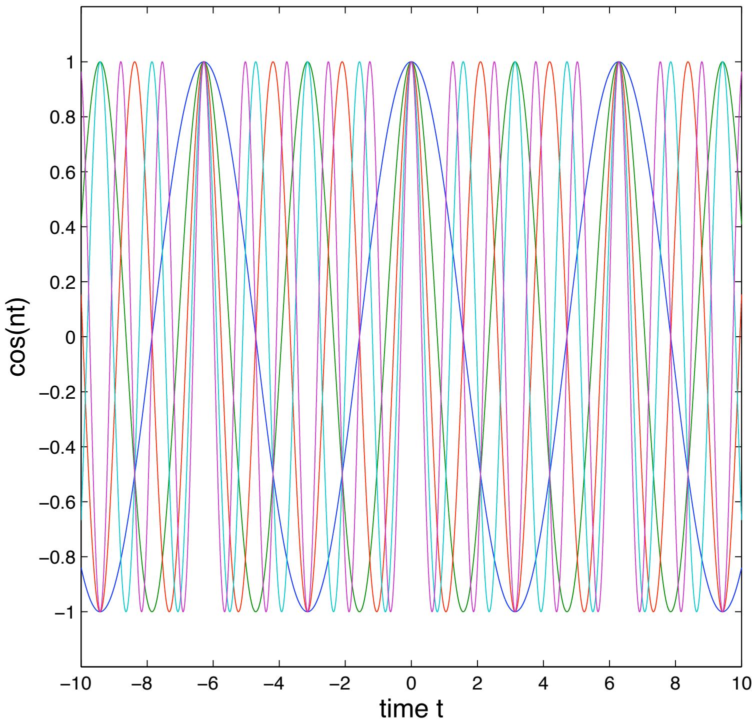

Figure 2.9: The first five members of the family of functions $\cos(nt)$.

Thus if we have the force $F(t) = F_1(t) + F_2(t)$, then the displacement will be $x(t) = x_1(t) + x_2(t)$. This "superposition" of solution keeps working if we have more and more forces to add up, so if we think of the time–dependent force as being a sum of pulses then the displacement will be the sum of responses to the individual pulses, as promised.

Thinking in terms of pulses is a good idea, and we could develop it a little further, but not now.[6] Instead let's look at using sines and cosines (!). Somewhat remarkably, in the same way that we can make an arbitrary function out of many pulses, it turns out that we can make an arbitrary function by adding up sines and cosines. This is surprising because sines and cosines are periodic and extended—how then can we make little localized blips? There is a rigorous theory of all this, but what we need here is just to motivate the idea that sines and cosines are a sensible choice ... .

[6]In the 2008 version of the course, I spent a full lecture on this idea, and really should write up the notes. Stay tuned for revisions.

To understand that sines and cosines can be used to make any function we want, let's try to make a brief pulse. Let's start in a window of time that runs from $t = -10$ up to $t = +10$ (in some units). We can make functions like $\cos(t)$, $\cos(2t)$, and so on. Let's do this in MATLAB, just to be explicit. To do this on a computer we need to discrete time steps, so let's choose steps of size $dt = 0.001$:

```
dt = 0.001;
t = [-10:dt:10];
y= zeros(500,length(t));
for n=1:500;
   y(n,:)  = cos(n*t);
end;
```

This program will generate functions `y(n,t)` $= \cos(nt)$, as shown in Fig 2.9. Notice that all of these functions line up at $t = 0$, where they equal one, and then at other times they have values that have a chance of canceling out.

In fact if we add up all the functions in Fig 2.9, we get the results shown at left in Fig 2.10. If instead of looking at the first five terms, we sum up the first five hundred terms, we get the results shown in Fig 2.10. This should be starting to convince you that we can add up lots of cosines and get something that looks like a perfectly sharp pulse. The only problem is that in addition to a pulse at $t = 0$, we also have pulses at $t = \pm 2\pi$, and if we looked at a bigger window of time we would see pulses at $t = \pm 4\pi$, $t = \pm 6\pi$, etc.. We can start to fix this by including not just $\cos(t)$, $\cos(2t)$, $\cdots$, but also terms like $\cos(1.5t)$, $\cos(2.5t)$, $\cdots$. This will cancel out the pulses at $t = \pm 2\pi$. Then if we add not just halves, but also thirds, fourths, etc we can cancel the pulses at larger and larger times, until eventually all that's left will be the pulse at $t = 0$.

The arguments here are not rigorous, but hopefully give the sense that adding up sines and cosines allows us to make pulses. If we can make pulses, we can make anything. Thus any force vs time can be thought of as a sum of sines and cosines. This process is called Fourier analysis, and we'll see more about this in the spring. For now, we know from the discussion above that the displacement in response to this general force can be thought of as the sum of responses to the individual sine and cosine forces. Let's solve *one* of these problems and see how much we can learn.

The problem we want to solve is the damped harmonic oscillator driven by a force that depends on time as a cosine or sine at some frequency ω:

$$M\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \kappa x(t) = F_0\cos(\omega t). \tag{2.245}$$

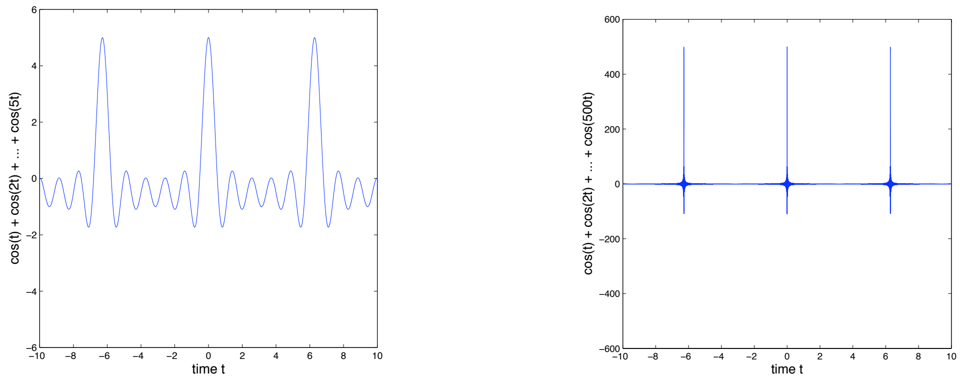

Figure 2.10: Left: The sum of the five functions in Fig 2.9. Right: The sum of five hundred such functions.

Actually we might want to do both cosine and sine, and let's call the motion in response to the sine $y(t)$:

$$M\frac{d^2y(t)}{dt^2} + \gamma\frac{dy(t)}{dt} + \kappa y(t) = F_0\sin(\omega t). \tag{2.246}$$

Now these two equations must have the same information hidden in them, since the difference between sine and cosine is just our choice of the point where $t = 0$.

We're going to do something a bit weird, which is to combine the two equations, multiplying the equation for $y(t)$ by a factor of i and then adding the equation for $x(t)$ (!):

$$\begin{aligned} M\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \kappa x(t) &= F_0\cos(\omega t) \\ +i \times \left[M\frac{d^2y(t)}{dt^2} + \gamma\frac{dy(t)}{dt} + \kappa y(t)\right] &= +i \times [F_0\sin(\omega t)] \end{aligned} \tag{2.247}$$

$$\begin{aligned} \Rightarrow M\frac{d^2[x(t)+iy(t)]}{dt^2} + \gamma\frac{d[x(t)+iy(t)]}{dt} &+ \kappa[x(t)+iy(t)] \\ &= F_0[\cos(\omega t) + i\sin(\omega t)]. \end{aligned} \tag{2.248}$$

Now we identify $z(t) = x(t) + iy(t)$, and remember that $\cos(\omega t) + i\sin(\omega t) = \exp(i\omega t)$, so that

$$M\frac{d^2z(t)}{dt^2} + \gamma\frac{dz(t)}{dt} + \kappa z(t) = F_0e^{i\omega t}. \tag{2.249}$$

Notice that the solution to our original physical problem is $x(t) = \mathrm{Re}[z(t)]$.

By now you can anticipate that what we will do is to look for a solution of the form $z(t) = z_0 e^{\lambda t}$. As usual this means that $dz/dt = \lambda z_0 e^{\lambda t}$ and $d^2z/dt^2 = \lambda^2 z_0 e^{\lambda t}$. Substituting, we have

$$M\lambda^2 z_0 e^{\lambda t} + \gamma\lambda z_0 e^{\lambda t} + \kappa z_0 e^{\lambda t} = F_0 e^{i\omega t}. \tag{2.250}$$

Notice that all the terms on the left have a common factor of $z_0 e^{\lambda t}$ (this should look familiar!) so we can group them together:

$$\left[M\lambda^2 + \gamma\lambda + \kappa\right] z_0 e^{\lambda t} = F_0 e^{i\omega t}. \tag{2.251}$$

Now the terms in brackets are just numbers, independent of time. If we want the two sides of the equation to be equal *at all times* then we have to have $e^{\lambda t} = e^{i\omega t}$, or $\lambda = i\omega$. Thus the time dependence of $z(t)$ has to be a complex exponential with the same frequency as the applied force. This means that when we apply a sinusoidal force with frequency ω, the displacement $x(t)$ also will vary as a sine or cosine at with frequency ω.

Once we recognize that $\lambda = i\omega$, we can cancel these exponentials from both sides of the equation and substitute for λ wherever it appears:

$$\begin{aligned}
\left[M\lambda^2 + \gamma\lambda + \kappa\right] z_0 e^{\lambda t} &= F_0 e^{i\omega t} \\
\left[M\lambda^2 + \gamma\lambda + \kappa\right] z_0 &= F_0 && (2.252) \\
\left[M(i\omega)^2 + \gamma(i\omega) + \kappa\right] z_0 &= F_0 && (2.253) \\
\left[-M\omega^2 + i\gamma\omega + \kappa\right] z_0 &= F_0 && (2.254) \\
z_0 &= \frac{F_0}{-M\omega^2 + i\gamma\omega + \kappa}. && (2.255)
\end{aligned}$$

So we have made it quite far: The position as a function of time $x(t)$ is the real part of $z(t)$, the time dependence is set by $z(t) = z_0 e^{i\omega t}$, and now we have an expression for z_0.

To get a bit further let's recall that, as with any complex number, we can write

$$z_0 = |z_0| e^{i\phi}. \tag{2.256}$$

Then we have

$$\begin{aligned}
x(t) &= \mathrm{Re}[z(t)] = \mathrm{Re}[z_0 e^{i\omega t}] && (2.257) \\
&= \mathrm{Re}[|z_0| e^{i\phi} e^{i\omega t}] && (2.258) \\
&= \mathrm{Re}[|z_0| e^{i(\omega t + \phi)}] && (2.259) \\
&= |z_0| \cos(\omega t + \phi) && (2.260)
\end{aligned}$$

So we see explicitly that the displacement is a cosine function of time, with an amplitude $|z_0|$ and a phase shift ϕ relative to the driving force. For more on the phase shift see the fourth problem set; for now let's look at the amplitude $|z_0|$.

To compute $|z_0|$ we use the definition $|z| = \sqrt{z^* z}$, where z^* is the complex conjugate of z. Now since F_0 is a real number (it's the actual applied force, and hence a physical quantity!),

$$\begin{aligned} z_0 &= \frac{F_0}{-M\omega^2 + i\gamma\omega + \kappa} \\ \Rightarrow z_0^* &= \frac{F_0}{-M\omega^2 - i\gamma\omega + \kappa}. \end{aligned} \tag{2.261}$$

Putting these together we have

$$|z_0|^2 = \frac{F_0}{-M\omega^2 + i\gamma\omega + \kappa} \cdot \frac{F_0}{-M\omega^2 - i\gamma\omega + \kappa} \tag{2.262}$$

$$= \frac{F_0^2}{(-M\omega^2 + i\gamma\omega + \kappa)(-M\omega^2 - i\gamma\omega + \kappa)} \tag{2.263}$$

$$= \frac{F_0^2}{(-M\omega^2 + \kappa)^2 + (\gamma\omega)^2}. \tag{2.264}$$

Thus the amplitude of oscillations in response to a force at frequency ω is given by

$$|z_0| = \frac{F_0}{\sqrt{(-M\omega^2 + \kappa)^2 + (\gamma\omega)^2}} \tag{2.265}$$

We see that the amplitude is proportional to the magnitude of the force F_0, which means that the whole system is *linear*; this follows from the fact that the differential equation is linear. Thus it makes sense to measure the coefficient which relates the magnitude of the force to the magnitude of the displacement:

$$G(\omega) \equiv \frac{|z_0|}{F_0} = \frac{1}{\sqrt{(-M\omega^2 + \kappa)^2 + (\gamma\omega)^2}}. \tag{2.266}$$

This is plotted in Fig 2.11, for examples of underdamped and overdamped oscillators. Here we look at some simple limits to get a feeling for how it should behave.

Notice first that at zero frequency we have

$$G(\omega) = \frac{1}{\kappa}. \tag{2.267}$$

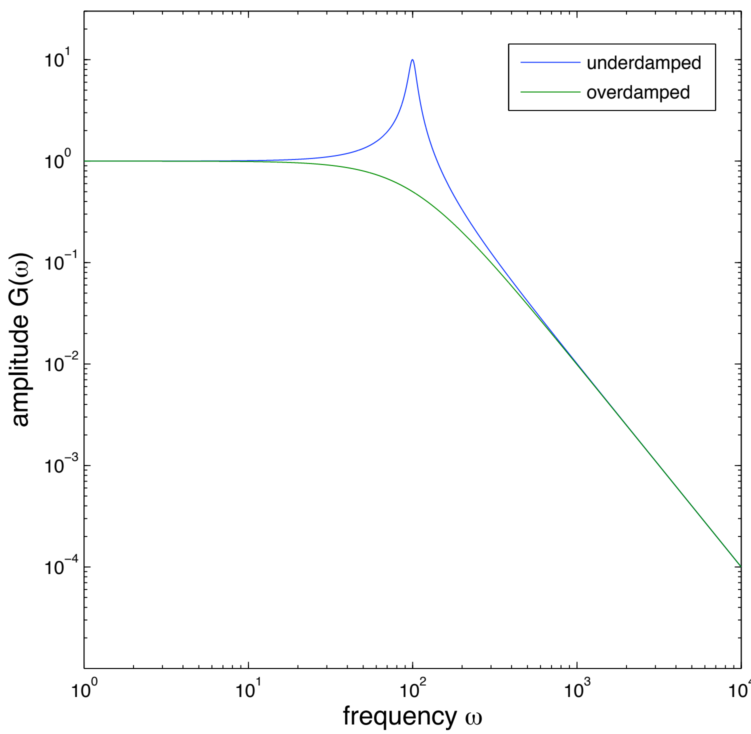

Figure 2.11: Comparing the response $G(\omega)$ for overdamped and underdamped oscillators. In both cases we choose units where $\kappa = 1$ and $\omega_0 = 100$. The underdamped case corresponds to $\gamma = 10$ and the overdamped case $\gamma = 200$. Note the constant behavior $G(\omega \to 0) = 1/\kappa$ at low frequencies, and the asymptotic $G(\omega \to \infty) \approx 1/(M\omega^2)$ at high frequencies. The latter behavior shows up as a line of slope two on this log–log plot.

This makes sense: zero frequency corresponds to applying a constant force, and if we do this we expect to stretch the spring by a constant amount—since nothing is changing in time mass and drag are irrelevant. The proportionality between force and displacement is the stiffness κ, which appears here as $1/\kappa$ because we ask how much displacement you get for a fixed force, rather than the other way around.

At very high frequencies, the $M\omega^2$ term is bigger than all the others, and so we find

$$G(\omega \to \infty) \approx \frac{1}{M\omega^2}. \tag{2.268}$$

This actually means that if we are pushing the system at very high frequencies we hardly feel the stiffness or damping at all. What we feel instead is the inertia provided by the mass, and the applied force goes into accelerating this mass.

Finally we notice that, in the denominator of the expression for $G(\omega)$ [Eq (2.266)] there is the combination $(-M\omega^2 + \kappa)^2$. This can never be negative, but it becomes zero when $M\omega^2 = \kappa$, which is the same as $\omega = \omega_0$, where we recall that $\omega_0 = \sqrt{\kappa/M}$ is the natural frequency of the oscillator. Thus when we drive the system with a force that oscillates at the natural frequency, the denominator of $G(\omega)$ can become small, and if γ is small enough this should result in a very large response. This large response is called a *resonance*.

The plot of $G(\omega)$ in Fig 2.11 makes clear how the resonance looks: a peak in the amplitude of oscillations as a function of frequency. What is interesting is that condition for seeing this peak is the same as the condition for underdamping in the motion with no force. Thus the response of the system to applied forces is very closely related to it's "free" decay in the absence of forces. For underdamped oscillators there is a resonant peak and for overdamped oscillators the response just gets smaller as the frequency gets higher, monotonically.

Problem 47: For the driven harmonic oscillator,

$$M\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \kappa x(t) = F(t), \tag{2.269}$$

we showed that if $F(t) = F\cos(\omega t)$, then the position as a function of time can be written as

$$x(t) = \mathrm{Re}[A\exp(i\omega t)], \tag{2.270}$$

where A is given by

$$A = \frac{F}{-M\omega^2 + i\gamma\omega + \kappa}. \tag{2.271}$$

Recall that, as with any complex number, we can write $A = |A|\exp(i\phi_A)$.

(a.) Be sure you understand how to go from Eq (2.271) to the expression for $|A|$, as covered in the lecture. Then derive an expression for ϕ_A, showing explicitly how it depends on the driving frequency ω.

(b.) Does the phase shift ϕ_A have simple behaviors at low frequency ($\omega \to 0$) or at high frequency ($\omega \to \infty$)? Can you give an intuitive explanation for these limiting results?

(c.) Is there anything special about the phase shift at the resonance point where $\omega = \omega_0 = \sqrt{\kappa/M}$? Does this depend on whether the oscillator is underdamped or overdamped?
