# Dual Methods for Total Variation-Based Image Restoration

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## Outline

1. **Introduction to image processing**

2. Total variation (TV) regularization
   (a) Primal
   (b) Dual

3. Solving dual TV
   (a) Barrier relaxation method

4. Summary, future work, and acknowledgements

## Digital Images

**Digital Image** An image $f(x, y)$ discretized in both spatial coordinates and in brightness

**Digital Image = Matrix**

- row and column indices = point in image
- matrix element value = gray level at that point
- **pixel** = element of digital array or **picture element**

## Digital Image Processing

**Digital Image Processing** Set of techniques for the manipulation, correction, and enhancement of digital images

### Methods

- Fourier/wavelet transforms
- Stochastic/statistical methods
- **Partial differential equations (PDEs) and differential/geometric models**
  - Systematic treatment of geometric features of images (shape, contour, curvature)
  - Wealth of techniques for PDEs and computational fluid dynamics

## Examples of Digital Image Processing

**Smoothing** Removing bad data

**Sharpening** Highlighting **edges** (discontinuities)

**Restoration** Determination of unknown original image from given noisy image

- Ill-conditioned inverse problem
- No unique solution
- **Regularization techniques** impose desirable properties on the solution

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## Digital Image Restoration

- **Wanted:** De-noised image $u$ with edges

- **Given**
  - Observed image $z$
  - User-chosen weight $\alpha$

- **Need**
  - Regularity functional $R(u)$
  - Function space $S(\Omega)$

- **Tikhonov regularization**

$$\min_{u \in S(\Omega)} \alpha R(u) + \frac{1}{2} \underbrace{\|u - z\|^2}_{\text{error}}$$

## Total Variation (TV) Regularization

$$\min_{u \in S(\Omega)} \alpha R(u) + \frac{1}{2}\|u - z\|^2$$

- **Total Variation [Rudin-Osher-Fatemi 92]**

$$R(u) = \int_\Omega |\nabla u| \, dx \, dy \quad \text{where}$$
$$|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$$
$$S(\Omega) = W^{1,1}(\Omega), \text{first derivative in } L^1$$

- **Features**
  - $u$ need not be differentiable
  - Discontinuities allowed
  - Derivatives considered in the weak sense

# Euler-Lagrange Equation

- 1st-order necessary condition for the minimizer $u$

$$\min_u \int_\Omega \alpha|\nabla u| + \frac{1}{2}(u-z)^2\,dx\,dy$$

- **Theory**

$$-\alpha\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) + u - z = 0$$

- Degenerate when $|\nabla u| = 0$

- **Practice**

$$\boxed{-\alpha\nabla\cdot\left(\frac{\nabla u}{\sqrt{|\nabla u|^2+\beta}}\right) + u - z = 0}$$

small $\beta > 0$

## Previous Work

$$-\alpha\nabla\cdot\left(\frac{\nabla u}{\sqrt{|\nabla u|^2+\beta}}\right)+u-z=0$$

**Rudin-Osher-Fatemi 1992** Time marching to steady state with gradient descent. Improvement in **Marquina-Osher 1999**.

**Chan-Chan-Zhou 1995** Continuation procedure on $\beta$.

**Vogel-Oman 1996** Fixed point iteration.

## Dependence on $\beta$

$$\boxed{-\alpha\nabla\cdot\left(\frac{\nabla u}{\sqrt{|\nabla u|^2+\beta}}\right)+u-z=0} \qquad (1)$$

$\beta$ **large** Smeared edges.

$\beta$ **small** PDE nearly degenerate.

**No** $\beta$ if we rewrite (1) in terms of new variable...

## Introduce Dual Variable $w$

$$TV(u) = \int_\Omega |\nabla u| \, dx \, dy$$

$$TV(u) = \begin{cases} \displaystyle\int_\Omega \max_{|w|\le 1} (\nabla u \cdot w) \, dx \, dy & u \text{ smooth} \\ \displaystyle\int_\Omega \max_{|w|\le 1} u(\nabla \cdot w) \, dx \, dy & u \text{ non-smooth} \end{cases}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad w_i \in C_0^\infty(\Omega), \quad \text{and} \quad |w|_\infty \le 1$$

**(Giusti 1984, Chan-Golub-Mulet 1995)**

## Interpretation

$$w = \begin{cases} \frac{\nabla u}{|\nabla u|} & u \text{ smooth and } |\nabla u| \neq 0 \\ \text{not unique} & \text{otherwise} \end{cases}$$

## Deriving Dual Formulation

$$\min_u \int_\Omega \alpha|\nabla u| + \frac{1}{2}(u-z)^2\,dx\,dy$$

- "Weak" definition of Total Variation

$$= \min_u \int_\Omega \alpha \max_{|w|\le 1} u(\nabla\cdot w) + \frac{1}{2}(u-z)^2\,dx\,dy$$

- Interchange max and min

$$= \max_{|w|\le 1} \min_u \underbrace{\int_\Omega \alpha u(\nabla\cdot w) + \frac{1}{2}(u-z)^2\,dx\,dy}_{\Psi(u)}$$

- Quadratic function of $u$

$$\nabla\Psi(u) = \vec{0} \Longleftrightarrow u = z + \alpha(\nabla\cdot w)$$

- Write $u$ in terms of $w$

$$\max_{|w|\le 1} \int_\Omega \alpha\,\underbrace{(z+\alpha(\nabla\cdot w))}_{u}(\nabla\cdot w) + \frac{1}{2}(\underbrace{z+\alpha(\nabla\cdot w)}_{u} - z)^2\,dx\,dy$$

## Dual Total Variation Regularization

$$\max_{|w|\leq 1} \int_\Omega \alpha z(\nabla \cdot w) + \frac{3\alpha^2}{2}(\nabla \cdot w)^2 \, dx \, dy$$

- **Advantages**
  - Quadratic objective function in $w$
  - No need for $\beta$
  - $u = z + \alpha(\nabla \cdot w)$
- **Disadvantages**
  - Constrained optimization problem
  - One constraint per pixel

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## Dual Total Variation Algorithms

**Dissertation of C. 2001**

- Primal-Dual Interior Point (Developed by Mulet.)

- **Relaxation (Coordinate Descent) Methods: Easy to code**
  - Dual
  - Hybrid
  - **Barrier: Constrained → unconstrained Suggested by Vandenberghe.**

## Discrete Dual Total Variation Regularization

Discrete image has $N = n \times n$ pixels

- **Continuous**

$$\max_{|w| \le 1} \int_\Omega \alpha z (\nabla \cdot w) + \frac{3\alpha^2}{2} (\nabla \cdot w)^2 \, dx \, dy$$

$$w : \Omega \to \mathbb{R}^2$$

- **Discrete**

$$\max_{\substack{|w_i| \le 1 \\ i=1,\dots,N}} \; \alpha z^T A w + \frac{3\alpha^2}{2} \|Aw\|^2$$

$Aw$ represents $\nabla \cdot w$

$$w_i = \begin{pmatrix} w_i^x \\ w_i^y \end{pmatrix} \in \mathbb{R}^2 \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix} \in \mathbb{R}^{2N}$$

# Barrier Method

- **Detailed**

$$\max_{\substack{|w_i|\le 1\\ i=1,\ldots,N}} \alpha z^T A w + \frac{3\alpha^2}{2}\|Aw\|^2$$

- **General**
  - **Constrained**

$$\min_{\substack{g_i(w)>0\\ i=1,\ldots,N}} f(w)$$

  - **Unconstrained**

$$\min_w B(w,\mu)$$

where

$$B(w,\mu) = f(w) - \mu \sum_{i=1}^{N} \log g_i(w)$$

## Barrier Method

Constrained problem $\rightarrow$ sequence of unconstrained problems

- **Barrier Function**

$$B(w,\mu) = f(w) - \underbrace{\mu \sum_{i=1}^{N} \log g_i(w)}_{\substack{\text{infinite penalty} \\ \text{for violating feasibility} \\ \text{as } \mu \to 0}}$$

- **Barrier Method: Solves sequence of**

$$\min_w B(w,\mu_k) \quad \text{for } \mu_k \searrow 0$$

- Use **Relaxation Method** to solve each

$$\min_w B(w,\mu_k)$$

for fixed $\mu_k$.

## Relaxation Method

$$\min_{w} B(w, \mu_k)$$

- **Implementation**

1. Fix all components of $w$ except for the $i$th component.

2. Minimize $B(w, \mu_k)$ with respect to $w_i \in \mathbb{R}^2$:

   $$\min_{w_i} B(w_i, \mu_k) + \text{terms independent of } i$$

   Newton's method with backtracking line search

3. Update $w_i$ **(Gauss-Seidel implementation converges for convex unconstrained problems)**

4. Repeat procedure for $i = 1, \ldots, N$

5. Iterate until convergence

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## Summary

- Dual Total Variation problem solved
- No smoothing parameter required

## Future Work

- Deeper understanding of $w$ for non-smooth $u$
- More realistic images
- Barrier relaxation algorithm
  - Tighter control of stopping criteria
  - Multigrid implementation

## Collaborators

- **Tony F. Chan (UCLA)**
  - Professor, Department of Mathematics
  - Dean, Division of Physical Sciences, College of Letters and Science
- **Pep Mulet (University of València, Spain)**
  - Professor, Department of Applied Mathematics
- **Lieven Vandenberghe (UCLA)**
  - Associate Professor, Department of Electrical Engineering

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- **Institute for Mathematics and its Applications**

- **Audience**