Invariants of fast solutions of KdV-Burgers Equations

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1 Introduction

We are concerned with invariants of the solutions of

$$\frac{dU_k}{dt} = U_k(U_{k-1} - U_{k+1}) \tag{1}$$

with either cyclic boundary conditions

$$U_0 = U_N,\ U_{N+1} = U_1 \tag{2}$$

or zero boundary conditions

$$U_0 = 0,\ U_{N+1} = 0 \tag{3}$$

The zero boundary condition case is discussed in Moser [3] for the related problem

$$\frac{dA_k}{dt} = A_k(A_{k-1}^2 - A_{k+1}^2) \tag{4}$$

which is obtained from eq. (1) by replacing U_k with A_k^2 and rescaling t, while the cyclic boundary condition case is discussed in Goodman & Lax [2] for even N.

Moser notes the existence of $\lceil N/2 \rceil$ polynomials of A_k^2 that are invariant and states that they are independent. They are the coefficients of the characteristic equation of an $N+1$ by $N+1$ zero-diagonal symmetric Jacobi matrix whose terms depend on the A_k. Goodman & Lax discuss $N/2$ invariants expressed as the traces of powers of an N by N zero-diagonal symmetric Jacobi matrix and mention an additional invariant and this has recently been discussed in [1]. Here we give an explicit form of the invariants in both cases (including the cyclic case for odd N), show that this form is equivalent to the forms in the cited work, and show that they are functionally independent.

2 An Explicit Form of the Invariants

From now on we will consider indices to be restricted to the range $1, \cdots, N$. References outside that range (that is, to 0 or $N+1$) will be handled by applying eq. (2) or eq. (3) depending on the boundary condition.

We define a pair of integers to be *non adjacent* if they differ by at least 2. In the cyclic boundary condition case, integers 1 and N are considered to be adjacent. We define an **I**-set of the integers to be any non-null set of integers that are mutually non-adjacent.

In particular, any singleton integer is an **I**-set, no **I**-set can contain more than $\lfloor N/2 \rfloor$ members ($\lceil N/2 \rceil$ in the zero boundary case). There are either one, two, or three of these largest **I**-sets, depending on the boundary condition and the evenness of N[1].

It will be convenient to associate each **I**-set with an N-bit string that has a 1 entry in the i-th position if i is in the **I**-set. In this notation, an **I**-set is simply a N-bit string that has at least one 0 between each 1 entry and, in the cyclic case, a 0 on at least one end of the string. There is a (1-1) correspondence between each such string and each **I**-set, so we will use the notations interchangeably.

Two **I**-sets are *distinct* if and only if at least one member is different. We define an $\mathbf{I}_j$-set to be an **I**-set with exactly j members, and $\mathbf{S}_j$ to be the set of all distinct $\mathbf{I_j}$-sets.

Theorem

The following polynomials are invariants of eq. (1)

$$\phi_j(U_1, U_2, \cdots, U_N) = \sum_{\mathbf{I} \in \mathbf{S}_j} \prod_{i \in \mathbf{I}} U_i \tag{5}$$

.

Proof

Differentiating eq. (5) and using eq. (1) we get

$$\frac{d\phi_j}{dt} = \sum_{\mathbf{I} \in \mathbf{S}_j} \sum_{k \in \mathbf{I}} \prod_{i \in \mathbf{I}} U_i (U_{k-1} - U_{k+1}) \tag{6}$$

Working with the bit-string representation, differentiation doubles the number of products due to the $(U_{k-1} - U_{k+1})$ factor and changes a zero to a one, once on the left and once on the right of an existing one in an **I** set since there is at least one zero between any pair of ones in an **I**. There are two cases to consider: a single zero between a pair of ones, and two or more zeros. In the single zero case where part of the string contains $[\cdots 01010 \cdots]$ differentiation of the U represented by the first 1 introduces the term $[\cdots 01110 \cdots]$ with a negative sign while differentiation of the U represented by the second 1 introduces the term $[\cdots 01110 \cdots]$ with a positive sign. Hence they cancel. In the multiple zero case, where part of the string contains $[\cdots 0100 \cdots]$ differentiation of the U represented by the its 1 introduces the term $[\cdots 0110 \cdots]$ with a negative sign. However, since all non-adjacent combinations are present in S_j the term $[\cdots 0010 \cdots]$ is also present and differentiation of its U introduces the term $[\cdots 0110 \cdots]$ with a positive sign. Hence all terms cancel and the expression is invariant. QED

[1]For the cyclic case and even N the two sets are $\{i{+}1,\ i{+}3,\ i{+}5,\ \cdots,\ i + N - 1\}$ for $i = 0$ or 1. If N is odd the three largest sets are $\{i{+}1,\ i{+}3,\ i{+}5,\ \cdots,\ i + N - 2\}$ for $i = 0$, 1, or 2. For the zero boundary case the largest is $\{i{+}1,\ i{+}3,\ i{+}5,\ \cdots,\ i + N - 1\}$ for $i = 0$ or 1 when N is even, or $\{1, 3, 5, \cdots, N\}$ when N is odd.

In the cyclic case, the function

$$\phi_0 = \prod_{i=1}^{N} U_i \tag{7}$$

is also invariant, as can be seen by direct computation.

Theorem
The invariants eq. (5) (plus eq. (7) in the cyclic case) are functionally independent.

Proof
If the invariants were functionally dependent then there exists an F such that

$$F(\{\phi_j\}) \equiv 0$$

Differentiating w.r.t. U_i we have

$$\sum_j \frac{\partial F}{\partial \phi_j} \frac{\partial \phi_j}{\partial U_i} = 0, \ i = 1, \cdots, N$$

In other words, the Jacobian $J = \partial\phi_j/\partial U_i$ would not have full rank. We prove the theorem by showing that J has full rank. We do this by setting $U_i = \epsilon^{i-1}$, deleting columns of J, and showing that there exists an ϵ_0 such that the remaining square matrix is non-singular for all $\epsilon < \epsilon_0 > 0$.

In the cyclic case we have the additional invariant ϕ_0. We place this in the last position so that it determines the last row of J, but in the presentation of the proof below this row is assumed absent unless it is specifically stated otherwise.

Each entry in the j-th row of J is the sum of products of $j-1$ U_i's so is a sum of powers of ϵ. We want to identify the smallest power present in each term. Clearly it comes from the entries with the lowest indexed U_i's. In the j-th row the lowest term that can be present is $U_1 U_3 \cdots U_{2j-3}$ which will lead to a term $\epsilon^{(j-2)(j-1)}$ and this cannot appear in any column to the left of the $2j-1$-st column. All elements to the left of this entry in this row will have a higher power of ϵ and no element to the right of it will have a lower power. (For $j = 1$ this is a null statement since all elements in the first row are 1.) In the cyclic case, the last row of J has $\epsilon^{N(N-1)/2+1-j}$ in the j-th column, so its lowest power is in the last position.

Divide the j-th row by the lowest power of ϵ present in that row. This does not change the rank of the matrix. Now the matrix consisting of the odd-numbered columns (plus the last column in the cyclic even-N case) has the following property: In each row elements to the left of the diagonal are $\mathrm{O}(\epsilon)$, the diagonal elements are $1 + \mathrm{O}(\epsilon)$, while elements to the right of the diagonal are no larger than $1 + \mathrm{O}(\epsilon)$. Hence, as $\epsilon \to 0$ the determinant of this matrix $\to 1$. Hence, there exists ϵ_0 such that the matrix is non-singular for $0 \leq \epsilon \leq \epsilon_0$. Hence the invariants are independent.

3 Equivalence of Invariants to those of Moser and Goodman-Lax

Defining $A_j = +\sqrt{U_j}$, Moser considers the $N{+}1$ by $N{+}1$ Jacobi matrix

$$L = \begin{pmatrix} 0 & A_1 & 0 & & & \\ A_1 & 0 & A_2 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & A_{N-1} & 0 & A_N \\ & & & 0 & A_N & 0 \end{pmatrix} \tag{8}$$

and shows that eq. (1) is an isospectral transformation of L and hence that the coefficients of L's characteristic polynomial are invariants. Goodman & Lax consider the N by N matrix

$$L = \begin{pmatrix} 0 & A_1 & 0 & & & A_N \\ A_1 & 0 & A_2 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & A_{N-2} & 0 & A_{N-1} \\ A_N & & & 0 & A_{N-1} & 0 \end{pmatrix} \tag{9}$$

for even N and show that the trace of the powers of L are invariant. Note that because of the structure of L only the even powers of L have non-zero traces and only the coefficients of λ^{N-2n}, $n = 1, 2, \cdots$ in the characteristic polynomial of L yield meaningful invariants. (The above statement is not true for odd N in the cyclic case.) These are, of course, related conditions on L since the trace of a matrix is the sum of its eigenvalues.

It is convenient to re-order the rows and columns of L placing the odd-numbered rows and columns first. When we do this for eq. (8) we get

$$L = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix} \tag{10}$$

where the matrix B is

$$B = \begin{pmatrix} A_1 & A_2 & 0 & & \\ 0 & A_3 & A_4 & & \\ & \cdot & \cdot & \cdot & \\ & & & A_{N-1} & A_N \end{pmatrix} \tag{11}$$

if N is even, or

$$B = \begin{pmatrix} A_1 & A_2 & 0 & & \\ 0 & A_3 & A_4 & & \\ & \cdot & \cdot & \cdot & \\ & & & A_{N-2} & A_{N-1} \\ & & & & A_N \end{pmatrix} \tag{12}$$

if N is odd.

When we apply the renumbering to eq. (9) for even N we get a similar structure with

$$B = \begin{pmatrix} A_1 & A_2 & 0 & & 0 \\ 0 & A_3 & A_4 & & \\ & . & . & . & \\ & & & A_{N-3} & A_{N-2} \\ A_N & & & & A_{N-1} \end{pmatrix} \tag{13}$$

Note that L^2 is block triangular with two blocks, BB^T and B^TB. B is either square or has one more column than row. BB^T and B^TB have the same eigenvalues if B is square, otherwise B^TB has an additional zero eigenvalue. Hence the eigenvalue set of L^2 consists of pairs of the eigenvalues of BB^T plus a zero value if N is odd. (The corresponding eigenvalues of L occur in alternating sign pairs.) Thus, the Moser invariants are just the invariancy of the characteristic polynomial $C(BB^T)$. We will show that the coefficients of $C(BB^T)$ are the polynomials ϕ_j given in the previous section.

In the cyclic even N case, Goodman and Lax note that the traces of L^{2n} are invariants for $n = 1, 2, \cdots N/2$. Since

$$tr(L^{2n}) = 2\sum_{i=1}^{N/2} \lambda_i^n(L^2) \tag{14}$$

where $\{\lambda_i(L^2)\}$ is the set of $N/2$ eigenvalues of L^2, this is equivalent to the conditions that the eigenvalues of L^2 are invariant, and hence that $C(BB^T)$ is invariant.

4 Coefficients of the Characteristic Polynomial

While the polynomials given in the first section are invariants for all cases, including a cyclic boundary and odd N, here we exclude that case so that we can study the $C(L^2)$ by studying $C(BB^T)$, If N is even, $W = BB^T$ is

$$W = \begin{pmatrix} A_1^2 + A_2^2 & A_2A_3 & 0 & \cdots & 0 & 0 & cA_NA_1 \\ A_2A_3 & A_3^2 + A_4^2 & A_4A_5 & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & A_{N-4}A_{N-3} & A_{N-3}^2 + A_{N-2}^2 & A_{N-2}A_{N-1} \\ cA_NA_1 & 0 & 0 & \cdots & 0 & A_{N-2}A_{N-1} & A_{N-1}^2 + A_N^2 \end{pmatrix} \tag{15}$$

where $c = 1$ for the cyclic case, 0 otherwise. If N is odd (the zero boundary case) BB^T is

$$W = \begin{pmatrix} A_1^2 + A_1^2 & A_2A_3 & 0 & \cdots & 0 & 0 & 0 \\ A_2A_3 & A_3^2 + A_4^2 & A_4A_5 & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & A_{N-3}A_{N-2} & A_{N-2}^2 + A_{N-1}^2 & A_{N-1}A_N \\ 0 & 0 & 0 & \cdots & 0 & A_{N-1}A_N & A_N^2 \end{pmatrix} \tag{16}$$

We write the characteristic polynomial of W as

$$C(W) = \det(\lambda I - W) = \sum_{j=0}^{M} (-1)^j \lambda^{M-j} \sum_q \{P_{jq}\} \tag{17}$$

where M is the dimension of W and $\{P_{jq}\},\ q = 1, \cdots$ is the set of all principal minors of W of size j. There is a principal minor of size j corresponding to each set of j different integers from $1, \cdots, M$.

Theorem

$$\sum_q \{P_{jq}\} = \sum_{\mathbf{I} \in \mathbf{W}_j} \prod_{i \in \mathbf{I}} U_i + 2\delta_{nj} \prod_{i=1}^{N} A_i \tag{18}$$

where δ_{jk} is the Kroneker delta.

Proof
A principal minor, P_{jq}, of W of size j has the form

$$P_{j.} = \det \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & T_k \end{pmatrix} \tag{19}$$

Except for two special cases, each T_i is a tridiagonal matrix of the form in eq. (15) with $c = 0$ and indices ranging from $2p_j - 1$ to $2q_j$ rather than from 1 to N. The first special case is the zero boundary case with odd N where T_k could have the form of eq. (16) with an initial index of $2p_j$ instead of 1. The second is the cyclic boundary with $j = N/2$ when the principal minor is the whole matrix $\det(W)$ where W is given in eq. (15) with $c = 1$. Let us dispose of this case first. Define $d(c) = \det(W)$ for the W in eq. (15). We have

$$d(1) = d(0) + 2 \prod_{i=1}^{N} A_i \tag{20}$$

This explains the second term on the *rhs* of eq. (18). From now on we can just consider the zero boundary case.

From eq. (19) we have $P_{j.} = \prod_{i=1,\cdots k} \det(T_i)$. An important observation is that each $\det(T_i)$ in the product is a polynomial in a set of adjacent $\{A_m\}$ that are non-adjacent to all $\{A_m\}$ in any other $\det(T_k)$ in that product. This can be seen by considering the case $N = 8$ for the zero boundary case and examining the $P_{3.}$ obtained by removing the third row and column to get

$$P_{3.} = \det \begin{pmatrix} A_1^2 + A_2^2 & A_2 A_3 & 0 \\ A_2 A_3 & A_3^2 + A_4^2 & 0 \\ 0 & 0 & A_7^2 + A_8^2 \end{pmatrix} \tag{21}$$

When there is a "break" in the integer sequence that determines the principal minor, the off diagonal element is missing and the indices of the A's in the next block are at least 3 larger.

The sum of the dimensions of the T blocks is j as each T_i corresponds to each consecutive group of integers in the selection of j from $1, \cdots, M$ that determines the particular $P_{j.}$. Note that in the cyclic case, a T_i may "wrap around" the end of W. For example, if $N = 8$ ($M = 4$) with the third row and column removed, the sole T after a reordering of row and columns is

$$T_1 = \begin{pmatrix} A_7^2 + A_8^2 & A_8 A_1 & 0 \\ A_8 A_1 & A_1^2 + A_2^2 & A_2 A_3 \\ 0 & A_2 A_3 & A_3^2 + A_4^2 \end{pmatrix} \tag{22}$$

We will show that if the dimension of T_i is m_i then $\det(T_i)$ consists of all products without adjacent members of m_i different A_k^2 for the A_k's occurring in T_i. Since $P_{j.}$ is the product of a set of $\det(T_i)$ where the A_k members of different T_i are non-adjacent, P_j consists of products of $j = \sum_i m_i$ non-adjacent A_k^2's where the k's are the combination of j integers from $1, \cdots, M$ that determine $P_{j.}$. Since eq. (18) sums over all combinations, all products of j non-adjacent $U_i = A_i^2$ are present, and each appears only once.

To complete the proof we need to show the

Lemma *If the M by M matrix W is as given in eq. (15) with $c = 0$ or eq. (16) then*

$$\det(W) = \sum_{I \in I_M} \prod_{i \in I} U_i \tag{23}$$

where I_M is a non-adjacent set of M integers from $1, \cdots, N$.

Proof

We proceed by induction. Let $w(M)$ be the $\det(W)$ when W is M by M Defining $w(0)$ as 1 and the product of zero U_i's as 1, it is clearly true for $M = 0$. It is also trivially true for $M = 1$ where $w(1)$ is either $U_1 + U_2$ or U_1 corresponding to eq. (15) and eq. (16) respectively. For general M we have

$$w(M) = W_{MM} w(M-1) - W_{M-1,M} W_{M,M-1} w(M-2) \tag{24}$$

where W_{MM} is either $A_{N-1}^2 + A_N^2$ or A_N^2 and $W_{M-1,M}^2$ is either $A_{N-2}^2 A_{N-1}^2$ or $A_{N-1}^2 A_N^2$. In either case, the first term on the *rhs* of eq. (24) generates all products of M different U_j that are non-adjacent, but it also generates terms with a single adjacency, namely between U_L and U_{L-1} where L is either $N-1$ or N. That is exactly the set of terms subtracted by the second term on the *rhs* of eq. (24). Thus $w(M)$ has the desired property. QED

This shows that the Goodman Lax invariants (the traces of even powers of L) plus the product of all U_i) are equivalent to the simple algebraic invariants given earlier when N is even. These algebraic expressions are also invariant when N is odd although there appears to be no corresponding matrix formulation.

In the zero boundary case we have shown that there is a simple algebraic representation of the invariants given by Moser.

References

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