Using Imprecise Measures to Study Component and System Reliability

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Outline

- Binary coherent systems with precise classification
  - Definition
  - Reliability
- Binary state system with imprecise classification
  - Membership functions and probability of fuzzy sets
  - Reliability of degraded components and systems

Binary Coherent Systems

- Range of states, S={0, 1}
- State of the ith component,

$$X_i = \begin{cases} 1, & \text{if component } i \text{ functions} \\ 0, & \text{if component } i \text{ fails} \end{cases}$$

- Structure function,

$$\phi = \phi(\mathbf{X}) = \begin{cases} 1, & \text{if the system functions} \\ 0, & \text{if the system fails.} \end{cases}$$

Binary Coherent Systems (cont.)

- ϕ is a *binary coherent system* if
  1. ϕ is nondecreasing
  2. each component is relevant, i.e. $\forall i$, $\exists(\cdot_i, \mathbf{X}) \ni$

$$\phi(1_i, \mathbf{X}) = 1 \text{ and } \phi(0_i, \mathbf{X}) = 0.$$

Binary Coherent Systems (cont.)

Basic examples of binary coherent systems:

- Series:

$$\phi(\mathbf{X}) = \prod_{i=1}^{n} X_i$$

- Parallel (not stand-by):

$$\phi(\mathbf{X}) = \coprod_{i=1}^{n} X_i = 1 - \prod_{i=1}^{n}(1 - X_i)$$

- k-out-of-n:

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i \geq k \\ 0, & \text{if } \sum_{i=1}^{n} X_i < k \end{cases}$$

Reliability of Binary Coherent Systems

- $X_i \mid p_i \sim \text{Bernoulli}(p_i)$
  - $p_i = P(X_i = 1 \mid p_i)$– reliability of component i
  - $h(\mathbf{p}) = P(\phi(\mathbf{X}) = 1 \mid \mathbf{p})$– reliability of ϕ
  - System reliability is determined by the reliability of its components
- Assumptions:
  - Given $\mathbf{p}$, $X_i \perp X_j \ \forall i \neq j$.
  - Given p_i, $X_i \perp p_j \ \forall j \neq i$.

Reliability of Binary Coherent Systems (cont.)

- Series system:

$$h_S(\mathbf{p}) = P(\phi(\mathbf{X}) = 1 \mid \mathbf{p}) = \prod_{i=1}^{n} p_i$$

- Parallel (not stand-by) system:

$$h_P(\mathbf{p}) = \coprod_{i=1}^{n} p_i = 1 - \prod_{i=1}^{n} (1 - p_i)$$

- k-out-of-n system:

$$h_K(\tilde{p}) = \sum_{j=k}^{n} \binom{n}{j} \tilde{p}^j (1 - \tilde{p})^{n-j}$$

if all components are identical with reliability $p_i \equiv \tilde{p}$.

Binary State Systems with Imprecise Classification

- Membership functions and probability of fuzzy sets
- Reliability of degraded components and systems

Membership Function, $\mu_A(x)$

- For each x, $0 \leq \mu_A(x) \leq 1$ describes a belief of containment (membership) in a category A.
- If $\mu_A(x) = 0$ or 1, then A is a precise (i.e., crisp) set. Otherwise, A is a fuzzy set.

Ex. 1: $A_1 = \{x \in (1, 2, \ldots, 10) \mid x \geq 7\}$

x	1	2	3	4	5	6	7	8	9	10
$\mu_{A_1}(x)$	0	0	0	0	0	0	1	1	1	1

Ex. 2: $A_2 = \{x \in (1, 2, \ldots, 10) \mid x \text{ is large}\}$

x	1	2	3	4	5	6	7	8	9	10
$\mu_{A_2}(x)$	0	0	0	0	0	0.2	0.5	0.9	1	1

Membership Function, $\mu_A(x)$

- Operations:
  1. $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$
  2. $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$
  3. $\mu_{A^C}(x) = 1 - \mu_A(x)$
- Definitions parallel Lukasiewicz's (1930) definitions for the conjunction, disjunction and complement of many-valued propositions
- Bellman and Giertz (1973) showed that Operations 1 and 2 are unique with respect to constructing an algebra of sets; Operation 3 is not unique but "appears reasonable".

Probability of Fuzzy Sets

Due to Bement et. al. (2000)

- Based on two premises: $\mu_{\tilde{A}}(x)$ is interpreted as a likelihood function, and fuzzy sets arise from boundary uncertainty
- $\pi_D(x \in \tilde{A})$ is D's probability that $x \in \tilde{A}$ if it is known that $X = x$
- $\mu_{\tilde{A}}(x)$ is additional information (expert testimony) provided by expert, Z

$$P_D(X \in \tilde{A}; \mu_{\tilde{A}}(x)) \propto \int_x \mu_{\tilde{A}}(x) \pi_D(x \in \tilde{A}) \mathrm{d}P_D(x) \tag{1}$$

Probability of Fuzzy Sets (cont.)

- D needs to specify two probability measures, P_D and π_D
  - $(\Omega, \mathcal{F}, P_D)$, where Ω is the set of all possible outcomes of the random phenomenon under study
  - $(\Omega, \mathcal{F}, \pi_D)$, where Ω consists of two outcomes $x \in \tilde{A}$ or $x \notin \tilde{A}$
  - $\pi_D(x \in \tilde{A}) = 1 - \pi_D(x \notin \tilde{A})$
- To evaluate the constant of proportionality, D needs to evaluate

$$P_D(X \notin \tilde{A}; \mu_{\tilde{A}}(x)) \propto \int_x \mathcal{L}_D(X \notin \tilde{A}; \mu_{\tilde{A}}(x)) \pi_D(x \notin \tilde{A}) \mathrm{d}P_D(x) \quad (2)$$

- $\mathcal{L}_D(X \notin \tilde{A}; \mu_{\tilde{A}}(x))$ not necessarily $1 - \mu_{\tilde{A}}(x)$

Components in Vague Binary States

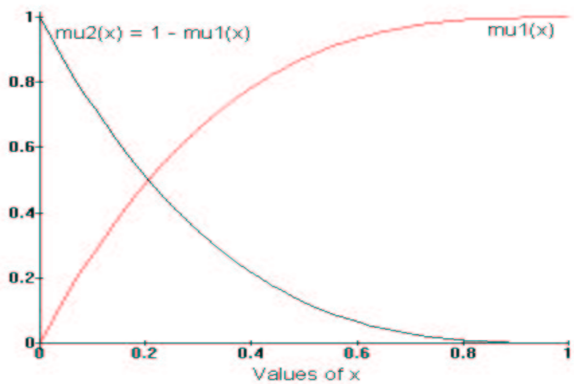
- Let X = the state of a component at time $\tau > 0$
- X takes values in $\mathcal{S} = \{x;\ 0 \leq x \leq 1\}$
- Consider $\mathcal{S}_1 \subset \mathcal{S}$, where $\mathcal{S}_1$=$\{x;\ x$ is a "desirable" state$\}$ and $\mu_1(x)$ membership function
- If $\mathcal{S}_2 \subset \mathcal{S}$ was defined as

$$\mathcal{S}_2 = \{x; x \text{ is an "undesirable" state}\},$$

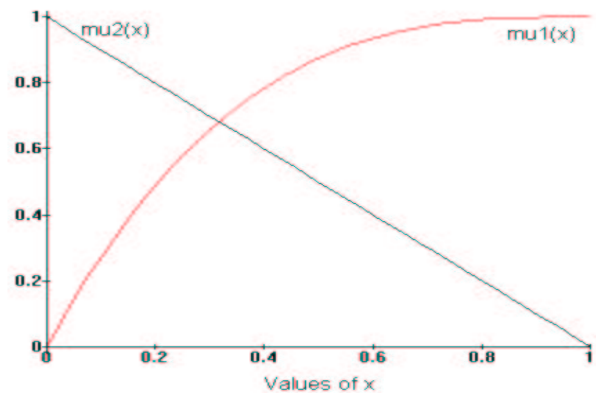
then $\mathcal{S}_1^C$ need not be $\mathcal{S}_2$ unless $\mu_2(x) = 1 - \mu_1(x)$, where $\mu_2(x)$ is the membership function of $\mathcal{S}_2$

Components in Vague Binary States (cont.)

- $\mu_2(x)$ need not be symmetrical to $\mu_1(x)$



Symmetric case



Asymmetric case

- In general, $\mathcal{S}_1^C$ need not be $\mathcal{S}_2$ and vice versa, unless $\mathcal{S}_1$ and $\mathcal{S}_2$ are precise sets.

Components in Vague Binary States (cont.)

- For $\mathcal{S}_1$=$\{x;\ x$ is a "desirable" state$\}$ and $\mu_1(x)$ specified, we define component reliability as

$$P_D(X \in \mathcal{S}_1; \mu_1(x)) \propto \int_x \mu_1(x)\pi_D(x \in \mathcal{S}_1)\mathrm{d}P_D(x). \qquad (3)$$

- With $\mathcal{S}_2$=$\{x;\ x$ is an "undesirable" state$\}$, and $\mu_2(x)$ specified, we define component unreliability as

$$P_D(X \in \mathcal{S}_2; \mu_2(x)) \propto \int_x \mu_2(x)\pi_D(x \in \mathcal{S}_2)\mathrm{d}P_D(x). \qquad (4)$$

Components in Vague Binary States (cont.)

- We could define component unreliability as $P_D(X \in \mathcal{S}_1^C; \mu_1(x))$, where $\mathcal{S}_1^C$ has membership function, $1 - \mu_1(x)$. In this case, the component unreliability is

$$P_D(X \in \mathcal{S}_1^C; \mu_1(x)) \propto \int_x (1 - \mu_1(x))[1 - \pi_D(x \in \mathcal{S}_1)]\mathrm{d}P_D(x). \quad (5)$$

- When a component's state is vague and binary, its unreliability is not the complement of its reliability!

Vague Binary Systems

- Claim: the structure functions of binary state coherent systems with precise classification are membership functions of precise sets
- Let X_i = state of component i, taking values in $\mathcal{S} = \{x;\ 0 \leq x \leq 1\}$
- $\mathcal{S}_1^{[i]} \subset \mathcal{S}$, where $\mathcal{S}_1^{[i]} = \{x_i;\ x_i \text{ is a "desirable" state}\}$ with associated membership function, $\mu_1^{[i]}(x)$, $i = 1, \ldots, n$
- Consider $\mathcal{S}_1^{[i]} \subset \mathcal{S}$ precise $\forall i$. e.g., $\forall i$, $\exists\ x_i \ni$ $\mu_1^{[i]}(x) = \begin{cases} 1 & x \geq x_i^* \\ 0 & x < x_i^* \end{cases}$

Vague Binary Systems

- Let $\mathbf{X} = (X_1, \ldots, X_n)$ and suppose that the n components are connected in series. Thus $\phi_S(\mathbf{X}) = 1$ if and only if $x \geq x_i^*$ for all i, $i = 1, \ldots, n$, implying

$$\phi_S(\mathbf{X}) = \prod_{i=1}^{n} \mu_1^{[i]}(X_i) = \min_i [\mu_1^{[i]}(X_i)], \qquad (6)$$

- Similarly, if n components were connected in parallel redundancy, then

$$\phi_P(\mathbf{X}) = \coprod_{i=1}^{n} \mu_1^{[i]}(X_i) = \max_i [\mu_1^{[i]}(X_i)] \qquad (7)$$

- For a k-out-of-n system, $\phi_K(\mathbf{X}) = \mu_1^{[(n-k+1)]}(\mathbf{X})$ where $\mu_1^{[(n-k+1)]}(\mathbf{X})$ represents the $(n-k+1)$st membership function when $\mu_1^{[i]}(X_i)$'s are ordered

Vague Binary Systems

Motivated by the above, we define system structure functions as

$$\begin{aligned}\phi_S(\mathbf{X}) &= \min_i[\mu_1^{[i]}(X_i)] = \mu_1^{[(1)]}(\mathbf{X}), \\ \phi_P(\mathbf{X}) &= \max_i[\mu_1^{[i]}(X_i)] = \mu_1^{[(n)]}(\mathbf{X}), \text{ and} \\ \phi_K(\mathbf{X}) &= \mu_1^{[(n-k+1)]}(\mathbf{X})\end{aligned}$$

Vague Binary Systems

There are two different strategies for defining the reliability of a vague coherent system.

1. Assume that a system is reliable if the necessary components are in a desirable state
   - The reliability of a series system is $\prod_{i=1}^{n}[P_D(X_i \in \mathcal{S}_1^{[i]}; \mu_1^{[i]}(x_i))]$ assuming independent X_i's, where

$$P_D(X_i \in \mathcal{S}_1^{[i]}; \mu_1^{[i]}(x_i)) \propto \int_{x_i} \mu_1^{[i]}(x_i) \pi_D(x_i \in \mathcal{S}_1^{[i]}) \mathrm{d}P_D^{[i]}(x_i), \qquad (8)$$

   - Assuming independent X_i's, the reliability of a parallel redundant system is

$$P_D(\bigcup_{i=1}^{n}\{X_i \in \mathcal{S}_1^{[i]}\}; \mu_1^{[i]}(x_i),\ i = 1, \ldots, n)$$

   - The case of k-out-of-n systems follows along similar lines

Vague Binary Systems

2. $\mathcal{S}^*_{\phi(\mathbf{X})} = \{x;\ \phi(\mathbf{X}) = x$ is a "desirable" state of the system$\} \subset \mathcal{S}$ with associated membership function $\mu^*_{\phi(\mathbf{X})}(x)$.

- The reliability of a series system is

$$P_D(\phi_S(\mathbf{X}) \in \mathcal{S}^*_{\phi_S(\mathbf{X})}; \mu^*_{\phi_S(\mathbf{X})}(x)) \propto \int_x \mu^*_{\phi_S(\mathbf{X})}(x)\pi_D(x \in \mathcal{S}^*_{\phi_S(\mathbf{X})})\mathrm{d}P_D(x),$$

where $\mathrm{d}P_D(x)$ is obtained from

$$P_D(\phi_S(\mathbf{X}) \geq x) = \prod_{i=1}^{n} P_D(X_i \geq \mu_1^{[i]^{-1}}(x)),$$

assuming independent X_i's; $\mu_1^{[i]^{-1}}(x)$ denotes the inverse of $\mu_1^{[i]}(x)$.

Vague Binary Systems (cont.)

- Similarly for parallel systems,

$$P_D(\phi_P(\mathbf{X}) \in \mathcal{S}^*_{\phi_P(\mathbf{X})}; \mu^*_{\phi_P(\mathbf{X})}(x)) \propto \int_x \mu^*_{\phi_P(\mathbf{X})}(x)\pi_D(x \in \mathcal{S}^*_{\phi_P(\mathbf{X})})\mathrm{d}P_D(x),$$

  where $\mathrm{d}P_D(x)$ is obtained via $\prod_{i=1}^n P_D(X_i \leq \mu_1^{[i]^{-1}}(x))$ assuming independent X_i's.

- The case of $(n-k+1)$-out-of-n follows by considering the distribution of the kth order membership function, $\mu_1^{[(k)]}(\mathbf{X})$.

Ongoing Research

- Reliability of $(m+1)$-level degraded systems
- Other applications
  - Quality of life
  - Disclosure limitation

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