“Random Walks, Trees and Extensions of Riordan Group Techniques”

by
Naiomi T. Cameron
Howard University
Department of Mathematics

Advisor: Dr. Louis Shapiro
Department of Mathematics

Annual Joint Mathematics Meetings
Baltimore, MD

January 2003
1. Introduction

- The Catalan Numbers and Their Interpretations/Functions Related to the Catalan Function
- The Ternary Numbers and Generalized Dyck Paths
- The Riordan Group

2. Main Results

- The Chung-Feller Theorem
- Analogues to Catalan-Related Functions
- The Expected Number of Returns
- Area Under Ternary Paths
3. Future Work

- Determinant Sequences of the Ternary Numbers

4. Conclusion
A path, $P$, of length $n$ from $(x_0, y_0)$ to $(x, y)$ with step set $S$ is a sequence of points in the plane,

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = (x, y)$$

such that all $(x_{i+1} - x_i, y_{i+1} - y_i) \in S$.

The points are called vertices. The height of a vertex, $v$, is the ordinate of that point. A path, $P$, is positive if each of its vertices has nonnegative height.

Dyck paths are positive paths from $(0,0)$ to $(2n,0)$ with $S = \{(1,1), (1,-1)\}$. Below are the five Dyck paths of length 6.
A (rooted) **plane tree** or **tree**, $T$, is a connected graph with no cycles, where one vertex is designated as the root.

The five plane trees on 4 vertices:

A plane $m$-ary tree is a plane tree in which every vertex (including the root) has degree 0 or $m$.

The five binary (or 2-ary) trees with 6 edges:
Let $D_n$ denote the set of all Dyck paths of length $2n$. Let $A_n$ denote the set of all plane trees with $n$ edges.

$$|D_n| = |A_n| = \frac{1}{n+1} \binom{2n}{n}$$  \hspace{1cm} (1)

The $n$th Catalan number, $\frac{1}{n+1} \binom{2n}{n}$, is denoted $c_n$, and

$$C(z) = \sum_{n=0}^{\infty} c_n z^n$$

is called the Catalan function. We have that

$$C(z) = 1 + zC^2(z)$$

and

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$
The $n$th Catalan number, $c_n = \frac{1}{n+1} \binom{2n}{n}$, is the number of:

- triangulations of a convex $(n+2)$-gon into $n$ triangles with $n-1$ nonintersecting (except at a vertex) diagonals,

- planted (i.e., degree of the root is 1) plane trees with $2n+2$ vertices where every non-root vertex has degree 0 or 2,

- Dyck paths from $(0,0)$ to $(2n+2,0)$ with no peak at height 2,

- noncrossing partitions of $[n]$. 
Definition 1. The Central Binomial function, $B(z)$, is the generating function for the central binomial coefficients, $\binom{2n}{n}$. That is,

$$B(z) = \sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1 - 4z}}$$

Definition 2. The Fine function, $F(z)$, is the generating function for the Fine numbers,

$$1, 0, 1, 2, 6, 18, 57, 186, ...$$

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = \frac{1 - \sqrt{1 - 4z}}{z(3 - \sqrt{1 - 4z})}$$

Definition 3. The Motzkin function, $M(z)$, is the generating function for the Motzkin numbers $1, 1, 2, 4, 9, 21, 51, 127, 323, ...$

$$M(z) = \sum_{n=0}^{\infty} m_n z^n = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$
**Identity 1.** \( B(z) = 1 + 2zC(z)B(z) \)

**Identity 2.** \( F(z) = \frac{C(z)}{1+zC(z)} \)

**Identity 3.** \( M(z) = \frac{1}{1-z}C\left(\frac{z^2}{(1-z)^2}\right) \)

\( B(z) \) counts paths from \((0,0)\) to \((2n,0)\) with \( S = \{(1,1), (1,-1)\} \).

\( F(z) \) counts Dyck paths with no hills and plane trees with no leaf at height 1.

\( M(z) \) counts the number of positive paths from \((0,0)\) to \((n,0)\) using \( S = \{(1,1), (1,0), (1,-1)\} \), and the number of plane trees with \( n \) edges where every vertex has degree \( \leq 2 \).
Let $\mathcal{T}_n$ denote the set of all positive paths from $(0, 0)$ to some $(3n, 0)$ with $S = \{(1, 1), (1, -2)\}$. We call these paths ternary paths.

Let $t_n = |\mathcal{T}_n|$.

**Theorem 1.** Let $T(z) = \sum_{n=0}^{\infty} t_n z^n$. Then:

(i.) $T(z) = 1 + z T^3(z)$

(ii.) $t_n = \frac{1}{2n+1} \binom{3n}{n}$

A generalized $t$-Dyck path is a positive path from $(0, 0)$ to $((t+1)n, 0)$ with $S = \{(1, 1), (1, -t)\}$. 
The ternary numbers, \( \frac{1}{2n+1} \binom{3n}{n} \), also count:

- positive paths starting at \((0, 0)\) and ending at \((2n, 0)\) using steps in \(\{(1, 1), (1, -1), (1, -3), (1, -5), (1, -7), \ldots, \}\);

- rooted plane trees with \(2n\) edges where every vertex (including the root) has even degree (referred to as “even trees’’);

- dissections of a convex \(2(n + 1)\)-gon into \(n\) quadrangles by drawing \(n - 1\) diagonals;

- noncrossing trees on \(n + 1\) points;
• rooted plane trees with $3n$ edges where every node (including the root) has degree zero or three (called ternary trees)

The three ternary trees with 6 edges:
Definition 4. An infinite lower triangular matrix, \( L = (l_{n,k})_{n,k \geq 0} \) is a Riordan matrix if there exist generating functions \( g(z) = \sum g_n z^n \), \( f(z) = \sum f_n z^n \), \( f_0 = 0 \), \( f_1 \neq 0 \) such that \( l_{n,0} = g_n \) and \( \sum_{n \geq k} l_{n,k} z^n = g(z)(f(z))^k \).

\( L \) is called **Riordan** and we write \( L = (g(z), f(z)) \), or simply \( L = (g, f) \).

**Example 1.**

\[
\begin{pmatrix}
1 \\
1 & 1 \\
2 & 2 & 1 \\
5 & 5 & 3 & 1 \\
14 & 14 & 9 & 4 & 1 \\
42 & 42 & 28 & 14 & 5 & 1 \\
132 & 132 & 90 & 48 & 20 & 6 & 1 \\
\vdots
\end{pmatrix}
\]

\((C(z), zC(z)) = \)
Theorem 2. (Chung, Feller) Let $c_{n,k}$ denote the number of paths from $(0,0)$ to $(2n,0)$ with steps in $\{(1,1), (1,-1)\}$ such that $k$ up steps lie above the $x$-axis, $k = 0, 1, \ldots, n$. Then $c_{n,k} = c_n = \frac{1}{n+1} \binom{2n}{n}$, independently of $k$.

Theorem 3. Let $u_{n,k}$ be the number of paths from $(0,0)$ to $(3n,0)$ with steps in $\{(1,1), (1,-2)\}$ such that $k$ up steps lie above the $x$-axis, $k = 0, 1, \ldots, 2n$. Then $u_{n,k} = t_n = \frac{1}{2n+1} \binom{3n}{n}$.

Proof:

$z \leftrightarrow$ an up step

$y \leftrightarrow$ an up step above the $x$-axis

$$U(y, z) = \sum_{0 \leq k \leq n} u_{n,k} y^k z^n$$

$$U(y, z) = \frac{T(z^2)}{1 - \Psi(y, z)}$$

where

$$\Psi(y, z) = y^2 z^2 T^2(y^2 z^2) T(z^2) + y z^2 T(y^2 z^2) T^2(z^2)$$
Theorem 4. The function $N(z) = \sum \binom{3n}{n} z^n$ counts

(i.) paths in $\mathbb{Z} \times \mathbb{Z}$ starting at $(0,0)$ and ending at $(3n,0)$ using steps in $\{(1,1),(1,-2)\}$, and

(ii.) even trees with $2n$ edges and one distinguished vertex.

Theorem 5.

(i.) $N(z) = 1 + 3zT^2(z)N(z)$

(ii.) $N(z)T^s(z) = \sum_{n=0}^{\infty} \binom{3n+s}{n} z^n$

Proof: (i) Note $\frac{d}{dz} \left(zT(z^2)\right) = N(z^2)$
Theorem 6. (i.) The “Motzkin analogue”, $M_T(z)$, satisfies

$$M_T(z) = 1 + z + 2z^2 + 5z^3 + 13z^4 + 36z^5 + 104z^6 + \ldots$$

$$= 1 + z M_T(z) + z^2 M_T^2(z) + z^3 M_T^3(z)$$

(ii.) $M_T(z)$ counts positive paths from $(0, 0)$ to $(n, 0)$ using $\{(1, 1), (1, -1), (1, -2), (1, 0)\}$.

Theorem 7. (i.) The “Fine Analogue,” $F_T(z)$ satisfies

$$F_T(z) = \frac{1}{1 + z - z T^2(z)}$$

$$T(z) = \frac{F_T(z)}{1 - z F_T(z)}$$

(ii.) $F_T(z)$ counts ternary paths with no bumps.
Theorem 8. The expected number of returns for generalized Dyck paths from \((0,0)\) to \(((t + 1)n, 0)\) is

\[
\frac{(t + 2)n}{tn + 2}
\]

with variance

\[
\frac{2tn(n - 1) ((t + 1)n + 1)}{(tn + 2)^2(tn + 3)}
\]

Proof: Let \(Y(n)\) denote the number of returns. Let \(p_m(n)\) denote the probability that a randomly chosen path of length \((t+1)n\) has \(m\) returns. Then

\[
P_Y(n)(z) := \sum_{m=0}^{\infty} p_m(n) z^n
\]

\[
= \frac{t(tn + 1)}{(t + 1)((t + 1)n - 1)} zF(2, 1-n; 2-(t+1)n; z)
\]

The limiting distribution of \(Y(n)\) is \(\text{negbin}(2, \frac{t}{t+1})\).
Theorem 9. (Woan, Rogers, Shapiro) The sum of the areas of all strict Dyck paths of length $2n$ is $4^{n-1}$.

Theorem 10. (Kreweras/Merlini et al./Chapman) The sum of the areas of all Dyck paths of length $2n$ is $4^n - \frac{1}{2}\binom{2n+2}{n+1}$.

Let $D_n$ denote the set of Dyck paths of length $2n$. Define the area of $D \in D_n$, $a(D)$, to be the area of the region bounded by $D$ and the $x$-axis.

Let $B_n$ denote the set of all plane binary trees with $2n$ edges. Let $B \in B_n$. Define the total weight of $B$, $\omega(B)$ by

$$\omega(B) := \sum_{v \in B} hgt(v)$$

Theorem 11. For all $n \in \mathbb{N}$,

$$\sum_{B \in B_n} \omega(B) = 2 \sum_{D \in D_n} a(D)$$
Let $S_n$ denote the set of all ternary paths in $T_n$ with exactly 1 return (strict ternary paths).

Let $a_n^s$ (resp. $a_n^t$) denote the total area of the region bounded by the paths of $S_n$ (resp. $T_n$) and the $x$-axis.

Let $h_{n,k}^s$ (resp. $h_{n,k}^t$) denote the number of points in $S_n \cap \mathbb{Z} \times \mathbb{Z}$ (resp. $T_n \cap \mathbb{Z} \times \mathbb{Z}$) which have height $k$. 
Theorem 12. Let $H_s^k(z) = \sum_{n=0}^{\infty} h_{n,k}^s z^n$ and $H_t^k(z) = \sum_{n=0}^{\infty} h_{n,k}^t z^n$.

(i.)

$$H_s^k(z) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k-j}{j} (zT^3(z))^{k-j}$$

$$= (zT^3(z))^k \frac{z^2 T^3(\frac{z^2}{1-z})}{z^2 T^3(\frac{z^2}{1-z}) - 1}$$

(ii.)

$$H_t^k(z) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k-j}{j} (zT^3(z))^{k-j} T^2(z)$$

$$= (zT^3(z))^{k+2} \frac{z^2 T^3(\frac{z^2}{1-z})}{z^2 T^3(\frac{z^2}{1-z}) - 1}$$
\[
\begin{bmatrix}
1 \\
0 & 1 & 1 \\
0 & 3 & 4 & 2 & 1 \\
0 & 12 & 18 & 13 & 9 & 3 & 1 \\
\vdots
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
3 \\
21 \\
144 \\
981 \\
6663 \\
\vdots \\
a_n^s \\
\vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
2 & 1 & 1 \\
7 & 5 & 6 & 2 & 1 \\
30 & 25 & 33 & 17 & 11 & 3 & 1 \\
\vdots
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
3 \\
27 \\
207 \\
1506 \\
10692 \\
\vdots \\
a_n^t \\
\vdots
\end{bmatrix}
\]
Let $\mathcal{W}_n$ denote the set of all ternary trees with $3n$ edges.

**Conjecture 1.** For all $n \in \mathbb{N}$,

$$\sum_{W \in \mathcal{W}_n} \omega(W) = \sum_{T \in \mathcal{T}_n} a(T)$$

Let $t(k, n)$ denote the total number of vertices at height $k$ in $\mathcal{W}_n$ and $\tau(y, z) = \sum t(k, n)y^kz^n$. Then

$$\frac{d}{dy}(\tau(y, z))|_{y=1} = \frac{3zT^3(z)}{(1 - 3zT^2(z))^2}$$

$$= 3z + 27z^2 + 207z^3 + 1506z^4 + 10692z^5 + \ldots$$

21
Given a sequence \( \{a_n\}_{n=0}^{\infty} \), define

\[
A_n^k := \begin{vmatrix}
    a_k & a_{k+1} & a_{k+2} & \cdots & a_{n+k-1} \\
    a_{k+1} & a_{k+2} & a_{k+3} & \cdots & a_{n+k} \\
    a_{k+2} & a_{k+3} & a_{k+4} & \cdots & a_{n+k+1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n+k-1} & a_{n+k} & a_{n+k+1} & \cdots & a_{2n+k-2}
\end{vmatrix}
\]

**Theorem 13.** (Gronau et al.) Let \( p_{ij} \) be the number of paths leading from \( a_i \) to \( b_j \) in \( G \), let \( p^+ \) be the number of disjoint path systems \( W \) in \( (G,A,B) \) for which \( \sigma(W) \) is an even permutation, and let \( p^- \) be the number of such systems with for which \( \sigma(W) \) is odd. Then \( \det(p_{ij}) = p^+ - p^- \).

Let \( Q_n^k \) be the family of all sets of pairwise vertex disjoint paths in \( G \), \( \xi_0, \xi_1, \ldots, \xi_{n-1} \), such that \( \xi_i \) joins \((-3i,0)\) with \((3(i+k),0)\), \( i = 0, 1, \ldots, n-1 \). The theorem implies that

\[
A_n^k = |Q_n^k|
\]
When \( k = 1, \ n = 3 \), we have

\[
\begin{vmatrix}
Q^1_3 \\
\end{vmatrix} = A^1_3 =
\begin{vmatrix}
1 & 3 & 12 \\
3 & 12 & 55 \\
12 & 55 & 273 \\
\end{vmatrix} = 26
\]

The corresponding vertex disjoint path systems are

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
5 \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
12 \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
9 \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]
A theorem stated in the text is paraphrased here:

**Theorem 14.** (U. Tamm)

\[
A_0^n = \prod_{j=0}^{n-1} \frac{(3j + 1)(6j)!(2j)!}{(4j + 1)(4j)!}
\]

\[
A_1^n = \prod_{j=1}^{n} \frac{(6j-2)}{2(4j-1)}
\]

**Question 1.** What about \(A_k^n\) for \(k \geq 2\)?
Definition 5. An alternating sign matrix is a square matrix of 0s, 1s, and -1s for which

- the entries of each row and each column sum to 1,
- the non-zero entries of each row and each column alternate in sign.

Example 2.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]
Conjecture 2. (Stanley/Mills, Robbins, Rumsey)

Let $F(2n+1)$ denote the number of $(2n+1) \times (2n+1)$ alternating sign matrices which are invariant under a reflection about the vertical axis.

$$F(2n+1) = \prod_{j=1}^{n} \frac{(6j-2)}{2} \frac{2j}{(4j-1)}$$

Question 2. Is there a bijection between vertex disjoint ternary path systems and alternating sign matrices invariant under vertical reflection?