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Contagion as a Wealth Effect

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ABSTRACT

Financial contagion is described as a wealth effect in a continuous-time model with two risky assets and three types of traders. Noise traders trade randomly in one market. Long-term investors provide liquidity using a linear rule based on fundamentals. Convergence traders with logarithmic utility trade optimally in both markets. Asset price dynamics are endogenously determined (numerically) as functions of endogenous wealth and exogenous noise. When convergence traders lose money, they liquidate positions in both markets. This creates contagion, in that returns become more volatile and more correlated. Contagion reduces benefits from portfolio diversification and raises issues for risk management.

During the financial panic associated with the default of the Russian government in August 1998 and the subsequent collapse of the hedge fund Long Term Capital Management, numerous hedge funds, banks, and securities firms tried simultaneously to reduce exposures to a variety of financial instruments, such as Russian bonds, Brazilian stocks, U.S. mortgages, spreads between on-the-run and off-the-run government securities, and spreads between swaps and U.S. Treasuries. Although the fundamental values of these positions would appear to have little correlation, during this financial crisis, the asset prices in these markets exhibited the following common empirical pattern:

1. Financial intermediaries suffered losses as prices moved against their positions;
2. Market depth and liquidity decreased simultaneously in several markets;
3. The volatility of prices increased simultaneously in several markets; and,
4. Correlation of price changes of seemingly independent positions of financial intermediaries increased.

Instead of using the term "panic" to describe the crisis, market commentators blamed the market behavior on increased "risk aversion" on the part of traders who follow "short-term" trading strategies. The commentators de-

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scribed "contagion" as the rapid spread from one market to another of declining prices, declining liquidity, increased volatility, and increased correlation associated with the financial intermediaries' own effect on the markets in which they trade.

The purpose of this paper is to explain contagion with a theoretical model in which increased risk aversion is based on a wealth effect of financial intermediaries. Financial intermediaries are modeled as a group of perfectly competitive convergence traders who speculate that the transitory effect of noise trading on asset prices will induce temporary deviations of prices from their long-term mean. Convergence traders trade in markets for two risky assets. When convergence traders suffer trading losses, they have a reduced capacity for bearing risks. This motivates them to liquidate positions in both markets, resulting in reduced market liquidity, increased price volatility in both markets, and increased correlation. Through this mechanism, the wealth effect leads to contagion. This mechanism is consistent with the report published by Bank for International Settlements (BIS; 1999) and empirical studies of Kaminsky and Reinhart (2000).

This paper describes a continuous-time model in which convergence traders follow short-term (but rational) trading strategies. Two risky assets have constant fundamental risk measured in units of the numeraire good (consumption). Three types of investors, noise traders, long-term value-based investors, and short-term convergence traders, exchange these two risky assets for a safe asset. Noise traders trade one of the risky assets randomly, but their position in this risky asset exhibits mean reversion. Long-term investors are prudent but not fully rational. They follow a robust long-term investment strategy holding both risky assets proportionally to the spread between the asset prices and their fundamental values. This spread represents the present value of trading profits to long-term investors in a worst-case scenario in which they have no opportunities to take profits early but instead hold these assets forever, collecting all the future cash flows. Convergence traders aggressively exploit short-term opportunities by taking the other side of noise trading. They are rational in the sense that they correctly take into account the effect of all market participants on price dynamics in both markets. Convergence traders are assumed to be perfect competitors with logarithmic utility. Logarithmic utility implies a trading strategy in which both the expected trading profits and the percentage variance of the portfolio equal the short-term (instantaneous) squared Sharpe ratio in the market. Logarithmic utility also implies a risk management strategy that prevents wealth from dropping to zero through dynamic portfolio rebalancing.

Xiong (2001) develops a continuous-time equilibrium model of convergence trading with one risky asset. Xiong shows that the convergence traders' wealth effect can act as an amplification mechanism that increases price volatility and may cause convergence trading to be price destabilizing in extreme circumstances. This paper shows that in an otherwise similar framework with two risky assets, the volatility amplification generates empirical patterns which characterize contagion. In both Xiong (2001) and this paper, it is assumed that there is no capital inflow to the convergence-trading industry through entry of new convergence traders or capital inflows to existing convergence traders. This assumption is consistent with Shleifer and Vishny's (1997) argument that asymmetric information and moral hazard can cause agency problems for professional traders, therefore resulting in imperfect capital flows to the convergence-trading industry.

In equilibrium, the asset price dynamics and convergence traders' wealth dynamics are simultaneously determined. This introduces endogenous risks into our model in the sense that means and variances of asset returns are endogenous functions of two state variables: the wealth of convergence traders and the positions of noise traders. There are three sources of risks: innovations in fundamentals in each of the two markets and innovations in noise-trading supply in one market. With only two risky assets, markets are incomplete. In equilibrium, the trading strategy of a representative convergence trader solves a fixed-point problem. This fixed-point problem is equivalent to a system of two second-order partial differential equations. A numerical solution of the equilibrium (using a projection technique) makes it possible to quantify, for particular parameter choices, the patterns of volatility, liquidity, correlation, and convergence-traders' wealth associated with contagion.

Severe "contagion" happens when noise trading deviates significantly from its mean and convergence traders' wealth is at some intermediate level. In this situation, convergence traders take large positions, and these positions need to be reduced in response to shocks that reduce wealth. The position rebalancing of convergence traders leads to increased volatility in both markets, increased price correlation across the two markets, and reduced market liquidity.

To understand the mechanism in convergence trading, it is useful to describe the response to innovations in fundamentals separately from innovations in noise trading. Fundamental shocks (in either market) cause a wealth effect. In response to unfavorable fundamental innovations which reduce wealth, convergence traders liquidate positions in a manner that tends to magnify volatility and create correlation in the returns between the two assets. In response to innovations in noise trading that increase the positions of noise traders, two forces are at work. In addition to the wealth effect, which motivates the convergence traders to reduce positions due to reduced wealth, they have an opposite incentive (substitution effect) to add to positions because these positions become more profitable as noise-trading innovations push prices further out of line. Usually, the wealth effect is smaller than the substitution effect and convergence traders respond to noise-trading shocks by taking the other side in a manner that reduces volatility and adds to liquidity. In certain extreme cases, however, when convergence traders have unusually large positions, the wealth effect dominates the substitution effect and convergence traders respond to noise-trading shocks by liquidating positions. This exacerbates price volatility and consumes some of the liquidity provided by long-term investors. It happens exactly when contagion is most severe.

In our model, the risks facing an individual convergence trader are endogenously determined by the trading of all market participants. The model implies that it is important for a risk management system to take into account the additional risks such as contagion and volatility amplification created by the wealth effect of convergence traders. This view is also expressed by Morris and Shin (2000), who study the potential coordination failure of market participants. The existence of these endogenous risks presents a challenge to risk-management systems based on applying statistical tools to historical returns. The weaknesses of these statistical models have been discussed by Danielsson (2000). Our economic model, based on assumptions about the liquidity provided by long-term investors and the behavior of convergence traders, suggests risk managers measure risks after considering the capitalization and positions of other traders.

There is a growing literature in economics and finance studying contagion. Dornbusch, Park, and Claessens (2000) provide a lengthy review. Schinasi and Smith (1999) suggest that the combination of leverage and value-at-risk portfolio management rules can induce contagion, but they do not provide an equilibrium model of the wealth effect. Gromb and Vayanos (2000) study an equilibrium model of arbitrage trading with margin constraints, and they show a similar contagion effect to our model. The wealth effect studied in this paper is related to the papers on portfolio insurance by Grossman and Zhou (1996) and Basak (1995). Since these models are set up in complete markets with only market risks, they are not suitable for explaining contagion, the transmission of idiosyncratic risk from one small market niche to another, for example, from Russian bond markets to U.S. mortgage-backed securities markets.

An alternative approach to our paper studies financial contagion as information transmission. The idea here is that the fundamental risks are correlated across assets. Thus, when one asset declines in price because of noise trading, rational traders reduce the prices of all assets if they cannot distinguish declines based on fundamentals from declines based on noise trading. King and Wadhwani (1990) use this approach to explain the uniformity of price declines in world stock markets during the 1987 crash. Calvo (1999) and Yuan (1999) study the behavior of uninformed investors when informed traders can be margin constrained. Since uninformed rational investors cannot distinguish between selling based on liquidity shocks and shocks to fundamentals, they suggest that it is possible for contagion to result from confused uninformed investors. Their studies are complementary to ours, since the wealth effect in our model operates even when asset fundamentals are uncorrelated across markets.

Fleming, Kirby, and Ostdiek (1998) find empirical evidence that cross-market hedging is associated with transmission of volatility across bond and stock markets. Kodres and Pritsker (1998) develop a theoretical model of financial contagion based on cross-market hedging with asymmetric information. Calvo and Mendoza (2000) suggest that information cost and relative performance compensation can induce rational herding behavior of investors, thus resulting in contagion. Contagion can also be modeled as self-fulfilling sunspot equilibrium as in Masson (1998) and its references. Rochet and Tirole (1996) study the propagation of financial distress through interbank lending. Lagunoff and Schreft (1998) and Allen and Gale (2000) study the fragility of financial markets through a chain reaction of banks or financial intermediaries to withdraw from illiquid investments. Caballero and Krishnamurthy (2000) show that the weakening of a country's international collateral can induce the fire sale of emerging-market assets due to imperfect international credit markets. Goldstein and Pauzner (2000) study contagion also as a wealth effect of investors with decreasing absolute risk aversion, but in a framework of self-fulfilling banking crisis.

On the empirical side, the implications of our model are consistent with the following empirical regularities identified in the literature: (1) Not all asset-price volatility is explained by fundamentals; (2) conditional correlations between asset returns are not constant; and, (3) variations in conditional correlations are not explained by fundamentals. The empirical literature includes the following: Campbell and Kyle (1993) show that the excess volatility literature is consistent with the idea that noise trading increases price volatility. Shiller (1989) and Pindyck and Rotemberg (1990, 1993) find evidence of excess correlation in asset price comovements. Longin and Solnik (1995) find that conditional correlations of world stock markets are not constant. Hamao, Masulis, and Ng (1990) and Lin, Engle, and Ito (1994) find evidence of volatility spillover in international stock markets. Karolyi and Stultz (1996) and Connolly and Wang (1998) find that macroeconomic announcements and other public information do not affect comovements of Japanese and American stock markets. King, Sentana, and Wadhwani (1994) find that observable economic variables explain only a small fraction of international stock market comovements. Balyeat and Muthuswamy (1999) find a U-shaped relationship between correlations of stock returns and the level of market movement. Forbes and Rigobon (1999) discuss the econometric issues of heteroskedasticity and endogeneity related to the contagion tests. Bae, Karolyi, and Stultz (2000) use a new statistical method to measure contagion. Ang and Chen (2000), Connolly and Wang (2000), and Longin and Solnik (2001) find correlation to be large in market downturns.

The paper proceeds as follows. Section I introduces the structure of the model. Section II derives the equilibrium as a fixed-point problem. Section III illustrates the equilibrium using a numerical example and discusses the implications of the model. Section IV discusses the implications for risk management. Section V concludes the paper.

The Model

The model is set up in a continuous-time framework with two risky assets and a riskless asset. There are three types of traders: noise traders, convergence traders and long-term value-based investors. The two risky assets have independent fundamental processes. One of the assets is subject to stochastic and mean-reverting supply caused by noise traders. The other asset has a fixed supply. Convergence traders are fully rational with logarithmic utility and an infinite trading horizon. They trade in both assets and exploit the short-term opportunity created by noise traders. Long-term investors hold the assets based on the spread between the prices and the fundamentals. Long-term investors are not fully rational in the sense that they ignore the short-term opportunity caused by noise traders, but their strategy is very robust to the risks of model misspecification. The trading of long-term investors provides convergence traders an exit strategy during crises.

A. Asset Fundamentals

We assume that traders in the financial markets exchange a safe asset with constant interest rate r for two risky assets, which we call asset A and asset B. In the context of convergence trading, each of these two risky assets can be thought of as a spread position between other assets. To model how fundamental uncertainty about future cash flows is revealed to the markets, we assume that the cash flows of these two assets are observable, mean-reverting stochastic processes D^A and D^B with constant instantaneous volatilities σ^A and σ^B, constant rates of mean reversion λ^A and λ^B, and known long-term means $\bar{D}^A$ and $\bar{D}^B$. Thus, the cash-flow processes can be written

$$dD^A = -\lambda^A(D^A - \bar{D}^A)\,dt + \sigma^A dz^A, \tag{1}$$

$$dD^B = -\lambda^B(D^B - \bar{D}^B)\,dt + \sigma^A dz^B. \tag{2}$$

We assume for simplicity that the two cash-flow processes are independent. The fundamental values P_F^A and P_F^B of the two risky assets (not to be confused with the market prices P^A and P^B described later) are defined as their expected payoffs to a risk neutral investor discounted at the risk-free rate of interest (using variations of Gordon's growth formula):

$$P_F^A = \frac{\bar{D}^A}{r} + \frac{D^A - \bar{D}^A}{r + \lambda^A}, \tag{3}$$

$$P_F^B = \frac{\bar{D}^B}{r} + \frac{D^B - \bar{D}^B}{r + \lambda^B}. \tag{4}$$

The risk-neutral returns processes dQ_F^A and dQ_F^B corresponding to the fundamental values (not to be confused with the actual returns processes dQ^A and dQ^B discussed later) are given by the hypothetical mark-to-market profits of a fully levered one-share portfolio, which collects the dividend and pays the risk-free rate of interest:

$$dQ_F^A = dP_F^A + (D^A - rP_F^A)\,dt, \tag{5}$$

$$dQ_F^B = dP_F^B + (D^B - rP_F^B)\,dt. \tag{6}$$

Using the cash-flow processes and the fundamental processes above, it is straightforward to show that the risk neutral mark-to-market profits on asset A and B follow Brownian motions with constant volatility, which we define as σ_F^A and σ_F^B:

$$dQ_F^A = \frac{\sigma^A}{r + \lambda^A}\, dz^A = \sigma_F^A\, dz^A, \tag{7}$$

$$dQ_F^B = \frac{\sigma^B}{r + \lambda^B}\, dz^B = \sigma_F^B\, dz^B. \tag{8}$$

The equilibrium discussed below depends on the fundamental cash-flow process only through the parameters σ_F^A and σ_F^B. In other words, the specific rates of mean reversion and the long-term means of cash flows do not affect the equilibrium except through their effect on σ_F^A and σ_F^B. Furthermore, the risky assets can be scaled arbitrarily (as in a stock split) to give any level of fundamental volatility, without changing the equilibrium. Thus, in what follows, we assume without loss of generality that fundamental volatility is the same for both assets and is defined by $\sigma_F = \sigma_F^A = \sigma_F^B$.

The fact that σ_F is constant implies that fundamental volatility is constant when measured in dollars per share. Without loss of generality, we can think of convergence trading positions as spread positions. The constant volatility assumption better describes the fundamental risks of typical spread positions. These positions have distinct long and short legs. Therefore, they do not have natural up and down directions that justify the log-normal process associated with the concept of constant percentage volatility.

B. Market Clearing Conditions

The equilibrium prices for the two risky assets (as opposed to the fundamental value discussed above) arise from trading by the three different types of market participants: long-term investors, convergence traders, and noise traders. Noise traders are assumed to trade only in market A. This assumption is made to reduce the number of state variables needed to characterize the equilibrium. Following Campbell and Kyle (1993) and Wang (1993), we assume the supply of noise traders to follow an exogenous mean-reverting process

$$d\theta = -\lambda_\theta(\theta - \bar{\theta})\,dt + \sigma_\theta\, dz_\theta, \tag{9}$$

with $\bar{\theta}$ as the long-term mean, λ_θ as the mean-reversion parameter, and σ_θ as the innovation standard deviation. This process is also assumed to be independent from the fundamental cash-flow processes D^A and D^B. Asset B has a fixed supply of $\bar{\theta}^B$. We denote long-term investors' demand as X_L^A and X_L^B, and convergence traders' demand as X^A and X^B. The market clearing conditions (which hold at every point in time) can be written as

$$X_L^A + X^A = \theta, \tag{10}$$

$$X_L^B + X^B = \bar{\theta}^B. \tag{11}$$

C. Long-term Investors

Long-term investors are assumed to have the following demand curve for the two risky assets:

$$X_L^A = \frac{1}{k^A}(P_F^A - P^A), \tag{12}$$

$$X_L^B = \frac{1}{k^B}(P_F^B - P^B), \tag{13}$$

where $k^A > 0$ and $k^B > 0$ denote the slopes of the downward-sloping demand curves. These demand curves are proportional to the spreads between the fundamental values and the actual prices. Graham (1973) calls this spread a safety margin. It represents the net present value of profits to long-term investors when they hold the assets forever and collect all the future cash flow. This is a worst-case scenario, which happens if the safety margin does not change over time. If we assume long-term investors have exponential utility and assume (incorrectly) that the safety margin is constant over time, they would use this (suboptimal) strategy. The slope of the demand curve is then given by

$$k^A = \phi\sigma_F^2, \tag{14}$$

where ϕ is the long-term investors' absolute risk aversion and σ_F^2 is the variance of fundamental shocks.

If we think of the same long-term investors as participating in both markets, then the demand in one market does not depend on prices in the other because the fundamentals of the two markets are uncorrelated. Therefore, under this assumption, we should assume $k^A = k^B$ because the fundamental volatility in the two markets is identical. However, if long-term investors are segmented in a similar manner to Merton (1987), with one population of long-term investors trading in market A and another in market B, the parameters k^A and k^B can have different values.

According to these demand curves, long-term investors always provide liquidity to the market. When the price falls below the fundamental value in either market, long-term investors will buy the asset. When the price falls further below the fundamental value, long-term investors will buy more. The slopes of the demand curves k^A and k^B measure the liquidity provided by long-term investors. Larger k^A or k^B mean steeper demand curves, and thus represent less liquidity from long-term investors. Notice that long-term investors have no wealth effects. Implicitly, they are assumed to have deep pockets (consistent with exponential utility). As shown later, the liquidity provided by long-term investors provides an exit strategy for convergence traders during crises.

While this long-term strategy is profitable, it is not optimal. Because the inventory of noise traders θ changes randomly in a mean-reverting manner, a short-term strategy can improve the portfolio performance of long-term investors. A short-term strategy implies trading more aggressively against noise trading than the long-term strategy. This creates an opportunity for convergence traders to prosper in the market by using a short-term strategy.

The rationale behind the long-term strategy is its robustness. Graham (1973) noticed a long time ago that a short-term strategy that improves upon the long-term strategy for a given noise-trading process can be subject to large model specification risks. Therefore, he advocates a long-term strategy to exploit long-term opportunities (measured by the safety margins) in the market. This view is consistent with recent studies on the aversion to model uncertainty by Epstein and Wang (1994) and Hansen, Sargent, and Tallarini (1999). Since the focus of our model is on the effect of convergence traders, we simplify matters by assuming the simplistic trading rule of long-term investors.

D. Convergence Traders

Convergence traders behave optimally in response to a given noise-trading process. Intuitively, this means that they make profits not only by purchasing risky assets when they are priced below fundamentals, but they also make short-term profits by taking the other side of transitory noise trading. Due to the aggressive nature of convergence trading, convergence traders are subject to large wealth fluctuation with the leverage they may be induced to use. This makes their wealth effect an important variable in determining their asset demand. In order to capture the dependence of their demand on both short-term opportunity and wealth, convergence traders are assumed to be a continuum of perfect competitors who maximize an additively separable logarithmic utility function with an infinite time horizon and a time-preference parameter ρ:

$$J(t) = \max \mathrm{E}_t \int_0^\infty \mathrm{e}^{-\rho s} \ln(C_{t+s})\, ds. \tag{15}$$

With logarithmic utility, convergence traders have decreasing absolute risk aversion. As their wealth gets close to zero, convergence traders become infinitely risk averse. To prevent their wealth from becoming negative, convergence traders will use the liquidity provided by long-term investors to liquidate their risky positions as their wealth decreases. Note that without long-term investors, there can be no equilibrium with only convergence traders and noise traders, because wealth cannot be guaranteed to stay positive for convergence traders when fundamentals have a normal distribution.

Since logarithmic utility gives convergence traders an incentive to keep their wealth from falling below zero, there are no bankruptcy risks, and creditors are always willing to lend money to them at the risk-free rate r.

The trading opportunities to convergence traders are the excess return processes:

$$dQ^A = dP^A + (D^A - rP^A)\,dt, \tag{16}$$

$$dQ^B = dP^B + (D^B - rP^B)\,dt, \tag{17}$$

with P^A and P^B denoting the prices of the risky assets (not the fundamental values P_F^A and P_F^B). The processes dQ^A and dQ^B represent the cash flow to a fully levered portfolio long one share of the risky asset A or B. The convergence traders' budget constraint is

$$dW = X^A dQ^A + X^B dQ^B + (rW - C)\,dt, \tag{18}$$

where W denotes their wealth, C denotes their consumption, and X denotes their demand for the risky asset in shares. Consumption C can also be interpreted as a dividend paid to investors in the convergence traders' funds. The convergence traders' demand X^A, X^B, and consumption C are derived from their utility optimization problem.

II. The Equilibrium

This paper studies a symmetric and perfectly competitive equilibrium. In this equilibrium, each individual convergence trader is a price-taker, and given everyone else's trading strategy, each individual convergence trader will optimally choose the same strategy. This equilibrium condition implies that a representative convergence trader's trading strategy solves a fixed-point problem.

There are three sources of uncertainty, the fundamental shock in asset A (dz^A), the fundamental shock in asset B (dz^B), and the noise-trading shock in asset A (dz_θ). Since there are only two risky assets, markets are incomplete. There are also two state variables: the level of noise trading θ and the aggregate wealth of convergence traders W. Due to logarithmic utility, the total wealth of all convergence traders can be aggregated together to represent their aggregate risk-bearing capacity. Unlike models with constant absolute risk aversion, the exact number of convergence traders is not important for the equilibrium.

The fundamental variables D^A and D^B are not state variables. Due to the normal distribution assumption for the cash-flow processes, the fundamental risks are constant for these two assets and the dividends only measure the levels of fundamental values. Since long-term investors trade on long-term opportunities (safety margins) measured by the difference between the prices and fundamentals, while convergence traders trade on short-term opportunity measured by the Sharpe ratios (as shown later by the model), variables D^A and D^B have no effects on the trading strategies of either long-term investors or convergence traders. Therefore, they have no effect for the equilibrium.

The equilibrium can be characterized by three functions: convergence traders' demand functions for the two risky assets $X^A(\theta,W)$ and $X^B(\theta,W)$, and convergence traders' consumption function $C(\theta,W)$. These three functions solve the convergence traders' utility optimization problem. At the same time, they always satisfy the market clearing conditions.

Given convergence traders' demand functions X^A and X^B, the price functions of the risky assets can be derived by plugging the long-term investors' demand functions into the market clearing conditions:

$$P^A = P_F^A - k^A(\theta - X^A(\theta,W)), \tag{19}$$

$$P^B = P_F^B - k^B(\bar{\theta}^B - X^B(\theta,W)). \tag{20}$$

These equations reveal the key feature of our model that convergence traders' wealth dynamics influence the price dynamics of both risky assets, and can potentially cause correlation between the two asset prices although they are fundamentally uncorrelated. Actually, the wealth dynamics and asset price dynamics need to be determined simultaneously in the equilibrium.

The equilibrium can be set up in three steps. In the first step, the two excess return processes are derived given convergence traders' demand and consumption functions. In the second step, convergence traders' optimal investment and consumption policies are derived given the excess return processes. Finally, the equilibrium is shown to solve a fixed-point problem that is a system of two nonlinear second-order partial differential equations. These equations can be solved numerically.

A. Excess Return Processes

Given the convergence traders' demand and consumption functions, we can use Ito's lemma to express the excess return processes dQ^A and dQ^B (equations (16) and (17)) in terms of a drift term and innovation terms associated with the three sources of uncertainty dz^A, dz^B, and dz_θ. Let μ^A denote the drift and σ_A^A, σ_B^A, σ_θ^A denote the loadings on the innovations in the markets. The drift and loadings on innovations are functions of the two state variables W and θ. Using analogous notation for asset B, we have

$$dQ^A = \mu^A(\theta,W)dt + \sigma_A^A(\theta,W)dz^A + \sigma_B^A(\theta,W)dz^B + \sigma_\theta^A(\theta,W)dz_\theta, \tag{21}$$

$$dQ^B = \mu^B(\theta,W)dt + \sigma_A^B(\theta,W)dz^A + \sigma_B^B(\theta,W)dz^B + \sigma_\theta^B(\theta,W)dz_\theta. \tag{22}$$

We can think of several of these innovation coefficients as representing contagion. In equation (21), σ_B^A measure the effect of an innovation in the fundamentals of asset B on returns to asset A, that is, it captures fundamental contagion going from market B to market A. In equation (22), σ_A^B and σ_θ^B measure the effects of innovations in fundamentals in market A and noise trading in market A on returns in market B, that is, these terms capture fundamental contagion and noise-trading contagion going from market A to market B. There is no noise-trading contagion going from market B to market A because there is assumed to be no noise trading in market B.

The wealth effect shows up through the simultaneous relationship between convergence traders' wealth process W and the excess return processes dQ^A and dQ^B. On the one hand, shocks to the two return processes change the aggregate wealth of convergence traders through their budget constraint (equation (18)) when they are taking positions on these two assets. One the other hand, the changes of convergence traders' wealth cause fluctuations in their risk-bearing capacity, and thus induce them to rebalance their portfolio. Their portfolio rebalancing can change the prices of the two assets through the market clearing conditions (equations (19) and (20)). In equilibrium, any shock to any one of these assets feeds back to itself through the convergence traders' wealth, potentially amplifying the shock. The shock will also be transmitted to the other asset through the same wealth channel, thus resulting in a contagion effect. In this way, the wealth effect can act as both an amplification mechanism and a contagion mechanism.

In the expressions for drifts μ^A, μ^B and innovation sensitivities σ_A^A, σ_B^A, σ_θ^A, σ_A^B, σ_B^B, and σ_θ^B, it is shown in Appendix A that the wealth effect appears as a factor $A(\theta, W)$ defined by

$$A = \frac{1}{1 - k^A X^A X_W^A - k^B X^B X_W^B}. \tag{23}$$

The subscripts θ or W denote the derivatives of a function with respect to noise trading θ or wealth W, that is, X_W^A is the derivative of the demand for asset A with respect to wealth. The factor A has an intuitive interpretation, which is explained as follows: Let dW' denote a hypothetical change in convergence traders' wealth that would occur in response to an exogenous shock (e.g., dz_D^A, dz_D^B, or dz_θ^Z) if convergence traders did not update their positions in response to the changes in wealth. Let dW denote the actual change in wealth that would occur when convergence traders do update their positions in response to the exogenous shocks. As a result of the initial shock, convergence traders rebalance their portfolio by reducing their positions in both assets A and B by $X_W^A dW$ and $X_W^B dW$, respectively. To induce long-term investors to pick up the positions liquidated by convergence traders, the prices of both assets need to drop by $k^A X_W^A dW$ and $k^B X_W^B dW$. When the prices fall, the convergence traders' wealth will further drop by $X^A \cdot k^A X_W^A dW + X^B \cdot k^B X_W^B dW$. Therefore, an initial wealth drop of dW' can cause a total wealth drop of

$$dW = dW' + (k^A X^A X_W^A + k^B X^B X_W^B) dW. \tag{24}$$

This equation gives

$$dW = A \cdot dW', \tag{25}$$

which suggests that factor A measures the magnitude of amplification caused by the wealth effect. As shown later, this amplification factor $A(\theta, W)$ is always larger than 1.

It is shown in Appendix A that the coefficients of the dz terms in equations (21) and (22) for dQ^A and dQ^B are given by

$$\sigma_A^A = \sigma_F(1 - k^B X^B X_W^B) A(\theta, W), \tag{26}$$

$$\sigma_B^A = \sigma_F k^A X^B X_W^A A(\theta, W), \tag{27}$$

$$\sigma_\theta^A = -k^A \sigma_\theta[(1 - X_\theta^A)(1 - k^B X^B X_W^B) - k^B X_W^A X^B X_\theta^B] A(\theta, W), \tag{28}$$

$$\sigma_A^B = \sigma_F k^B X^A X_W^B A(\theta, W), \tag{29}$$

$$\sigma_B^B = \sigma_F(1 - k^A X^A X_W^A) A(\theta, W), \tag{30}$$

$$\sigma_\theta^B = k^B \sigma_\theta[X_\theta^B(1 - k^A X^A X_W^A) - k^A X^A X_W^B(1 - X_\theta^A)] A(\theta, W). \tag{31}$$

Each of these terms can be explained in an intuitive way. Let us illustrate with the first term. By using the definition of $A(\theta, W)$ in equation (23), the term σ_A^A in equation (26) can be rewritten as

$$\sigma_A^A = \sigma_F[1 + k^A X^A X_W^A A(\theta, W)]. \tag{32}$$

When a fundamental shock dz^A hits market A, it causes an initial price change of $\sigma_F dz^A$ and an initial wealth shock of $X^A \cdot \sigma_F dz^A$. Through the wealth amplification mechanism discussed above, the wealth shock will be amplified by A, resulting in a total wealth shock of $X^A \cdot \sigma_F dz^A \cdot A$. Convergence traders rebalance their positions in asset A by $X_W^A \cdot X^A \cdot \sigma_F dz^A \cdot A$. In order to clear the market, the price of asset A needs to change further by $k^A \cdot X_W^A \cdot X^A \cdot \sigma_F dz^A \cdot A$ to attract long-term investors to take the other side of a rebalancing trade by convergence traders. In this way, an initial fundamental shock of dz^A can cause a total price change of $\sigma_F dz^A [1 + k^A X^A X_W^A A(\theta, W)]$ to asset A as indicated by equation (32). Similar intuitive explanations can be obtained for other volatility terms listed above in equations (27) through (31).

The drift values $\mu^A(\theta, W)$ and $\mu^B(\theta, W)$ are complicated expressions involving X^A, X^B, and derivatives to second order (see Appendix A). Appendix A also gives expressions for the aggregate wealth of the convergence traders W.

B. Optimal Strategy for Convergence Traders

Given the trading opportunities to a convergence trader defined by dQ^A and dQ^B, the value function J can be written as a function of wealth W^i and the two state variables W and θ:

$$J(W^i, \theta, W) = \max_{\{X^{iA}, X^{iB}, C^i\}} \mathrm{E}_t \int_0^\infty \mathrm{e}^{-\rho s} \ln(C_{t+s}^i)\, ds. \tag{33}$$

Notice that W^i measures the individual convergence trader's wealth, while W represents the aggregate wealth of all of convergence traders. By solving a Bellman equation, Appendix B shows the optimal consumption and portfolio rules to be

$$X^{i^A} = \frac{W^i}{1-\phi^2}\left[\frac{\mu^A}{(\sigma^A)^2} - \phi\,\frac{\mu^B}{\sigma^A\sigma^B}\right], \tag{34}$$

$$X^{i^B} = \frac{W^i}{1-\phi^2}\left[\frac{\mu^B}{(\sigma^B)^2} - \phi\,\frac{\mu^A}{\sigma^A\sigma^B}\right], \tag{35}$$

$$C^i = \rho\, W^i. \tag{36}$$

In the formula above, μ^A and μ^B are the instantaneous risk premia of the two assets, σ^A and σ^B are the instantaneous volatility, and ϕ is the instantaneous return correlation between the two assets. All these variables are functions of the two state variables θ and W endogenously determined by the equilibrium.

Consumption is a constant fraction of the wealth equal to the impatience level ρ. The consumption strategy can be interpreted as a constant dividend rate. The trading strategy is also proportional to the convergence trader's wealth, because logarithmic utility implies that the convergence trader's risk bearing capacity is proportional to wealth. This trading strategy prevents wealth from falling to zero through dynamic portfolio rebalancing. Whenever wealth drops, the convergence trader needs to liquidate some risky positions across the portfolio if the trading opportunities are unchanged. As wealth falls close to zero, the convergence trader becomes infinitely risk averse and takes almost zero positions. The existence of long-term investors in the market is crucial to the implementation of this strategy, because the liquidity from long-term investors provides an exit opportunity for the convergence traders when they need to get out of their positions.

The optimal trading strategy is short-term in the sense that it only depends upon the instantaneous risk premium and variance of the return processes. This contrasts with the long-term strategy used by long-term investors. This trading strategy is also myopic, that is, there is no hedging demand (against changes in the future investment opportunity set), as discussed in Merton (1971) and Breeden (1979). This is a well-known property of logarithmic utility, and it makes the model more tractable.

According to this optimal strategy, an individual convergence trader maintains an instantaneously mean-variance efficient portfolio involving the two risky assets. As shown in Appendix B, the expected trading profits in percentage terms and the instantaneous percentage variance of the convergence trader's portfolio is equal to the squared Sharpe ratio of the instantaneously mean-variance efficient portfolio. These features highlight the importance of Sharpe ratio for convergence traders. Also, we see the advantage of using logarithmic utility. Logarithmic utility implies an intuitive trading strategy in terms of Sharpe ratios, similar to the way in which Sharpe ratios are actually used in markets.

C. Fixed-point Problem

In equilibrium, the portfolio and consumption rules $X^A(\theta,W)$, $X^B(\theta,W)$, and $C(\theta,W)$ should solve the log-utility optimization problem and satisfy the market clearing conditions at the same time. Since optimal consumption and portfolio rules of an individual convergence trader are proportional to wealth, we can aggregate the rules of all convergence traders together by replacing the individual wealth variable W^i with aggregate wealth W. Let us denote these aggregate optimal rules by $X^{*A}(\theta,W)$, $X^{*B}(\theta,W)$, and $C^*(\theta,W)$. Notice that X^{*A}, X^{*B}, and C^* are functions of conjectured strategies X^A, X^B, and C, respectively, as derived explicitly in Appendix B. It is evident that this definition of equilibrium is equivalent to a fixed-point problem:

$$X^{*A}(\theta,W) = X^A(\theta,W), \tag{37}$$

$$X^{*B}(\theta,W) = X^B(\theta,W), \tag{38}$$

$$C^*(\theta,W) = C(\theta,W). \tag{39}$$

These fixed-point conditions represent that given the portfolio and consumption rules of all other convergence traders, a representative convergence trader will optimally choose the same rules. Thus, assuming a transversality condition holds, the calculation of equilibrium for the economy boils down to solving a fixed-point problem.

To make the equilibrium interesting, it is assumed that convergence traders' time preference ρ (also their consumption rate) is higher than the risk-free rate ($\rho > r$). If $\rho < r$, convergence traders gradually accumulate their wealth from investing in the risk-free asset, and eventually they will have infinite wealth in a stationary equilibrium. Infinite wealth of convergence traders will cause the risky assets to be priced in a risk-neutral manner. This is not an interesting case for us to study. The assumption of $\rho > r$ insures that there is only limited wealth for convergence traders in a stationary equilibrium. Thus, interesting implications can be derived from the dynamics of convergence traders' wealth process.

No theoretical existence or uniqueness results are available at this point. It is conjectured that the existence of an equilibrium with a stationary distribution of wealth is guaranteed by the assumption that long-term investors have a fixed, downward sloping demand curve for the risky asset. Without long-term investors, convergence traders may not be able to liquidate their positions in crises, resulting in no equilibrium. This paper uses a numerical method to find an equilibrium, that is, an approximate solution to the fixed-point problem, and discusses the implications for volatility and comovements of asset prices caused by convergence traders' wealth changes.

To solve the fixed-point problem, it is necessary to solve a set of two second-order partial differential equations with two state variables (W and θ). Although the solution for the optimal consumption rule is trivial ($C(W,\theta) = \rho W$), the equilibrium portfolio rules X^A and X^B need to solve two partial differential equations (shown in Appendix C). These equations are highly nonlinear and entangled together in such a way that it is hopeless to solve them analytically. Notice that this entanglement captures exactly the contagion modeled in this paper. We solve these equations using a numerical method.

While a numerical solution of the partial differential equations is necessary, the partial differential equations do satisfy easily described boundary conditions for $W = 0$ and $W = \infty$. When wealth is zero, convergence traders do not trade, so we have the boundary condition

$$X^A(\theta,0) = 0, \tag{40}$$

$$X^B(\theta,0) = 0. \tag{41}$$

On this bound, prices are given by $P^A = P_F^A - k^A\theta$ and $P^B = P_F^B - k^B\bar{\theta}^B$. The innovation on per-share returns for asset A is $\sigma_F dz^A + k^A\sigma_\theta dz_\theta$, and the innovation on per-share returns for asset B is $\sigma_F dz^B$. The volatility of per-share returns on asset A is $\sqrt{\sigma_F^2 + (k^A\sigma_\theta)^2}$, and volatility of per share returns on asset B is σ_F.

When wealth approaches infinity, risk premiums are driven toward zero, that is, assets are priced in a risk-neutral manner. This drives long-term investors out of the market, so that convergence traders absorb all of the asset supplies. This implies the following conditions:

$$X^A(\theta,\infty) = \theta, \tag{42}$$

$$X^B(\theta,\infty) = \bar{\theta}^B. \tag{43}$$

Prices are equal to the fundamental values $P^A = P_F^A$, $P^B = P_F^B$, where P_F^A and P_F^B are given in equations (3) and (4). The innovation of per-share returns for asset A is $\sigma_F dz^A$, and the innovation of per-share returns for asset B is $\sigma_F dz^B$. The volatility of per-share returns for both assets A and B is σ_F.

III. A Numerical Illustration of the Equilibrium

We solve the equilibrium numerically using a projection method. The basic idea is to approximate the equilibrium demand functions of convergence traders by rational functions using Chebyshev polynomials. Appendix D discusses the details of this numerical method. For different sets of parameter values, the calculated equilibria have similar qualitative properties. To illustrate the equilibrium, we choose the following values of the 10 parameters needed to describe the model:

$$\sigma_F = 0.268, \quad \bar{\theta} = 0.5, \quad \lambda_\theta = 0.5, \quad \sigma_\theta = 0.2, \quad \bar{\theta}^B = 0.5,$$
$$k^A = 1.0, \quad k^B = 4.0, \quad \phi_F = 0.0, \quad r = 6.00\%, \quad \rho = 8.00\%. \tag{44}$$

These 10 parameter choices describe 10 features of the model. The first seven features describe facts about the equilibrium when convergence traders have zero wealth:

1. Price changes in market A are uncorrelated with price changes in market B.
2. The Sharpe ratio available in market B is 0.448 ($r^B k^B \bar{\theta}^B / \sigma_F$, from Appendix A).
3. The average Sharpe ratio in market A is 0.090 (from Appendix A).
4. The standard deviation of the Sharpe ratio in market A is 0.335 (from Appendix A).
5. Noise traders make price volatility in market A ($\sqrt{\sigma_F^2 + (k^A \sigma_\theta)^2} =$ 0.334) 25 percent higher than it would be if there is no noise trading.
6. The half-life of noise trading is 1.39 years ($\ln(2)/\lambda_\theta$).
7. The liquidity provided by long-term traders to market A is four times the liquidity provided to market B (through parameters k^A and k^B) in the following sense: for long-term investors to increase their demands of assets A and B by same amount, the price of asset B has to drop four times as much as the price decrease of asset A.

The remaining three features scale units in terms of which quantities are measured:

8. The assumption $\sigma_F = 0.268$ scales the share units for both assets.
9. The assumption $r = 6$ percent scales the rate at which the present value is calculated.
10. Convergence traders' wealth decreases at a rate of 2 percent ($\rho - r$) per year, if they do not trade.

We describe the equilibrium with graphs depicting various relationships as functions of the two state variables, wealth W and noise trading θ. Notice that both state variables have been transformed into the region [−1,+1]. The domain of these graphs is a square in the transformed W, θ plane centered at the origin. These graphs fit into a rectangular box with this square as base. All the graphs are rotated so that the intersection of the graphs with vertical faces of the box indicate the behavior of the variable at extreme values of the state variables as follows:

Southeast face: Convergence traders have zero wealth.

Northwest face: Convergence traders have infinite wealth.

Northeast face: Noise traders have a four-standard-deviation short position.

Southwest face: Noise traders have a four-standard-deviation long position.

A. Demand Functions and Sharpe Ratios

Panels A and B of Figure 1 show the demand functions of convergence traders for the two risky assets. The intersection of both graphs and the southeast face are horizontal lines at zero, reflecting the boundary condition that convergence traders have a zero aggregate position for both assets when they have no wealth. In Panel A, the northwest face contains a 45 degree line, while in Panel B the northwest face contains a horizontal line at $\bar{\theta}^B$. Both indicate the boundary conditions that convergence traders absorb all the noise in market A and total supply in market B when they have infinite wealth.

Panels C and D of Figure 1 show the Sharpe ratios of assets A and B. Both Sharpe ratios are zero when wealth is infinite. When wealth is zero, Panel C shows that as the position of noise traders varies from long to short, the Sharpe ratio on asset A varies (linearly) over positive and negative values, indicating both long and short positions can be profitable trading opportunities by taking the opposite side of noise trading. Panel D shows that when wealth is zero, the Sharp ratio in market B is a positive constant (because supply is positive) that does not vary with noise trading in market A. Note, however, that for intermediate levels of wealth, the Sharpe ratio in market B is higher when noise trading in market A is not close to its mean. This is due to increased correlation between assets A and B when convergence traders have significant positions in both assets. Increased correlation between A and B discourages convergence traders from holding asset B when they have large positions in asset A. Therefore, a larger risk premium must be offered in market B to attract convergence traders.

B. Wealth Dynamics and Stationary Distribution

Panel A of Figure 2 shows the Sharpe ratio of the portfolio of convergence traders. As discussed earlier, log-utility maximizers choose a myopic instantaneously mean-variance-efficient portfolio, and the squared Sharpe ratio from this portfolio determines both the expected trading profits in percentage terms and the instantaneous variance of the convergence traders' portfolio. From the graph, the Sharpe ratio is zero when convergence traders have infinite wealth, indicating zero expected trading profits and also zero risk for their portfolio. When convergence traders have zero wealth, the Sharpe ratio is large, especially when the noise trading gets far away from its long-term mean. This indicates very profitable trading opportunities for convergence traders. At the same time convergence traders face large risks in their portfolio when they exploit these opportunities. Even though the

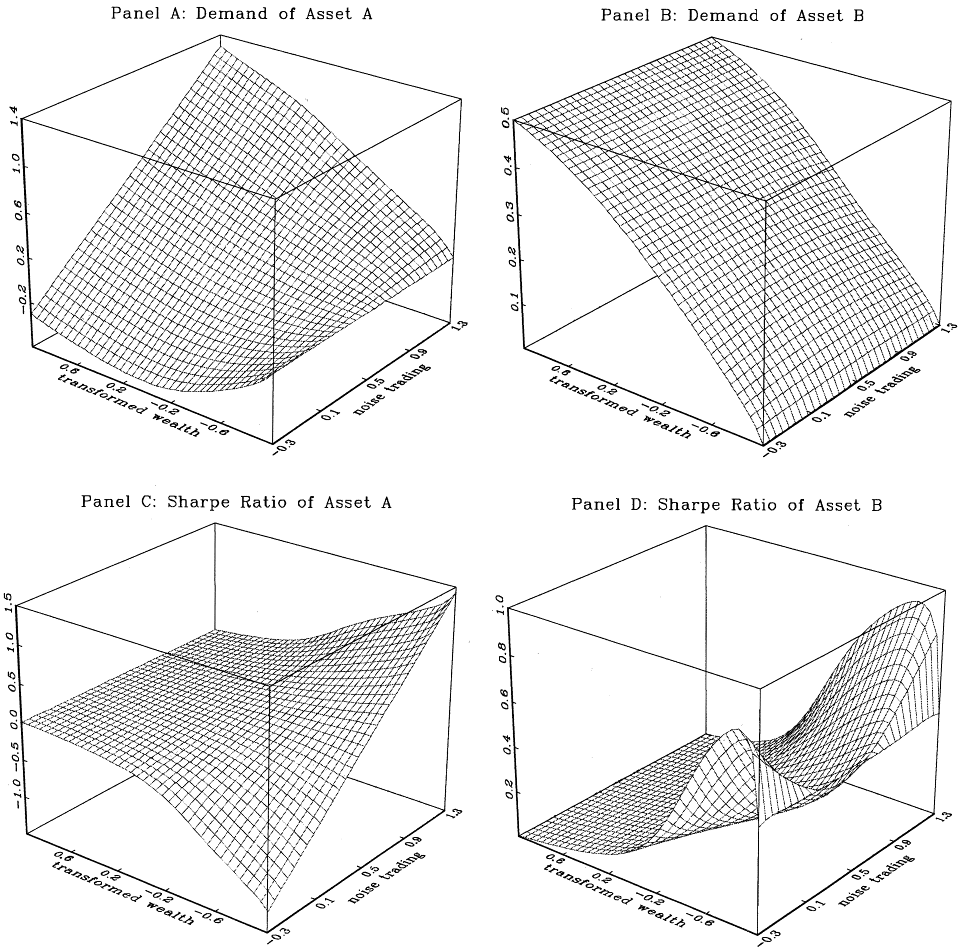

Figure 1. Demand functions and Sharpe ratios. The two independent variables are convergence traders' aggregate wealth and noise trading. Aggregate wealth has been transformed monotonically using $(W - 1)/(W + 1)$ from $(0,\infty)$ into $(-1,1)$. As the transformed wealth ranges from -1 to 1, the aggregate wealth ranges from zero to infinity. Noise trading ranges from four standard deviations below its mean to four standard deviations above its mean. Panels A and B are the equilibrium demands by convergence traders for assets A and B, respectively. Panels C and D are the equilibrium Sharpe ratios for assets A and B, respectively.

Sharpe ratio of asset A varies from positive to negative, the portfolio Sharpe ratio is always positive because convergence traders can take a short position in response to a negative Sharpe ratio.

For a given level of noise trading, the portfolio Sharpe ratio gradually decreases as convergence traders' wealth goes up from zero to infinity. This is the sense in which convergence trading makes markets efficient. The increase of risk bearing capacities among convergence traders reduces the

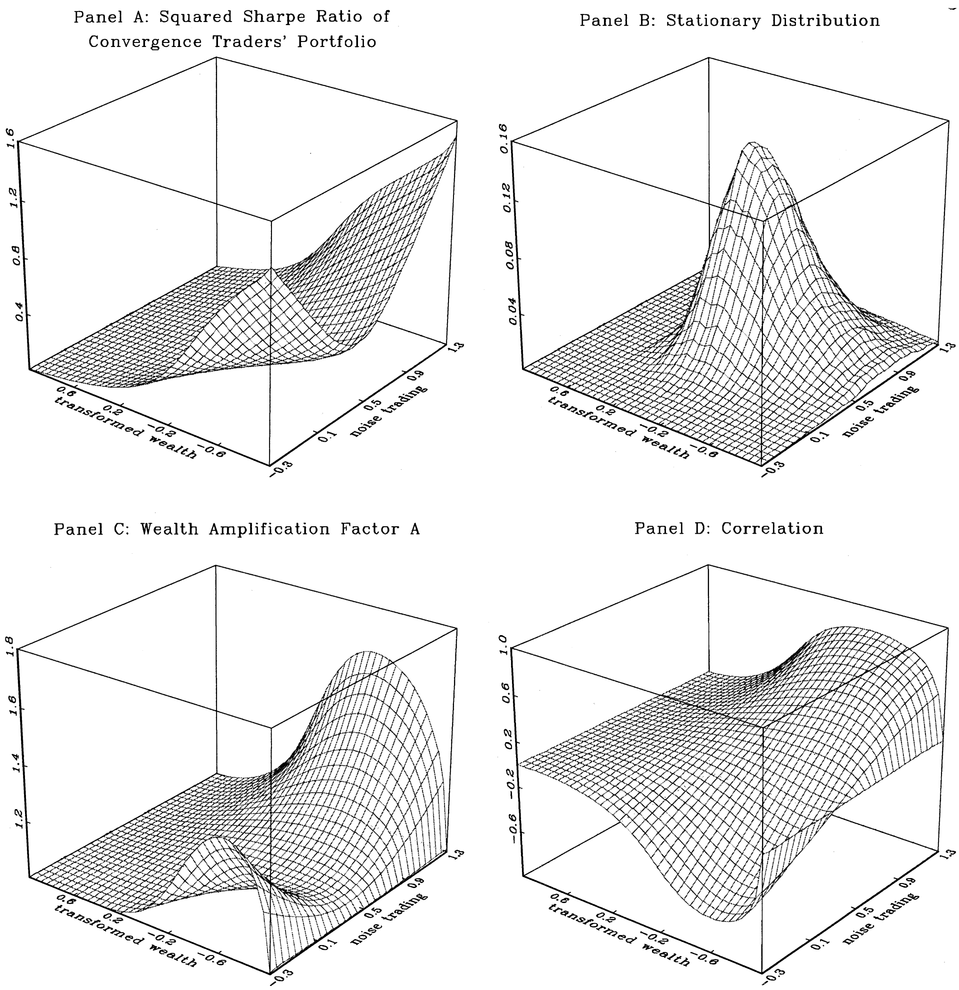

Figure 2. Sharpe ratio of convergence traders' portfolio, stationary distribution, wealth amplification factor and correlation. The two independent variables are convergence traders' aggregate wealth and noise trading. Aggregate wealth has been transformed monotonically using $(W - 1)/(W + 1)$ from $(0,\infty)$ into $(-1,1)$. As the transformed wealth ranges from -1 to 1, the aggregate wealth ranges from zero to infinity. Noise trading ranges from four standard deviations below its mean to four standard deviations above its mean. Panel A is the squared Sharpe ratio of the convergence traders' aggregate portfolio. Panel B is the stationary distribution density of the two state variables, estimated by simulating 20,000 years of equilibrium trading. Panel C is the amplification factor associated with the convergence traders' wealth effect. Panel D is the correlation between price changes of assets A and B.

equilibrium risk premium. This property of the Sharpe ratio results in a mean-reverting dynamics for the convergence traders' wealth process. On the one hand, when convergence traders' wealth is low, the trading is so profitable that their wealth is expected to go up. On the other hand, as their wealth becomes large, increased risk-bearing capacity will drive down the risk premium (or the portfolio Sharpe ratio), and they cannot make enough money from trading to make up for their consumption, so their wealth is expected to go down. As a result, the wealth process follows mean-reverting dynamics.

Since both of the two state variables (noise trading and convergence traders' wealth) follow mean-reverting processes, the equilibrium is stationary. The stationary distribution of the state variables is approximated through a simulation of 20,000 years of weekly data (using an Euler approximation) the results of which are shown in Panel B of Figure 2. This graph verifies that noise trading concentrates within two standard deviations around its long-run mean, and convergence traders' wealth is mostly between zero and some intermediate level.

C. Wealth Amplification Factor

Panel C of Figure 2 shows the wealth amplification factor A (equation (23)), which measures the magnitude of the wealth effect discussed earlier. The amplification factor equals one when convergence traders have either zero or infinite wealth. For intermediate values of wealth, the amplification factor is always larger than one, indicating that any shocks to convergence traders' wealth will be amplified by their portfolio rebalancing.

Also, notice that the wealth amplification become very large when the noise trading θ is very far away from its mean and convergence traders have some intermediate level of wealth. The reason is that two conditions are necessary for the wealth effect to be large. First, the trading opportunity should be great, so that convergence traders will be induced to take large levered positions relative to their wealth and therefore make their portfolio highly sensitive to shocks in the market. Second, the positions of convergence traders should be large so that their position rebalancing caused by exogenous shocks can generate large price impact. Combining these two conditions, the amplification effect is large when noise trading is large and convergence traders' wealth is in some intermediate level.

D. Volatility and Contagion

Due to the nature of the wealth amplification effect, any shocks to either market A or B will be amplified by the portfolio rebalancing of convergence traders. When the wealth amplification factor is relatively high, the volatility of price changes in both markets is relatively high, and the correlation between the two markets is relatively far away from zero.

Figure 3 shows the volatility of price changes in asset A and its three components corresponding to the three types of shock in the model. Figure 4 shows the volatility of price changes in asset B and its three components. All these volatility terms have been normalized by the volatility of fundamental shocks σ_F.

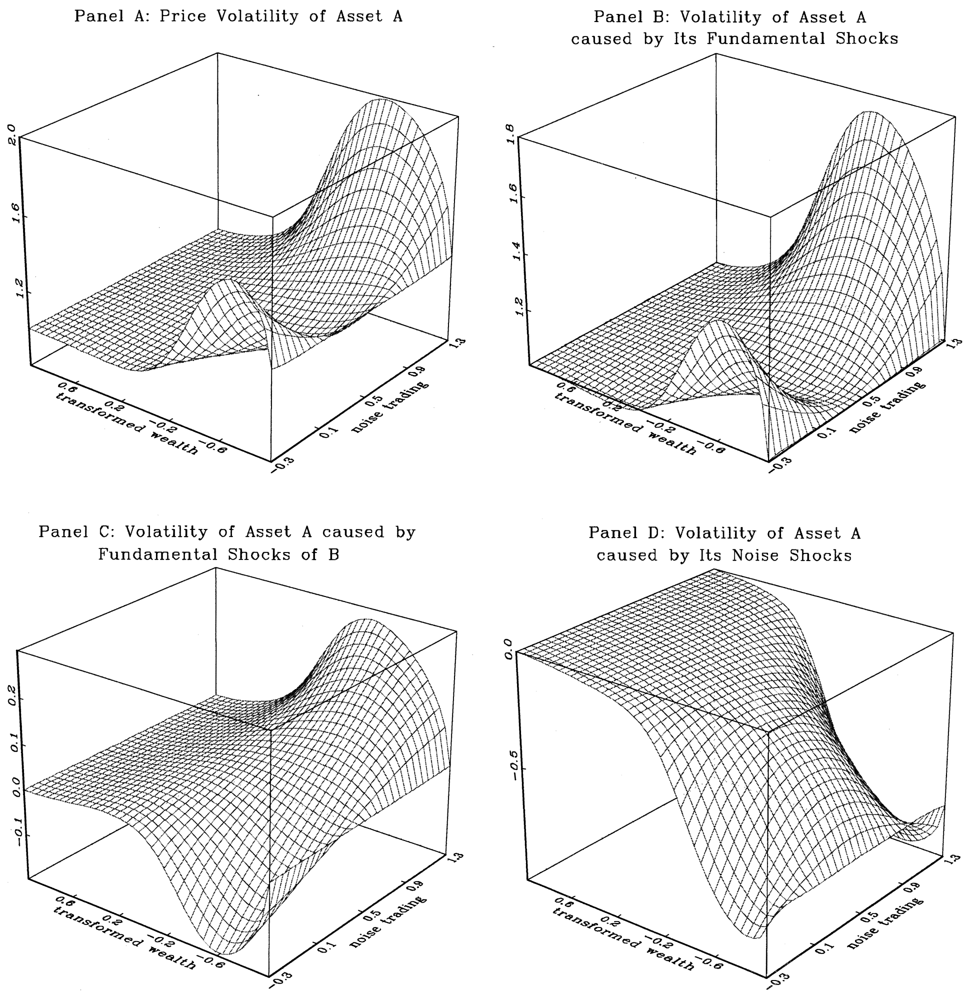

Figure 3. Volatility of asset A and its three components. The two independent variables are convergence traders' aggregate wealth and noise trading. Aggregate wealth has been transformed monotonically using $(W - 1)/(W + 1)$ from $(0,\infty)$ into $(-1,1)$. As the transformed wealth ranges from -1 to 1, the aggregate wealth ranges from zero to infinity. Noise trading ranges from four standard deviations below its mean to four standard deviations above its mean. Panel A is the volatility of the price of asset A, measured as the standard deviation of price changes. Panel B is the component of volatility due to innovations in the fundamental value of asset A. Panel C is the component of volatility due to innovations in the fundamental value of asset B. Panel D is the component of volatility due to innovations in noise trading. The magnitudes of these volatility variables have been normalized by the parameter σ_F.

Panel A of Figure 3 delivers interesting intuitions about the mechanism of convergence trading. When wealth is infinite, the volatility is a constant equal to the volatility of fundamental shocks σ_F. The noise-trading shocks have no impact on price volatility because they are perfectly absorbed by convergence traders. When wealth is zero, the volatility is a constant equal

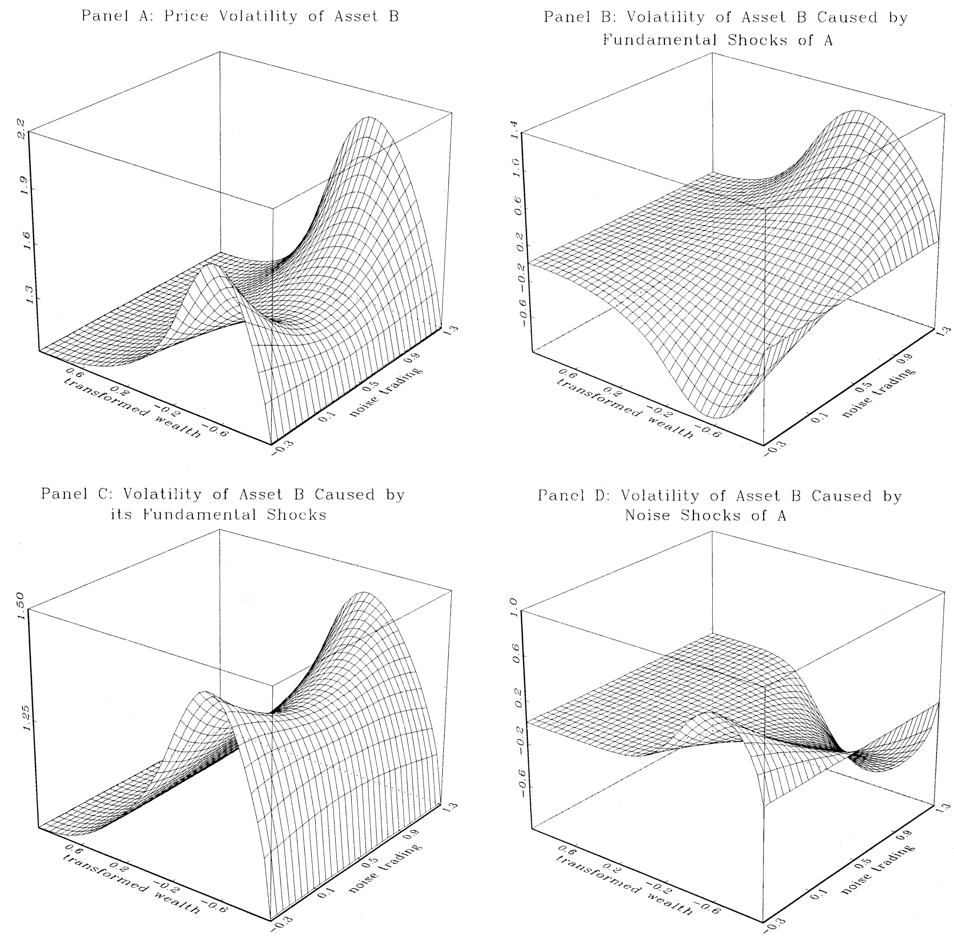

Figure 4. Volatility of asset B and its three components. The two independent variables are convergence traders' aggregate wealth and noise trading. Aggregate wealth has been transformed monotonically using $(W - 1)/(W + 1)$ from $(0,\infty)$ into $(-1,1)$. As the transformed wealth ranges from -1 to 1, the aggregate wealth ranges from zero to infinity. Noise trading ranges from four standard deviations below its mean to four standard deviations above its mean. Panel A is the volatility of the price of asset B, measured as the standard deviation of price changes. Panel B is the component of volatility due to innovations in the fundamental value of asset A. Panel C is the component of volatility due to innovations in the fundamental value of asset B. Panel C is the component of volatility due to innovations in noise trading. The magnitudes of these volatility variables have been normalized by the parameter σ_F.

to $\sqrt{\sigma_F^2 + k^2\sigma_\theta^2}$, which is higher than the volatility of the fundamental shocks because noise-trading shocks are only imperfectly buffered by long-term investors. These constant levels can be used as benchmarks to evaluate the effect of convergence traders on the price volatility of asset A. In the middle "valley", where noise trading is close to its mean ($\theta = 0.5$), the volatility declines monotonically from the higher benchmark to the lower benchmark as wealth increases from zero to infinity. This represents a substitution effect in convergence trading. As noise trading gets bigger, the price gets further out of line, and convergence traders are induced to take larger positions. As a result, their trading reduces price volatility and provides liquidity into the market. When noise trading is far away from its mean, the wealth effect can become dominant. In this situation, convergence traders are already taking large levered positions against noise trading, further movements of noise trading away from its mean can cause convergence traders to lose so much on their current positions that they liquidate positions even though the positions are expected to be more profitable. When this happens, convergence traders become price destabilizing in the sense they are induced by the wealth effect to trade in exactly the same direction as noise traders. This interaction between the wealth effect and the substitution effect is exactly revealed in Panel D of Figure 3, which shows that the impact of noise-trading shocks on the price volatility of asset A is greatest when noise traders take large positions and convergence traders have an intermediate level of wealth. A more detailed discussion about this interaction between the wealth effect and substitution effect is provided by Xiong (2001).

Panel A of Figure 4 shows the volatility of price changes per share in asset B, normalized by the volatility of fundamental shocks σ_F. The volatility is equal to σ_F on the two boundaries when wealth is either zero or infinity. In between, the volatility of asset B is always larger than σ_F, since there is no noise trading in asset B and fundamental shocks are always amplified by the wealth effect. Panel B of Figure 3 and Panel C of Figure 4 show the volatility of assets A and B caused by their own fundamental shocks. Their shapes are very similar to the wealth amplification factor.

Panel C of Figure 3 and Panels B and D of Figure 4 show the effect of volatility transmission (contagion effect) from one market to the other. Panel C of Figure 3 shows the normalized magnitude of σ_B^A, the instantaneous volatility of price changes in asset A resulting from an exogenous fundamental shock dz^B in market B. Panel B of Figure 4 shows the normalized magnitude of σ_A^B, the instantaneous volatility of price changes in asset B resulting from an exogenous fundamental shock dz^A in market A. Panel D of Figure 4 shows the normalized magnitude of σ_θ^B, the instantaneous volatility of price changes in asset B resulting from an exogenous noise-trading shock dz_θ in market A. The magnitude of all of the contagion factors is economically significant. For intermediate levels of wealth (where the effects are greatest), as noise trading varies from -4 to 4 standard deviations from its mean, the contagion factors vary, relative to the volatility of fundamental shocks σ_F, approximately as follows:

1. σ_B^A varies from -0.20 to 0.20.
2. σ_A^B varies from 0.80 to 1.20.
3. σ_θ^B varies from 0.08 to 0.18.

Also notice that the magnitude of σ_B^A is smaller than the magnitude of σ_A^B. The reason for this is that the Sharpe ratio of asset B is smaller in absolute value than that of asset A. Thus, convergence traders take large positions for asset A, resulting in larger risk exposure to shocks to asset A. Therefore, shocks in market A can cause larger wealth fluctuations to convergence traders' portfolio, which are transmitted to the price of asset B in larger magnitude than the shocks in market B are transmitted to the price of asset A.

Panel D of Figure 2 shows the correlation between the price changes of the two assets. Since the fundamentals and noise trading in the two markets are independently distributed, nonzero correlation is associated with contagion. When wealth is zero or infinity, correlation is zero. At intermediate values of wealth, correlation can be significantly different from zero. Correlation is positive when convergence traders are long in both markets, and negative when convergence traders are short in market A and long in market B. The magnitude of the correlation becomes large when noise trading in the market gets far away from its mean and convergence traders' wealth is at some intermediate level. These regions are exactly where the wealth amplification effect is large. As noise trading varies from -4 to 4 standard deviations away from its mean, correlation at intermediate levels of wealth ranges from -0.6 to $+ 0.8$.

These graphs suggest "crisis" situations when noise trading is far away from its mean and convergence traders' wealth is at some intermediate level. In these situations, the wealth effect can induce convergence traders to liquidate large amounts of positions across their whole portfolio in response to unfavorable shocks, resulting in large price volatility and greatly reduced liquidity in all markets, and large correlation between different markets. These graphs also confirm some of the stylized facts associated with asset price volatility and correlation between asset prices. First, asset price volatility is always larger than fundamental volatility. The additional volatility comes from both noise trading and the wealth amplification effect. Second, the correlation between asset prices is larger than the correlation between asset fundamentals. This occurs because the wealth dynamics of convergence traders introduce an additional common factor among asset prices. Third, asset price volatility and the correlation between asset prices are both time varying. The stochastic volatility and correlation result from the nonlinear dynamics of the convergence traders' wealth process.

IV. Implications for Risk Management

Our model has important implications for risk management. The key insight is that in equilibrium, the risks are endogenously determined by the trading of all market participants, and it may be dangerous to treat risks as exogenous in risk management. More specifically, the following cautions can be drawn from our model. First, risk managers should recognize the wealth effect of convergence traders who use a short-term trading strategy. Second, risk managers should appreciate the importance of market liquidity provided by long-term investors in periods of crisis. Third, risk managers should realize that correlation between assets tends to deviate from historical values and rise during crises in such a way that portfolio losses occur in all positions simultaneously. Failure to recognize these factors in risk management can result in underestimation of volatility and correlation between asset prices, especially when the wealth amplification effect is severe.

The importance of these factors has been illustrated by the financial crisis of hedge fund Long Term Capital Management (LTCM) during 1998. As recalled by one of LTCM's partners (Lewis, 1999, p. 31): "It was as if there was someone out there with our exact portfolio, only it was three times as large as ours, and they were liquidating all at once." This suggests a severe copycat problem during this period. Due to the success of LTCM before the crisis, its convergence trading strategies were popular among other hedge funds and proprietary trading desks at many investment banks. When LTCM ran into trouble and needed to liquidate some of their positions, other convergence traders with similar positions were simultaneously trying to dump their positions. When all of these convergence traders were trying to get out of their positions through the only exit, the liquidity provided by long-term investors, the door did not appear to be as wide as it once was. Furthermore, the liquidation of convergence traders' positions was not limited to only one asset, it was spread out among all assets in convergence traders' portfolios due to increased risk aversion. This caused the correlation between asset returns to be much higher than in usual periods, resulting in the failure of diversification to reduce risks as much as models based on historical returns may suggest. The report by the BIS (1999) provides a documentation of volatility and correlation across a wide range of financial markets for periods around the crisis of LTCM in 1998.

Our model suggests risk managers take into account the endogenous risks caused by the trading of other market participants. Since these market-created risks, such as contagion and volatility amplification by the convergence traders' wealth effect, are only evident in extreme scenarios, studying historical data of asset returns and volatility tends to overlook or underestimate these risks, unless extremely long series of data are used. Even if very long series of data are available, the potential changes in the structure of the market can make it hopeless to determine these extreme risks from historical data.

To avoid these problems, risk managers should not rely only on statistical methods. Our economic model in this paper, based on assumptions about the liquidity provided by long-term investors and the behavior of convergence traders, suggests that risk managers calculate their optimal risky positions after considering the capitalization and positions of other traders in the market. Therefore, it offers risk managers a different perspective for controlling these endogenous risks associated with the convergence traders' wealth effect in extreme situations.

V. Conclusion

In this paper, we develop an equilibrium model of contagion that operates through a wealth effect of convergence traders. Convergence traders specialize in trading a small number of assets in which they take large risky positions against noise trading. Wealth effect occurs when convergence traders suffer large capital losses due to unfavorable shocks and need to liquidate positions across their portfolio. Their position liquidation can cause the original shocks to be greatly amplified and transmitted from one asset to other assets.

In equilibrium, the asset price dynamics and convergence traders' wealth dynamics are simultaneously determined. This simultaneous relationship introduces endogenous risks into the model in the form of contagion and volatility amplification through the wealth effect of convergence traders. Our model cautions risk managers to take into account these endogenous risks. Failure to do so can cause much larger risks in trading than what is forecast by naive statistical tools. Our economic model, based on assumptions about the liquidity provided by long-term investors and the behavior of convergence traders, suggests a direction of future research which could lead to better tools for risk management.

Appendix A. Derivation of Asset Return Processes

Given the aggregate portfolio policies $X^A(\theta, W)$ and $X^B(\theta, W)$ for convergence traders, we derive asset return processes by applying Ito's lemma. The market clearing condition gives the price functions for the two assets:

$$P^A = F^A - k^A(\theta - X^A), \tag{A1}$$

$$P^B = F^B - k^B(\bar{\theta}^B - X^B). \tag{A2}$$

The excess return process for investing in one share of asset A is given by

$$\begin{aligned} dQ^A &= dP^A + (D^A - rP^A)dt \\ &= \sigma_F dz^A - k^A d\theta + k^A dX^A + rk^A(\theta - X^A)dt. \end{aligned} \tag{A3}$$

Similarly, the excess return for investing in one share of asset B is

$$dQ^B = \sigma_F dz^B + k^B dX^B + rk^B(\bar{\theta}^B - X^B)dt. \tag{A4}$$

As discussed in Section II, we assume without loss of generality that the fundamental process of the two assets have the same fundamental volatility σ_F. From Ito's lemma, we obtain

$$dX^A = X^A_\theta d\theta + 1/2X^A_{\theta\theta}\mathrm{E}(d\theta)^2 + X^A_W dW + 1/2X^A_{WW}\mathrm{E}(dW)^2 + X^A_{\theta W}\mathrm{E}(d\theta dW), \tag{A5}$$

$$dX^B = X^B_\theta d\theta + 1/2X^B_{\theta\theta}\mathrm{E}(d\theta)^2 + X^B_W dW + 1/2X^B_{WW}\mathrm{E}(dW)^2 + X^B_{\theta W}\mathrm{E}(d\theta dW). \tag{A6}$$

The asset return processes and convergence traders' wealth process are simultaneously determined in the equilibrium. Equations (A1) through (A4) show the dependence of return processes dQ^A and dQ^B on convergence traders' aggregate wealth W. On the other hand, convergence traders' wealth depends on the return processes through their budget constraint

$$dW = X^A dQ^A + X^B dQ^B + (rW - C)dt. \tag{A7}$$

To deal with this circular relationship, we first substitute equation (A7) into equations (A5) and (A6), then further substitute equations (A5) and (A6) into equations (A3) and (A4). Finally, we obtain a set of two linear equations for dQ^A and dQ^B:

$$\begin{aligned}(1 - k^A X^A X_W^A)dQ^A &- k^A X_W^A X^B dQ^B \\ &= \sigma_F dz_A - k^A(1 - X_\theta^A)d\theta + [rk^A(\theta - X^A) + k^A X_W^A(rW - C)]dt \\ &\quad + \frac{k^A}{2} X_{\theta\theta}^A \mathrm{E}(d\theta)^2 + \frac{k^A}{2} X_{WW}^A \mathrm{E}(dW)^2 + k^A X_{\theta W}^A \mathrm{E}(d\theta dW)\end{aligned} \tag{A8}$$

$$\begin{aligned}&- k^B X^A X_W^B dQ^A + (1 - k^B X^B X_W^B)dQ^B \\ &= \sigma_F dz_B + k^B X_\theta^B d\theta + [k^B X_W^B(rW - C) + rk^B(\bar{\theta}^B - X^B)]dt \\ &\quad + \frac{k^B}{2} X_{\theta\theta}^B \mathrm{E}(d\theta)^2 + \frac{k^B}{2} X_{WW}^B \mathrm{E}(dW)^2 + k^B X_{\theta W}^B \mathrm{E}(d\theta dW).\end{aligned} \tag{A9}$$

Solution to these linear equations gives us the following return processes:

$$dQ^A = \mu^A dt + \sigma_A^A dz^A + \sigma_B^A dz^B + \sigma_\theta^A dz_\theta, \tag{A10}$$

$$\sigma_A^A = \sigma_F(1 - k^B X^B X_W^B)A(\theta, W), \tag{A11}$$

$$\sigma_B^A = \sigma_F k^A X^B X_W^A A(\theta, W), \tag{A12}$$

$$\sigma_\theta^A = -k^A \sigma_\theta[(1 - X_\theta^A)(1 - k^B X^B X_W^B) - k^B X_W^A X^B X_\theta^B]A(\theta, W), \tag{A13}$$

$$\begin{aligned}\mu^A = A(\theta, W)\{&k^A \lambda_\theta(\theta - \bar{\theta})[(1 - X_\theta^A)(1 - k^B X^B X_W^B) - k^B X_W^A X^B X_\theta^B] \\ &+ k^A X_W^A(rW - C) + rk^A(1 - k^B X^B X_W^B)(\theta - X^A) \\ &+ rk^A k^B X_W^A X^B(\bar{\theta}^B - X^B)\} \\ &+ \frac{k^A \sigma_\theta^2}{2}[X_{\theta\theta}^A(1 - k^B X^B X_W^B) + k^B X_W^A X^B X_{\theta\theta}^B]A(\theta, W) \\ &+ \frac{k^A(\sigma^W)^2}{2}[X_{WW}^A(1 - k^B X^B X_W^B) + k^B X_W^A X^B X_{WW}^B]A(\theta, W) \\ &+ k^A \sigma_\theta \sigma_\theta^W[X_{\theta W}^A(1 - k^B X^B X_W^B) + k^B X_W^A X^B X_{\theta W}^B]A(\theta, W),\end{aligned} \tag{A14}$$

$$dQ^B = \mu^B dt + \sigma_A^B dz^A + \sigma_B^B dz^B + \sigma_\theta^B dz_\theta, \tag{A15}$$

$$\sigma_A^B = \sigma_F k^B X^A X_W^B A(\theta, W), \tag{A16}$$

$$\sigma_B^B = \sigma_F [1 - k^A X^A X_W^A] A(\theta, W), \tag{A17}$$

$$\sigma_\theta^B = k^B \sigma_\theta [X_\theta^B (1 - k^A X^A X_W^A) - k^A X^A X_W^B (1 - X_\theta^A)] A(\theta, W), \tag{A18}$$

$$\begin{aligned}\mu^B = {} & A(\theta,W)\{-k^B \lambda_\theta (\theta - \bar\theta)[X_\theta^B(1 - k^A X^A X_W^A) - k^A X^A X_W^B (1 - X_\theta^A)] \\ & + k^B X_W^B (rW - C) + rk^B (1 - k^A X^A X_W^A)(\bar\theta^B - X^B) \\ & + rk^A k^B X^A X_W^B (\theta - X^A)\} \\ & + \frac{k^B \sigma_\theta^2}{2} [X_{\theta\theta}^B (1 - k^A X^A X_W^A) + k^A X^A X_{\theta\theta}^A X_W^B] A(\theta, W) \\ & + \frac{k^B (\sigma^W)^2}{2} [X_{WW}^B (1 - k^A X^A X_W^A) + k^A X^A X_{WW}^A X_W^B] A(\theta, W) \\ & + k^B \sigma_\theta \sigma_\theta^W [X_{\theta W}^B (1 - k^A X^A X_W^A) + k^A X^A X_{\theta W}^A X_W^B] A(\theta, W).\end{aligned} \tag{A19}$$

In the expressions above, the common term

$$A(\theta, W) = \frac{1}{1 - k^A X^A X_W^A - k^B X^B X_W^B} \tag{A20}$$

represents the wealth amplification factor. The total volatility of these returns is

$$\sigma^A = \sqrt{(\sigma_A^A)^2 + (\sigma_B^A)^2 + 2\phi_F \sigma_A^A \sigma_B^A + (\sigma_\theta^A)^2}, \tag{A21}$$

$$\sigma^B = \sqrt{(\sigma_A^B)^2 + (\sigma_B^B)^2 + 2\phi_F \sigma_A^B \sigma_B^B + (\sigma_\theta^B)^2}. \tag{A22}$$

The instantaneous correlation between the two return processes is

$$\phi = \frac{1}{\sigma^A \sigma^B} [\sigma_A^A \sigma_A^B + \sigma_B^A \sigma_A^B + \phi_F (\sigma_A^A \sigma_B^B + \sigma_B^A \sigma_A^B) + \sigma_\theta^A \sigma_\theta^B]. \tag{A23}$$

From the budget constraints, we can derive the process for convergence traders' aggregate wealth:

$$dW = \mu^W dt + \sigma_A^W dz^A + \sigma_B^W dz^B + \sigma_\theta^W dz_\theta, \tag{A24}$$

$$\mu^W = X^A \mu^A + X^B \mu^B + rW - C, \tag{A25}$$

$$\sigma_A^W = X^A \sigma_A^A + X^B \sigma_A^B, \tag{A26}$$

$$\sigma_B^W = X^A \sigma_B^A + X^B \sigma_B^B, \tag{A27}$$

$$\sigma_\theta^W = X^A \sigma_\theta^A + X^B \sigma_\theta^B. \tag{A28}$$

The total volatility of the wealth process is

$$\sigma^W = \sqrt{(\sigma_A^W)^2 + (\sigma_B^W)^2 + 2\phi_F \sigma_A^W \sigma_B^W + (\sigma_\theta^W)^2}. \tag{A29}$$

It is useful to show the return processes when the effect of convergence traders is small ($X^A \to 0$, $X^B \to 0$). Under this situation, the excess return processes are

$$dQ^A = \sigma_F dz^A - k^A d\theta + rk^A \theta dt, \tag{A30}$$

$$dQ^B = \sigma_F dz^B + rk^B \bar{\theta}^B dt. \tag{A31}$$

Sharpe ratios of asset A and B are

$$\frac{\mu^A}{\sigma^A} = \frac{rk^A\theta + k^A\lambda_\theta(\theta - \bar{\theta})}{\sqrt{\sigma_F^2 + (k^A\sigma_\theta)^2}}, \tag{A32}$$

$$\frac{\mu^B}{\sigma^B} = \frac{rk^B\bar{\theta}^B}{\sigma_F}. \tag{A33}$$

The Sharpe ratio of asset A fluctuates with its supply θ, and the variance of the Sharpe ratio is

$$\mathrm{Var}\left(\frac{\mu^A}{\sigma^A}\right) = \frac{(r + \lambda_\theta)^2 (k^A)^2 \sigma_\theta^2}{2\lambda_\theta[\sigma_F^2 + (k^A\sigma_\theta)^2]}. \tag{A34}$$

These return processes represent the original trading opportunities when there are no convergence traders at all.

Appendix B. Derivation of Optimal Strategy

In this section, we derive convergence traders' optimal trading strategy given the return processes of assets A and B. We can write the asset return processes in the following form:

$$dQ^A = \mu^A(\theta, W)dt + \sigma_A^A(\theta, W)dz^A + \sigma_B^A(\theta, W)dz^B + \sigma_\theta^A(\theta, W)dz_\theta, \tag{B1}$$

$$dQ^B = \mu^B(\theta, W)dt + \sigma_A^B(\theta, W)dz^A + \sigma_B^B(\theta, W)dz^B + \sigma_\theta^B(\theta, W)dz_\theta. \tag{B2}$$

These return processes represent the trading opportunities to an individual convergence trader, and these processes depend on the two state variables θ and W. The parameter θ denotes the supply shock to asset A. It follows

$$d\theta = -\lambda_\theta(\theta - \bar{\theta})dt + \sigma_\theta dz_\theta.$$

The parameter W is the aggregate wealth of convergence traders, and it follows the process

$$dW = \mu^W(\theta,W)dt + \sigma_A^W(\theta,W)dz^A + \sigma_B^W(\theta,W)dz^B + \sigma_\theta^W dz_\theta. \tag{B3}$$

We denote an individual convergence trader's portfolio choices, consumption, and wealth as X^{i^A}, X^{i^B}, C^i, and W^i, respectively. The convergence trader's budget constraint is

$$d\overline{W} = X^{i^A}dQ^A + X^{i^B}dQ^B + (rW^i - C^i)dt. \tag{B4}$$

The convergence trader maximizes her lifetime utility given by

$$J(W^i,\theta,W) = \max_{\{X^{i^A},X^{i^B},C^i\}} \mathrm{E}_t \int_0^\infty \mathrm{e}^{-\rho s}\ln(C^i_{t+s})\,ds. \tag{B5}$$

We solve the portfolio and consumption policies through a Bellman equation as developed by Merton (1971). The Bellman equation can be derived as

$$\begin{aligned}\rho J(W^i,\theta,w) &= \max_{\{X^{i^A},X^{i^B},C^i\}} [\ln(C^i) + \mathcal{L}^0 J] \\ &= \max_{X^{i^A},X^{i^B},C^i} [\ln(C^i) + J_{W^i}(X^{i^A}\mu^A + X^{i^B}\mu^B + rW^i - C^i) \\ &\qquad + 1/2 J_{W^iW^i}((X^{i^A})^2(\sigma^A)^2 + (X^{i^B})^2(\sigma^B)^2 \\ &\qquad\qquad + 2X^{i^A}X^{i^B}\phi\sigma^A\sigma^B) \\ &\qquad + \lambda_\theta(\bar{\theta}-\theta)J_\theta + \mu^W J_W + 1/2\sigma_\theta^2 J_{\theta\theta} + 1/2\sigma_W^2 J_{WW} \\ &\qquad + J_{W^i\theta}\mathrm{E}(dW^i d\theta)/dt + J_{W^iw}\mathrm{E}(dW^i dW)/dt \\ &\qquad + J_{\theta w}\mathrm{E}(d\theta dW)/dt],\end{aligned} \tag{B6}$$

where $\mathcal{L}^0$ is the drift operator, and ϕ is the instantaneous correlation between dQ^A and dQ^B. The value function of a logarithmic utility maximizer can be specified as

$$J(W^i,\theta,w) = \frac{1}{\rho}\ln(W^i) + j(\theta,W). \tag{B7}$$

The first order condition of the Bellman equation gives the optimal portfolio and consumption policies:

$$X^{i^A} = \frac{W^i}{1-\phi^2}\left[\frac{\mu^A}{(\sigma^A)^2} - \phi\frac{\mu^B}{\sigma^A\sigma^B}\right], \tag{B8}$$

$$X^{i^B} = \frac{W^i}{1-\phi^2}\left[\frac{\mu^B}{(\sigma^B)^2} - \phi\frac{\mu^A}{\sigma^A\sigma^B}\right], \tag{B9}$$

$$C^i = \rho W^i. \tag{B10}$$

After substituting the optimal policies into the Bellman equation, W^i disappears from both sides of the equation, and the Bellman equation collapses into a partial differential equation in θ and W only:

$$\begin{aligned}\rho j(\theta, W) = \ln(\rho) + \rho(r-\rho) + \frac{\rho}{2(1-\phi^2)}\left(\frac{(\mu^A)^2}{(\sigma^A)^2} - 2\phi\,\frac{\mu^A\mu^B}{\sigma^A\sigma^B} + \frac{(\mu^B)^2}{(\sigma^B)^2}\right) \\ + \lambda_\theta(\bar{\theta}-\theta)j_\theta + \mu^W j_W + 1/2\sigma_\theta^2 j_{\theta\theta} + 1/2\sigma_W^2 j_{WW} + \sigma_\theta\sigma_\theta^W j_{\theta W}.\end{aligned} \tag{B11}$$

Therefore, the convergence trader's policy functions and value function become separated. The solution to the PDE of the value function exists under certain technical conditions. We will focus on the policy functions and discuss the equilibrium of asset markets.

It is a well-known result of logarithmic utility that log-utility maximizers do not have any hedging need. Their portfolio and consumption policies are solely determined by their current trading opportunities and their wealth. For a general utility maximizer, the hedging need is represented by the dependence of policy functions on the value function. The assumption of logarithmic utility for convergence traders greatly simplifies the problem without losing the key feature of our model, which is the wealth effect.

Notice that log-utility maximizers hold a locally mean-variance efficient portfolio. It is easy to derive the instantaneous expected trading profits and variance of this portfolio:

$$\begin{aligned}\mathrm{E}\left[\frac{X^{iA}}{W^i}\,dQ^A + \frac{X^{iB}}{W^i}\,dQ^B\right] &= \mathrm{var}\left[\frac{X^{iA}}{W^i}\,dQ^A + \frac{X^{iB}}{W^i}\,dQ^B\right] \\ &= \frac{1}{1-\phi^2}\left[\frac{(\mu^A)^2}{(\sigma^A)^2} + \frac{(\mu^B)^2}{(\sigma^B)^2} - 2\phi\,\frac{\mu^A\mu^B}{\sigma^A\sigma^B}\right].\end{aligned} \tag{B12}$$

This value is exactly the squared Sharpe ratio of the instantaneous mean-variance efficient portfolio.

Appendix C. Partial Differential Equations

Appendix C presents the partial differential equations from the fixed-point problem. Given convergence traders' aggregate portfolio and consumption functions $X(\theta, W)$ and $C(\theta, W)$, the optimal aggregate portfolio and consumption rules can be easily derived from equations (B8) through (B10) by replacing W^i by W:

$$X^{*A} = \frac{W}{1-\phi^2}\left[\frac{\mu^A}{(\sigma^A)^2} - \phi\,\frac{\mu^B}{\sigma^A\sigma^B}\right], \tag{C1}$$

$$X^{*B} = \frac{W}{1-\phi^2}\left[\frac{\mu^B}{(\sigma^B)^2} - \phi\,\frac{\mu^A}{\sigma^A\sigma^B}\right], \tag{C2}$$

$$C = \rho W. \tag{C3}$$

The equilibrium consumption rule is trivial ($C = \rho W$), and the equilibrium portfolio rules from the fixed-point conditions are

$$X^A = \frac{W}{1-\phi^2}\left[\frac{\mu^A}{(\sigma^A)^2} - \phi\,\frac{\mu^B}{\sigma^A\sigma^B}\right], \tag{C4}$$

$$X^B = \frac{W}{1-\phi^2}\left[\frac{\mu^B}{(\sigma^B)^2} - \phi\,\frac{\mu^A}{\sigma^A\sigma^B}\right]. \tag{C5}$$

By substituting all the necessary terms from equations (A10) through (A22) into equations (C4) and (C5), two partial differential equations are obtained. In order to save space, the terms A, ϕ, σ^A, σ^B, σ^W, and σ_θ^W are not substituted into the partial differential equations. Instead they are considered functions of $X^A(\theta, W)$ and $X^B(\theta, W)$ from equations (A20) through (A28). The two partial differential equations are

$$\begin{aligned}
\frac{X^A(1-\phi^2)}{WA(\theta,W)} = {} & \frac{1}{(\sigma^A)^2}\{k^A\lambda_\theta(\theta-\bar{\theta})[(1-X_\theta^A)(1-k^BX^BX_W^B) - k^BX_W^AX^BX_\theta^B] \\
& \quad + k^AX_W^A(rW-C) + rk^A(1-k^BX^BX_W^B)(\theta - X^A) \\
& \quad + rk^Ak^BX_W^AX^B(\bar{\theta}^B - X^B)\} \\
& + \frac{k^A\sigma_\theta^2}{2(\sigma^A)^2}[X_{\theta\theta}^A(1-k^BX^BX_W^B) + k^BX_W^AX^BX_{\theta\theta}^B] \\
& + \frac{k^A(\sigma^W)^2}{2(\sigma^A)^2}[X_{WW}^A(1-k^BX^BX_W^B) + k^BX_W^AX^BX_{WW}^B] \\
& + \frac{k^A\sigma_\theta\sigma_\theta^W}{(\sigma^A)^2}[X_{\theta W}^A(1-k^BX^BX_W^B) + k^BX_W^AX^BX_{\theta W}^B]) \\
& - \frac{\phi}{\sigma^A\sigma^B}\{-k^B\lambda_\theta(\theta-\bar{\theta})[X_\theta^B(1-k^AX^AX_W^A) - k^AX^AX_W^B(1-X_\theta^A)] \\
& \quad + k^BX_W^B(rW-C) + rk^B(1-k^AX^AX_W^A)(\bar{\theta}^B - X^B) \\
& \quad + rk^Ak^BX^AX_W^B(\theta - X^A)\} \\
& - \frac{k^B\phi\sigma_\theta^2}{2\sigma^A\sigma^B}[X_{\theta\theta}^B(1-k^AX^AX_W^A) + k^AX^AX_{\theta\theta}^AX_W^B] \\
& - \frac{k^B\phi(\sigma^W)^2}{2\sigma^A\sigma^B}[X_{WW}^B(1-k^AX^AX_W^A) + k^AX^AX_{WW}^AX_W^B] \\
& - \frac{k^B\phi\sigma_\theta\sigma_\theta^W}{\sigma^A\sigma^B}[X_{\theta W}^B(1-k^AX^AX_W^A) + k^AX^AX_{\theta W}^AX_W^B],
\end{aligned} \tag{C6}$$

$$\begin{aligned}
\frac{X^B(1-\phi^2)}{WA(\theta,W)} = {} & \frac{1}{(\sigma^B)^2}\{-k^B\lambda_\theta(\theta-\bar{\theta})[X^B_\theta(1-k^AX^AX^A_W)-k^AX^AX^B_W(1-X^A_\theta)] \\
& \quad + k^BX^B_W(rW-C)+rk^B(1-k^AX^AX^A_W)(\bar{\theta}^B-X^B) \\
& \quad + rk^Ak^BX^AX^B_W(\theta-X^A)\} \\
& + \frac{k^B\sigma_\theta^2}{2(\sigma^B)^2}[X^B_{\theta\theta}(1-k^AX^AX^A_W)+k^AX^AX^A_{\theta\theta}X^B_W] \\
& + \frac{k^B(\sigma^W)^2}{2(\sigma^B)^2}[X^B_{WW}(1-k^AX^AX^A_W)+k^AX^AX^A_{WW}X^B_W] \\
& + \frac{k^B\sigma_\theta\sigma_\theta^W}{(\sigma^B)^2}[X^B_{\theta W}(1-k^AX^AX^A_W)+k^AX^AX^A_{\theta W}X^B_W] \\
& - \frac{\phi}{\sigma^A\sigma^B}\{k^A\lambda_\theta(\theta-\bar{\theta})[(1-X^A_\theta)(1-k^BX^BX^B_W)-k^BX^A_WX^BX^B_\theta] \\
& \quad + k^AX^A_W(rW-C)+rk^A(1-k^BX^BX^B_W)(\theta-X^A) \\
& \quad + rk^Ak^BX^A_WX^B(\bar{\theta}^B-X^B)\} \\
& - \frac{k^A\phi\sigma_\theta^2}{2\sigma^A\sigma^B}[X^A_{\theta\theta}(1-k^BX^BX^B_W)+k^BX^A_WX^BX^B_{\theta\theta}] \\
& - \frac{k^A\phi(\sigma^W)^2}{2\sigma^A\sigma^B}[X^A_{WW}(1-k^BX^BX^B_W)+k^BX^A_WX^BX^B_{WW}] \\
& - \frac{k^A\phi\sigma_\theta\sigma_\theta^W}{\sigma^A\sigma^B}[X^A_{\theta W}(1-k^BX^BX^B_W)+k^BX^A_WX^BX^B_{\theta W}]. \qquad \text{(C7)}
\end{aligned}$$

The two partial differential equations in (C6) and (C7) are highly nonlinear and entangled together. In addition to the two unknown functions $X^A(\theta,W)$ and $X^B(\theta,W)$, the partial differential equations involve the first derivatives X^A_θ, X^A_W, X^B_θ, X^B_W and the second derivatives $X^A_{\theta\theta}$, $X^A_{\theta W}$, X^A_{WW}, $X^B_{\theta\theta}$, $X^B_{\theta W}$, X^B_{WW}. Due to the complexity of the partial differential equations, they are solved numerically.

Appendix D. Numerical Method to the Fixed-point Problem

To study the equilibrium, a numerical method is needed to solve the fixed-point problem. We use a projection method in which each of the demand functions X^A and X^B are approximated with rational functions, where both the numerators and denominators are polynomials of two state variables. The algorithm chooses coefficients of the polynomials so that the partial differential equations describing the equilibrium are approximately solved for a range of test values and so that the boundary conditions hold. Instead of ordinary polynomials, we use Chebyshev polynomials for reasons of numerical stability: With Chebyshev polynomials, the calculation of the values of polynomials is more stable, there is less "collinearity" among estimated coefficients, and the degree of Chebyshev polynomials can be reduced easily by truncating high-order terms. Also, the use of Chebyshev polynomials makes it easier to impose boundary conditions as described in Appendix E. For a detailed introduction to projection methods and Chebyshev polynomials, see Judd (1998) and Press et al. (1992).

To use Chebyshev polynomials, whose natural range is $[-1,+1]$, it is first necessary to transform the state variables W and θ to fit this range. To transform W, whose range is $(0,\infty)$, we introduce the transformed variable z, and define it (with exogenously specified scale parameter γ) by

$$z = \frac{W-\gamma}{W+\gamma}, \quad z \in (-1,1). \tag{D1}$$

To transform theta, whose natural range is $(-\infty,+\infty)$, we truncate the variable at four standard deviations and use a linear transformation to define a new state variable y by

$$y = \frac{\theta-\bar{\theta}}{4\sigma_\theta/\sqrt{2\lambda_\theta}}, \quad y \in [-1,1]. \tag{D2}$$

Both of these transformations are obviously monotonic and smooth. The reverse transformations are

$$\theta = \bar{\theta} + \frac{4\sigma_\theta}{\sqrt{2\lambda_\theta}}y, \tag{D3}$$

$$W = \gamma\frac{1+z}{1-z}. \tag{D4}$$

The derivatives to the two state variables θ and W can be transformed as

$$\frac{\partial}{\partial\theta} = \frac{4\sigma_\theta}{\sqrt{2\lambda_\theta}}\frac{\partial}{\partial y}, \tag{D5}$$

$$\frac{\partial^2}{\partial\theta^2} = \frac{8\sigma_\theta^2}{\lambda_\theta}\frac{\partial^2}{\partial y^2}, \tag{D6}$$

$$\frac{\partial}{\partial W} = \frac{(1-z)^2}{2\gamma}\frac{\partial}{\partial z}, \tag{D7}$$

$$\frac{\partial^2}{\partial W^2} = \frac{(1-z)^4}{4\gamma^2}\frac{\partial^2}{\partial z^2} - \frac{(1-z)^3}{2\gamma^2}\frac{\partial}{\partial z}, \tag{D8}$$

$$\frac{\partial^2}{\partial\theta\partial W} = \frac{\sqrt{2}\sigma_\theta(1-z)^2}{\sqrt{\lambda_\theta}}\frac{\partial^2}{\partial y\partial z}. \tag{D9}$$

With these formulas, we can transform the original PDEs of $X^A(\theta,W)$ and $X^B(\theta,W)$ in equations (C6) and (C7) into PDEs of $X^A(y,z)$ and $X^B(y,z)$.

We approximate the equilibrium demand functions $X^A(y,z)$ and $X^B(y,z)$ as

$$X^A(y,z) = \frac{\sum_{i+j\le n_u^A} a_u^A(i,j)T_i(y)T_j(z)}{\sum_{i+j\le n_d^A} a_d^A(i,j)T_i(y)T_j(z)}, \tag{D10}$$

$$X^B(y,z) = \frac{\sum_{i+j\le n_u^B} a_u^B(i,j)T_i(y)T_j(z)}{\sum_{i+j\le n_d^B} a_d^B(i,j)T_i(y)T_j(z)}, \tag{D11}$$

where $T_i()$ is ith order Chebyshev polynomial, and n_u^A, n_d^A, n_u^B, and n_d^B are the total orders of polynomials in the numerators and denominators of X^A and X^B. Let $\{a_u^A(i,j)\}_{i+j\le n_u^A}$, $\{a_d(i,j)^A\}_{i+j\le n_d^A}$, $\{a_u^B(i,j)\}_{i+j\le n_u^B}$, and $\{a_d(i,j)^B\}_{i+j\le n_d^B}$ denote the expansion coefficients. The total number of coefficient is

$$\frac{(n_u^A+1)(n_u^A+2)}{2}+\frac{(n_d^A+1)(n_d^A+2)}{2}+\frac{(n_u^B+1)(n_u^B+2)}{2}+\frac{(n_d^B+1)(n_d^B+2)}{2}.$$

In terms of these state variables, the boundary conditions now hold for $z=-1$ (zero wealth) and $z=+1$ (infinite wealth). Furthermore, the boundary conditions are actually linear in terms of the transformed state variables, so for the purpose of estimating the coefficient parameters, they can be implemented as linear constraints on the coefficients in the Chebyshev polynomials. Appendix E explains in detail how the boundary conditions are implemented.

To capture the nonlinearities in the demand functions and the interactions between the two state variables, we found it necessary to use high-order polynomials. Let n_u^A, n_u^B, n_d^A, n_d^B denote the total orders (maximum sum of powers of the two state variables) of the polynomials in the numerators (subscript u) and denominators (subscript d) of the estimated equilibrium demand functions X^A and X^B, respectively. Then the total number of coefficient parameters needed to specify both demand functions is

$$\frac{(n_u^A+1)(n_u^A+2)}{2}+\frac{(n_d^A+1)(n_d^A+2)}{2}+\frac{(n_u^B+1)(n_u^B+2)}{2}+\frac{(n_d^B+1)(n_d^B+2)}{2}.$$

The boundary conditions, implemented as linear constraints on the coefficients, reduce the number of coefficient parameters by $2n_u^A+2n_d^A+2n_u^B+2n_d^B+4$. In our numerical implementation, we let the degree of both numerator and denominator in X^A be 12 and the degree of both numerator and denominator of X^B be 10. This results in 314 coefficients. The constraints implied by the boundary conditions reduce this number by 92. Thus, we estimate 222 coefficient parameters.

To estimate the coefficients, we minimize the sum of squared errors in the partial differential equations over a fixed set of test points. We noticed that the demand functions appear to have more curvature near the boundaries $z = +1$ and $z = -1$, so instead of using a uniformly spaced grid of test points in the transformed state variables, we increased the number of points near the boundaries. We chose a grid size of 21 (for variable y) by 100 (for variable z), so the partial differential equations are evaluated at 2,100 points. Since there are 222 parameters to estimate, the system is overdetermined by a factor of roughly 10.

On each test point, two types of error functions have been used at the same time. One is defined as

$$\text{Error1} = \sqrt{\left(\frac{X^A - X^{*A}}{\sigma_\theta}\right)^2 + \left(\frac{X^B - X^{*B}}{\sigma_\theta}\right)^2}, \tag{D12}$$

the difference between the given strategy $\{X^A, X^B\}$ and the optimal strategy $\{X^{*A}, X^{*B}\}$ normalized by the volatility of noise trading and the total supply of asset B. Since the magnitude of demand is very small when convergence traders' capital is small, this way of calculating error underestimates the percentage errors to arbitrageurs' portfolio over the region where the arbitrageurs' wealth is small. The other error function is defined as

$$\text{Error2} = \frac{\sigma_F\sqrt{(X^A - X^{*A})^2 + (X^B - X^{*B})^2}}{W}, \tag{D13}$$

the difference between the percentage wealth volatility caused by the fundamental shocks using $\{X^A, X^B\}$ and $\{X^{*A}, X^{*B}\}$. Since this error function is defined by the percentage of wealth, it can correctly estimate errors over the region where arbitrageurs' wealth is small, but it may underestimate the absolute errors to market clearing condition over the region where arbitrageurs' wealth is large (where the demands are small relative to arbitrageurs' wealth W). To give precise estimates of the numerical errors over all regions, a combination of these two types of errors is used:

$$\text{Error} = \sqrt{\text{Error1}^2 + \text{Error2}^2}. \tag{D14}$$

To solve the minimization problem, a Levenberg–Marquart algorithm is used. Despite the use of Chebyshev polynomials, the Hessian in our problem is not well behaved because of the linear constraints. Therefore, a gradient method has the potential to work better than Newton's method. The Levenberg–Marquart algorithm is designed to adjust smoothly between these two methods and thus deals with this problem.

We also notice that as the degree of the polynomial expansion goes up, estimation works better if X^A is estimated holding X^B fixed and vice versa. We therefore estimate each demand function separately (holding the other fixed). From our experience, the algorithm converges to stable estimates for both demand functions after a few iterations. There seems to be a collinearity problem between X^A and X^B that this approach deals with effectively.

For the example described below, the numerical error of the fixed-point problem is about 10^{-3}, which is interpreted as the maximum difference between the given strategies of convergence traders X^A, X^B and optimal strategies X^{*A}, X^{*B}. We take this as an indication that our numerical algorithm has found an equilibrium.

Appendix E. Boundary Constraints

The boundary conditions are linear to the after-transformation state variables y:

$$X^A(y,1) = \bar{\theta} + \frac{4\sigma_\theta}{\sqrt{2\lambda_\theta}} T_1(y), \tag{E1}$$

$$X^B(y,1) = \bar{\theta}, \tag{E2}$$

$$X^A(y,-1) = 0, \tag{E3}$$

$$X^B(y,-1) = 0. \tag{E4}$$

Due to the properties of Chebyshev polynomials, we have $T_j(1) = 1$ and $T_j(-1) = (-1)^j$. The functions of X^A and X^B in equations (D10) and (D11) become an expansion of y only when $z = 1$ or $z = -1$. To match the coefficient of y on the two bounds with the boundary conditions (E1) through (E4), we obtain a series of constraints on expansion coefficients. These linear constraints can be implemented by determining the first two columns of the expansion coefficients from the rest of the columns of those expansion coefficients. For the sake of space, these constraints are not listed here. In this way, the total number of parameters is reduced by $2n_u^A + 2n_d^A + 2n_u^B + 2n_d^B + 4$ (first two columns of these four coefficient matrices). Therefore, the total number of parameters needed to specify both demand functions X^A and X^B is

$$\frac{n_u^A(n_u^A - 1)}{2} + \frac{n_d^A(n_d^A - 1)}{2} + \frac{n_u^B(n_u^B - 1)}{2} + \frac{n_d^B(n_d^B - 1)}{2}. \tag{E5}$$

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