3.3 Conservation of $\vec{\mathbf{P}}$ and $\vec{\mathbf{L}}$

We have discussed the conservation of energy, showing how the existence of a potential energy constrains the possible form of forces that enter Netwon's $F = ma$. In these lectures we'll discuss the conservation of momentum and angular momentum. We will start with a fairly conventional freshman physics point of view, namely that conservation of momentum follows from the law of action and reaction. Then we will see that there is a deeper view, namely that conservation of momentum follows from insisting that our description of the world in terms of a potential energy as a function of position(s) doesn't depend on where we choose to put the origin of coordinate system. This independence is called "invariance" to translations, and the idea that conservation laws follow from invariance principles is one of the fundamental ideas of modern physics. After the discussion of momentum we will turn to angular momentum, which introduces some complications but exposes the same link between invariance and conservation. As we shall see, invariance is the statement that our description of the world, in a coordinate system that we chose, should be the same as that obtained by a different person who might choose a different coordinate system. In this sense there is no special coordinate system in which one obtains uniquely correct answers. This seems simple enough, but it means that our personal, human point of view is not privileged.

Consider a system of particles in which the different particles apply forces to one another in pairs. Then when we write $F = ma$ for the i^{th} particle,

$$m_{\mathrm{i}}\vec{a}_{\mathrm{i}} \equiv \frac{d\vec{p}_{\mathrm{i}}}{dt} = \vec{F}_{\mathrm{i}} \tag{3.5}$$

$$= \sum_{\mathrm{j}} \vec{F}_{\mathrm{j}\to\mathrm{i}}, \tag{3.6}$$

where $\vec{p}_{\mathrm{i}} \equiv m_{\mathrm{i}}\vec{v}_{\mathrm{i}}$ is the *momentum* of the i^{th} particle, and$\vec{F}_{\mathrm{j}\to\mathrm{i}}$ is the force which particle j exerts on particle i. The law of action and reaction is then the statement that

$$\vec{F}_{\mathrm{j}\to\mathrm{i}} = -\vec{F}_{\mathrm{i}\to\mathrm{j}}. \tag{3.7}$$

This is enough to show that the total momentum

$$\vec{P}_{\mathrm{total}} = \sum_{\mathrm{i}} \vec{p}_{\mathrm{i}} \tag{3.8}$$

is conserved. To show this we just compute the time derivative of $\vec{P}$ by substitution:

$$\frac{d\vec{P}_{\rm total}}{dt} = \sum_{\rm i} \frac{d\vec{p}_{\rm i}}{dt} \tag{3.9}$$

$$= \sum_{\rm i} \vec{F}_{\rm i} \tag{3.10}$$

$$= \sum_{\rm i}\sum_{\rm j} \vec{F}_{\rm j\to i} \tag{3.11}$$

$$= \sum_{\rm all\ pairs\ ij} \vec{F}_{\rm j\to i} \tag{3.12}$$

But when we sum over all pairs, we must count (for example) the pair (1,7) and the pair (7,1). Thus the sum over all pairs includes both the term $\vec{F}_{1\to 7}$ and the term $\vec{F}_{7\to 1}$. But when we add these two terms we get zero, because of Eq. (3.7), and this is true for every pair ij. Thus

$$\sum_{\rm all\ pairs\ ij} \vec{F}_{\rm j\to i} = 0, \tag{3.13}$$

and hence

$$\frac{d\vec{P}_{\rm total}}{dt} = 0, \tag{3.14}$$

which means that momentum is conserved.

There is a different way to go at proving conservation of momentum, and this illustrates the general connection between conservation laws and symmetries or invariances. We know that forces can be found by taking derivatives of the potential energy. The potential energy is a function of the (vector) positions of all the particles:

$$V = V(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_{\rm N}), \tag{3.15}$$

where there are N particles in our system. It sounds reasonable that, if the potential energy is going to mean something, it should not depend on the coordinate system we use to define the locations of all the particles. In particular, if we take our coordinate system with us as we move one step, all of the position vectors $\vec{r}_{\rm i}$ will shift by a constant amount. To make things simple, let's focus on shifts along the x axis. Then it is useful to explicit about the components of the vector positions: $\vec{r}_{\rm i} \equiv (x_{\rm i}, y_{\rm i}, z_{\rm i})$. We can write the potential energy as

$$V = V(x_1, x_2, \cdots, x_{\rm N}; y_1, y_2, \cdots, y_{\rm N}; z_1, z_2, \cdots, z_{\rm N}), \tag{3.16}$$

and then if we shift our coordinate system by an distance d along the x axis the potential energy becomes

$$V \rightarrow V(x_1 + d, x_2 + d, \cdots, x_\mathrm{N} + d; y_1, y_2, \cdots, y_\mathrm{N}; z_1, z_2, \cdots, z_\mathrm{N}). \quad (3.17)$$

What we would like is that this transformation actually leaves the potential energy *unchanged*, so that

$$\begin{aligned} &V(x_1, x_2, \cdots, x_\mathrm{N}; y_1, y_2, \cdots, y_\mathrm{N}; z_1, z_2, \cdots, z_\mathrm{N}) \\ &\quad = V(x_1 + d, x_2 + d, \cdots, x_\mathrm{N} + d; y_1, y_2, \cdots, y_\mathrm{N}; z_1, z_2, \cdots, z_\mathrm{N}). \end{aligned} \quad (3.18)$$

Obviously not all the potential functions that we might write down have this property, so our possible models of the world are constrained by this invariance.

We'd like Eq (3.18) to a be global statement, valid for any value of $\vec{d}$. But as is often true in calculus, we can check for global constancy by considering only small shifts: the global statement that something is constant is the local statement that derivatives are zero, applied at every point. Thus we look at very small d, so we can use a Taylor series expansion. In doing this, it's important to remember here that we can do an expansion for x_1 while holding all the other x_i fixed, so that there really isn't anything special about our use of partial derivatives here! So, we have

$$\begin{aligned} &V(x_1 + d, x_2 + d, \cdots, x_\mathrm{N} + d; y_1, y_2, \cdots, y_\mathrm{N}; z_1, z_2, \cdots, z_\mathrm{N}) \\ &\quad \approx V(x_1, x_2, \cdots, x_\mathrm{N}; y_1, y_2, \cdots, y_\mathrm{N}; z_1, z_2, \cdots, z_\mathrm{N}) \\ &\qquad + \left[d\frac{\partial V}{\partial x_1} + d\frac{\partial V}{\partial x_2} + \cdots + d\frac{\partial V}{\partial x_N} \right] \end{aligned} \quad (3.19)$$

Then to make sure that V doesn't change, we must have

$$d\frac{\partial V}{\partial x_1} + d\frac{\partial V}{\partial x_2} + \cdots + d\frac{\partial V}{\partial x_N} = 0. \quad (3.20)$$

Notice that we can take out the common factor of d, so really this is

$$\frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2} + \cdots + \frac{\partial V}{\partial x_N} = 0. \quad (3.21)$$

But $-\partial V/\partial x_\mathrm{i}$ is just the force acting on particle i (in the x direction). So we have just shown that the sum of all the forces must be zero, independent of any assumptions about forces acting in paris or ideas about action and reaction: The result follows from the invariance of the potential energy with respect to translations of our coordinate system. Notice that we did this specifically for the x direction, but we could equally have looked along y or z, and hence we know that the sum of the vector forces also is zero.

To finish this discussion, let's be clear once more that the zero sum of all forces results in conservation of momentum. To keep things simple let's just focus on the x components:

$$m_{\rm i}\frac{d^2x_{\rm i}}{dt^2} \equiv \frac{dp_{\rm i}^x}{dt} = -\frac{\partial V}{\partial x_{\rm i}} \quad (3.22)$$

$$\Rightarrow \sum_{\rm i=1}^{N}\frac{dp_{\rm i}^x}{dt} = -\sum_{\rm i=1}^{N}\frac{\partial V}{\partial x_{\rm i}} \quad (3.23)$$

$$\frac{d}{dt}\left[\sum_{\rm i=1}^{N}p_{\rm i}^x\right] = -\sum_{\rm i=1}^{N}\frac{\partial V}{\partial x_{\rm i}} \quad (3.24)$$

$$= 0, \quad (3.25)$$

where in the last step we use Eq (3.21). So what we have shown is that the x component of the total momentum does not change with time. Again we could have done this for the y or z components, and so really we know that *demanding that the potential energy be unchanged under translations implies conservation of momentum.*

It is good to think a bit about this result, which really is quite remarkable. When we say that we want our description of the world to be the same no matter what we choose for the origin of our coordinate system, this sounds like a philosophical statement: we want our mathematical models to have certain "nice" properties. But this particular nice property implies the conservation of momentum, and this a statement about measurable quantities in the physical world around us!

Let's keep going with the idea of invariance under changes of coordinate system. In addition to invariance under translations, we'd also like to insist on invariance under rotations—it shouldn't matter which way we are looking when we decide to set up our coordinate system, just as it doesn't matter where we are standing. Suppose we start with some coordinate system in which the particles are at positions $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_{\rm N}$. Now we turn by some angle θ, and these positions become $\vec{r}\,'_1, \vec{r}\,'_2, \cdots, \vec{r}\,'_{\rm N}$ in our new coordinate system. We want to insist that

$$V(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_{\rm N}) = V(\vec{r}\,'_1, \vec{r}\,'_2, \cdots, \vec{r}\,'_{\rm N}). \quad (3.26)$$

Obviously in order to impose this condition we have to understand how the positions $\vec{r}_{\rm i}$ tranform into $\vec{r}\,'_{\rm i}$ when we rotate.

For simplicity let's work in two dimensional space. Then by the vector position of $\vec{r}_{\rm i}$ the i$^{\rm th}$ particle what we really mean is the pair $x_{\rm i}, y_{\rm i}$:

$$\vec{r}_{\rm i} = x_{\rm i}\hat{\mathbf{x}} + y_{\rm i}\hat{\mathbf{y}}, \tag{3.27}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors in the x and y directions, respectively (also called $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$). Now if we rotate our x and y coordinate system through an angle θ, then the coordinates of a particle will transform as

$$x_{\rm i} \to x'_{\rm i} = x_{\rm i}\cos\theta + y_{\rm i}\sin\theta \tag{3.28}$$
$$y_{\rm i} \to y'_{\rm i} = y_{\rm i}\cos\theta - x_{\rm i}\sin\theta \tag{3.29}$$

It will turn out that we want to know what happens when we make a *small* turn, so that θ is very small. In fact so small that we only want to keep terms linear in θ. Why? Because (as in the discussion of momentum conservation, where we took $\vec{d}$ to be small) we are going to impose our condition on the potential energy by insisting that the derivative of the potential energy with respect to the rotation angle θ is zero, since this is enough to make sure that V never changes no matter how much we rotate.

With this in mind, let's recall that for small θ,

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \cdots, \tag{3.30}$$
$$\sin\theta \approx \theta - \frac{1}{3!}\theta^3 + \cdots, \tag{3.31}$$

so if we want to keep linear terms in θ, we approximate $\cos\theta$ as being 1, and $\sin\theta$ as being θ:

$$x_{\rm i} \to x'_{\rm i} = x_{\rm i} + \theta y_{\rm i} \tag{3.32}$$
$$y_{\rm i} \to y'_{\rm i} = y_{\rm i} - \theta x_{\rm i}. \tag{3.33}$$

So, what happens to the potential energy under this transformation?

We start with

$$V(\vec{r}_1, \vec{r}_2, \cdots) = V(x_1, y_1, x_2, y_2, \cdots) \tag{3.34}$$

because we agree to work in two dimensions (you can do all of this in 3D but the algebra is more complicated!). Thus when we rotate our coordinate system by a small angle θ the potential transforms as

$$V(x_1, y_1, x_2, y_2, \cdots) \to V(x_1{+}\theta y_1, y_1{-}\theta x_1, x_2{+}\theta y_2, y_2{-}\theta x_2, \cdots); \tag{3.35}$$

remember that $\cdots$ means that we keep going until we have listed the coordinates of all N particles in the system.

Since θ is small we can use the Taylor series idea once more:

$$\begin{aligned} & V(x_1+\theta y_1, y_1-\theta x_1, x_2+\theta y_2, y_2-\theta x_2, \cdots) \\ & \quad \approx V(\vec{x}_1, \vec{x}_2, \cdots) + \theta y_1 \frac{\partial V}{\partial x_1} - \theta x_1 \frac{\partial V}{\partial y_1} + \theta y_2 \frac{\partial V}{\partial x_2} - \theta x_2 \frac{\partial V}{\partial y_2} + \cdots . \end{aligned} \tag{3.36}$$

Collecting all of the terms, we find the transformation of V under small rotations:

$$V \to V - \theta \sum_{\mathrm{i}=1}^{\mathrm{N}} \left(x_{\mathrm{i}} \frac{\partial V}{\partial y_{\mathrm{i}}} - y_{\mathrm{i}} \frac{\partial V}{\partial x_{\mathrm{i}}} \right). \tag{3.37}$$

Now what appears under the summation in Eq. (3.37) is an interesting combination of things. First, remember that forces are related to the derivatives of the potential energy:

$$\vec{F}_{\mathrm{i}} = \left(-\frac{\partial V}{\partial x_{\mathrm{i}}} \right) \hat{\mathbf{x}} + \left(-\frac{\partial V}{\partial y_{\mathrm{i}}} \right) \hat{\mathbf{y}}. \tag{3.38}$$

The second thing to remember is about cross products. If we have two vectors in two dimensional (xy) space,

$$\vec{a} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} \tag{3.39}$$

$$\vec{b} = b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}}, \tag{3.40}$$

then their cross product $\vec{a} \times \vec{b}$ is a vector pointing out of the xy plane in the z direction,

$$\vec{a} \times \vec{b} = (a_x b_y - a_y b_x) \hat{\mathbf{z}}. \tag{3.41}$$

Third, recall that $\vec{r}_{\mathrm{i}} = x_i \hat{\mathbf{x}} + y_{\mathrm{i}} \hat{\mathbf{y}}$. Now we can put these three things together to realize that

$$x_{\mathrm{i}} \frac{\partial V}{\partial y_{\mathrm{i}}} - y_{\mathrm{i}} \frac{\partial V}{\partial x_{\mathrm{i}}} = -\hat{\mathbf{z}} \cdot (\vec{r}_{\mathrm{i}} \times \vec{F}_{\mathrm{i}}). \tag{3.42}$$

[Be sure to check that you understand these steps!]

The object $\vec{\tau}_{\mathrm{i}} = \vec{r}_{\mathrm{i}} \times \vec{F}_{\mathrm{i}}$ is something you have seen in your high school physics courses: it is the torque on the i^{th} particle. Notice that the torque always points in the $\hat{\mathbf{z}}$ direction if our particles are confined to the xy plane, which is why sometimes we forget that torque is a vector. So it's interesting that when we ask how the potential energy transforms under rotations, the torque just pops out:

$$V \to V + \theta\hat{\mathbf{z}}\cdot\left(\sum_{\mathrm{i}=1}^{\mathrm{N}} \vec{\tau}_{\mathrm{i}}\right). \tag{3.43}$$

Furthermore, if we want to have *invariance* under rotations then the coefficient of θ needs to be zero, so we must have

$$\sum_{\mathrm{i}=1}^{\mathrm{N}} \vec{\tau}_{\mathrm{i}} = 0. \tag{3.44}$$

But now we are almost done. Let's look at Netwon's equations,

$$\frac{d\vec{p}_{\mathrm{i}}}{dt} = \vec{F}_{\mathrm{i}}. \tag{3.45}$$

Notice that we have one of these (vector) equations for each particle. Since we know something about torques, it makes sense to take the cross product of each side of the equation with the vector $\vec{r}_{\mathrm{i}}$:

$$\vec{r}_{\mathrm{i}}\times\frac{d\vec{p}_{\mathrm{i}}}{dt} = \vec{r}_{\mathrm{i}}\times\vec{F}_{\mathrm{i}} = \vec{\tau}_{\mathrm{i}}. \tag{3.46}$$

The combination on the left of this equation is a little awkward, so we can make it nicer by realizing that

$$\frac{d(\vec{r}_{\mathrm{i}}\times\vec{p}_{\mathrm{i}})}{dt} = \frac{d\vec{r}_{\mathrm{i}}}{dt}\times\vec{p}_{\mathrm{i}} + \vec{r}_{\mathrm{i}}\times\frac{d\vec{p}_{\mathrm{i}}}{dt}. \tag{3.47}$$

Since $\vec{p}_{\mathrm{i}} = m_{\mathrm{i}}(d\vec{r}_{\mathrm{i}}/dt)$, the term

$$\frac{d\vec{r}_{\mathrm{i}}}{dt}\times\vec{p}_{\mathrm{i}} = m_{\mathrm{i}}\frac{d\vec{r}_{\mathrm{i}}}{dt}\times\frac{d\vec{r}_{\mathrm{i}}}{dt} = 0, \tag{3.48}$$

since the cross product of a vector with itself is zero. Thus

$$\vec{r}_{\mathrm{i}}\times\frac{d\vec{p}_{\mathrm{i}}}{dt} = \frac{d(\vec{r}_{\mathrm{i}}\times\vec{p}_{\mathrm{i}})}{dt}, \tag{3.49}$$

and hence Newton's equations imply that

$$\frac{d(\vec{r}_{\mathrm{i}}\times\vec{p}_{\mathrm{i}})}{dt} = \vec{\tau}_{\mathrm{i}}. \tag{3.50}$$

As you know, we call the combination

$$\vec{L}_{\mathrm{i}} \equiv \vec{r}_{\mathrm{i}}\times\vec{p}_{\mathrm{i}} \tag{3.51}$$

the angular momentum of the i^{th} particle; Eq. (3.50) usually is written as

$$\frac{d\vec{L}_{\mathrm{i}}}{dt} = \vec{\tau}_{\mathrm{i}}. \tag{3.52}$$

Finally, let's add up all these equations (one for each particle):

$$\sum_{\mathrm{i}=1}^{\mathrm{N}} \frac{d\vec{L}_{\mathrm{i}}}{dt} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \vec{\tau}_{\mathrm{i}} \tag{3.53}$$

$$\frac{d\vec{L}_{\mathrm{total}}}{dt} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \vec{\tau}_{\mathrm{i}}, \tag{3.54}$$

where the total angular momentum is the sum of individual angular momenta, by analogy with the total (linear) momentum $\vec{P}$ defined above,

$$\vec{L}_{\mathrm{total}} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \vec{L}_{\mathrm{i}}. \tag{3.55}$$

Now put Eq. (3.54) together with Eq. (3.44), and we find

$$\frac{d\vec{L}_{\mathrm{total}}}{dt} = 0, \tag{3.56}$$

which means that angular momentum is conserved.

To recap:

- The equation

$$\frac{d\vec{L}_{\mathrm{i}}}{dt} = \vec{\tau}_{\mathrm{i}} \tag{3.57}$$

  is just $F = ma$ in disguise (for particle i), so there is nothing really new here.

- In principle the forces $\vec{F}_{\mathrm{i}}$ can be arbitrary functions of position.

- If we assume that

  a. forces are derived from a potential energy V,
  b. the potential energy does not depend on the coordinate system is which we measure the particle positions, and
  c. in particular V is invariant under rotations of our coordinate system,

  then the total angular momentum $\vec{L}_{\mathrm{total}} = \sum \vec{L}_{\mathrm{i}}$ is conserved,

$$\frac{d\vec{L}_{\mathrm{total}}}{dt} = 0. \tag{3.58}$$