Liquidity and Short-term Debt Crises*

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Abstract

We examine the role of deteriorating market liquidity in exacerbating debt crises. We extend Leland's structural credit risk model with two realistic features: illiquid secondary bond markets and a mix of short-term and long-term bonds in a firm's debt structure. As deteriorating market liquidity pushes down bond prices, it amplifies the conflict of interest between the debt and equity holders because, to avoid bankruptcy, the equity holders have to absorb all of the short-fall from rolling over maturing bonds at the reduced market values. As a result, the equity holders choose to default at a higher fundamental threshold even if there is no friction for firms to raise more equity. A greater fraction of short-term debt further exacerbates the debt crisis by forcing the equity holders to realize the rollover loss at a higher frequency. Our model illustrates the financial instability brought by overnight repos, an extreme form of short-term financing, to many financial firms, and provides a new explanation to the widely observed flight-to-quality phenomenon. We also examine a tradeoff between short-term debt's cheaper financing cost and higher future bankruptcy cost in determining firms' optimal debt maturity structure and liquidity management strategy.

Keywords: Rollover risk, Credit Risk, Debt Maturity Structure, Liquidity Management, Flight to Quality

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1 Introduction

The recent debt crisis on Wall Street illustrates an intertwined relationship between market liquidity and financial firms' credit risk. On one hand, as financial firms' credit risk worsens, they are reluctant/unable to provide market liquidity; on the other, as market liquidity deteriorates, financial firms' debt crisis also intensifies.[1] Understanding this intertwined relationship is probably one of the most important research topics confronting the finance literature. This paper aims to analyze the second part of this relationship—how does deteriorating market liquidity intensify a debt crisis?

The extant literature has proposed several mechanisms to this question. Market illiquidity, together with the difficulty of creditors in coordinating their rollover decisions of a firm's short-term debt, could lead to runs on financial firms, e.g., He and Xiong (2009) and Morris and Shin (2009). Deteriorating liquidity and rising volatility also motivate creditors to increase the required margins on their collateralized loans to the firms, which in turn could force the firms to liquidate their positions in illiquid markets, e.g., Brunnermeier and Pedersen (2009), Acharya, Gale, and Yorulmzer (2009), and Shleifer and Vishny (2009). These mechanisms all rely on an implicit assumption that firms are constrained from raising more equity during financial distresses. However, this assumption seems at odds with the observation that in the current crisis many financial firms paid a substantial amount of dividends despite their financial distresses and the angry creditors.[2] This indicates an intricate interaction between debt and equity holders, which is missing from the aforementioned theories.

We provide a theoretical model to explicitly analyze the conflict of interest between debt and equity holders in debt crises. We show that even in the absence of any constraint on raising more equity, deteriorating market liquidity could exacerbate the conflict and lead equity holders to choose default at a higher fundamental threshold.

Specifically, we build on the structural credit model of Leland (1994, 1998) and Leland and Toft (1996). We extend the Leland framework with two realistic features. First, bond holders are subject to random liquidity shocks. Upon the arrival of a liquidity shock, they have to sell their bond holdings at a cost. This trading cost represents illiquidity of the bond markets, and can be broadly interpreted either as market impact of trade, e.g., Kyle (1985),

[1]See Brunnermeier (2009), Diamond and Rajan (2009), Gorton (2009) and Krishnamurthy (2009) for comprehensive descriptions of the recent financial crisis.

[2]See Scharfstein and Stein (2008) for a discussion about the financial firms' dividend payout during the crisis.

or as bid-ask spread, e.g., Amihud and Mendelson (1986). Second, the firm uses a mix of short-term and long-term debt, in addition to equity, to finance its operation. The firm repays maturing bonds by issuing new bonds with identical maturity, principal value, coupon rate and seniority at the market prices. When the bond prices fall, the equity holders have to absorb the short-fall from rolling over maturing bonds to avoid bankruptcy. Otherwise, the firm will be liquidated at a firesale price to pay off the debt holders.

A key result of our model is that, even if there is no friction for the firm to raise more equity, as deteriorating bond market liquidity pushes down the firm's bond prices, equity value would become zero and equity holders would choose to default when the loss from rolling over maturing bonds becomes sufficiently high. The reason lies with the standard conflict of interest between debt and equity holders. The equity holders have to bear all the rollover loss to avoid bankruptcy, while debt holders get paid in full. This unequal sharing of losses makes the equity value sensitive to the drop in bond prices and ultimately causes the equity holders to trigger costly bankruptcy at a high fundamental threshold. This default mechanism, similar in spirit to the debt overhang problem suggested by Myers (1977), highlights the intrinsic conflict of interest between debt and equity holders in debt crises.

This endogenous default problem becomes even more severe when the maturity of the firm's short-term debt becomes shorter. As we observed in the current crisis, financial firms increasingly rely on overnight repos, an extreme form of short-term financing with a maturity of one day, to fund their investment positions. Right before the bankruptcy of Lehman Brothers, it had to roll over 25% of its debt every day through overnight repos. At such a rapid rollover frequency, our model shows that the equity holders' financial burden becomes highly sensitive to bond market liquidity, and that the firm could choose to default even when the firm fundamental is still solvent. Our model thus calls for more attention on firms' maturity structure in assessing their default risk, in addition to their high leverage, whose role in the ongoing debt crisis is highlighted by Adrian and Shin (2009), Brunnermeier and Pedersen (2009), and Geanakoplos (2009).

Our model provides an interesting implication about the impact of a market liquidity breakdown on different firms. It is intuitive that the liquidity breakdown, by pushing down bond prices and raising firms' endogenous default thresholds, has a greater impact on the credit spreads and default probabilities of firms with weaker fundamentals. This implication provides a new explanation to the widely observed flight-to-quality phenomenon: after major market liquidity disruptions, the prices of low quality bonds drop much more than those of high quality bonds. The bond market fluctuation during the ongoing financial crisis provides a nice illustration of this phenomenon. According to a BIS report by Fender, Ho and Hordahl (2009), in a two-month period around the bankruptcy of Lehman Brothers in September 2008, the US five-year CDX high yield index spread shot up from around 700 basis points to over 1500, while the increase in the corresponding investment grade index spread was more modest. Different from the existing explanations of this phenomenon based on the changes in investors' investment constraints and preferences, e.g., Vayanos (2004) and Caballero and Krishnamurthy (2008), our model predicts that a surge in investor demand for market liquidity not only leads to a higher liquidity premium in bond prices, but also higher bond default probabilities. This additional prediction is consistent with the quickly rising default rates of speculative-grade bonds from the very low levels (around 1%) in early 2008 to near 5% in March 2009.

Our model also provides an insight on firms' optimal maturity structure, based on two opposing forces. On one hand, it is cheaper for firms to issue short-term debt, because short-term debt tends to be more liquid, e.g., Bao, Pan, and Wang (2009), and thus has a lower liquidity premium. On the other hand, the higher rollover frequency of short-term debt imposes a heavier financial burden on the firm if bond prices fall and thus makes future bankruptcy more likely. By trading off the short-term debt's cheaper financing cost and higher expected bankruptcy cost, our model suggests that firms with lower asset volatility, higher bankruptcy recovery rates, and higher secondary market debt liquidity tend to use a greater fraction of short-term debt. Our focus on market liquidity and future financial stability is different from the existing theories of optimal debt maturity based on the disciplinary role of short-term debt in preventing managers' asset substitution, e.g., Flannery (1994) and Leland (1998), and the theories based on private information of borrowers about their future credit ratings, e.g., Flannery (1986) and Diamond (1991).

Our analysis shows that debt maturity structure should be used as part of a firm's liquidity management strategy. Despite its higher cost, long-term debt gives the firm more flexibility to delay realizing financial losses in adverse states, either when the firm's fundamental or market liquidity deteriorates. This benefit is analogous to the role of cash reserves, the standard tool for risk management, e.g., Holmstrom and Tirole (2001) and Bolton, Chen, and Wang (2009). This implication of our model also echoes a related point made by Brunnermeier and Yogo (2009).

Our model is related to the credit risk literature, e.g., Collin-Dufresne, Goldstein, and Martin (2001), Huang and Huang (2003), Longstaff, Mithal, and Neis (2005), Ericsson and Renault (2006) and Chen, Lesmond, and Wei (2007). These studies provide evidence for liquidity as an important factor in firms' credit spreads. Our model adds to their results by showing that a higher trading cost not only leads to a higher liquidity premium, but also a higher default probability through the endogenous default channel.

The paper is organized as follows. Section 2 presents the model setting. We derive the debt and equity valuation and the firm's endogenous bankruptcy boundary in Section 3. Section 4 discusses the implications of the model for debt crises. We analyze firms' optimal maturity structure in Section 5. Section 6 concludes the paper.

2 The Model

We extend the structural credit risk model of Leland (1994, 1998) and Leland and Toft (1996) with two realistic features. First, the bond markets are illiquid. When a bond holder suffers a liquidity shock, he has to sell his bond position at a proportional trading cost. Second, a firm uses a mix of short-term and long-term bonds, in addition to equity, to finance its operation. The setting of our model is generic and applies to both financial and non-financial firms, although the effects illustrated by our model are stronger for financial firms because they tend to use higher leverage and shorter debt maturity.

2.1 Firm Assets

The unlevered firm asset value $\{V_t\}$ follows a geometric Brownian motion in the risk-neutral probability measure:

$$\frac{dV_t}{V_t} = (r - \phi)\, dt + \sigma dZ_t. \tag{1}$$

where r is the risk-free rate in this economy, ϕ is the firm's cash payout rate, σ is the asset volatility, and $\{Z_t\}$ is a standard Brownian motion. Throughout the paper, we refer to V_t as the firm fundamental.

When the firm bankrupts, we assume that creditors can only recover α fraction of the firm's asset value from liquidation. The liquidation loss $1 - \alpha$ can be interpreted in different ways, such as the loss from selling the firm's real asset to second best users, loss of customers because of the bankruptcy, asset firesale, legal fees, etc. An important issue to keep in mind is that the liquidation loss represents a dead weight loss of bankruptcy ex ante to both debt and equity holders, but ex post is borne only by the debt holders.

2.2 Stationary Debt Structure

The firm maintains two classes of debts with maturities m_1 and m_2, respectively. Without loss of generality, we let class-1 debt to have a shorter maturity, i.e., $m_1 < m_2$. Each class of debt is the one studied in Leland and Toft (1996). At each moment in time, the i^{th} class debt has a constant principal P_i outstanding and a constant annual coupon payment of C_i. The expiration of each class of debt is uniformly spread out across time. That is, during a time interval $(t, t+dt)$, $\frac{1}{m_i}dt$ fraction of class-i debt matures and needs to be rolled over. Given the shorter maturity of class-1 debt, it has to be rolled over at a higher frequency $1/m_1$.

To focus on the firm's debt maturity structure and liquidity effects, we take the firm's total debt principal $P = \sum_i P_i$ and total coupon payment $C = \sum_i C_i$ as given. By taking the leverage level as given, we ignore many interesting issues related to the tradeoff between tax benefits and bankruptcy costs, which is analyzed by Leland (1994) and other following work such as Goldstein, Ju, and Leland (2001), Strebulaev (2007), and He (2009). For simplicity, we also assume that the principals and coupon payments of the two debt classes are in proportion:

$$C_i = \lambda_i C,\ \ P_i = \lambda_i P, \tag{2}$$

where λ_i represents the fraction of the i^{th} class debt, with $\lambda_1 + \lambda_2 = 1$.

Furthermore, following the Leland framework, we assume that the firm can commit to a stationary debt structure denoted by $(C, P, m_1, m_2, \lambda_1, \lambda_2)$. That is, the firm always maintains the initially specified debt level represented by C and P and the maturity structure specified by $(m_1, m_2, \lambda_1, \lambda_2)$. Thus, when a bond matures, the firm replaces it by issuing a new bond with identical maturity, principal value, coupon rate and seniority.

For the i^{th} class debt, there is a continuum of bonds with the remaining time-to-maturity ranging from 0 to m_i. We measure these bonds by m_i units. Then each unit has a principal value of

$$p_i = \frac{P_i}{m_i} = \frac{\lambda_i}{m_i} P, \tag{3}$$

and an annual coupon payment of

$$c_i = \frac{C_i}{m_i} = \frac{\lambda_i}{m_i} C. \tag{4}$$

These bonds only differ in the time-to-maturity $\tau \in [0, m_i]$.

The two classes of debts have the same priority in dividing the firm's asset during bankruptcy, i.e., the firm's liquidation value is divided among all debt holders on a pro rata basis.

This assumption simplifies a complication in reality that long-term bonds are often secured by firm assets. We believe this simplification is innocuous to our main results.

2.3 Debt Rollover and Endogenous Bankruptcy

When the firm pays off maturing bonds by issuing new bonds, the fluctuation in bond prices could generate a rollover gain/loss, which needs to be absorbed by the equity holders. Specifically, over a short time interval $(t, t+dt)$, the net cash flow to the equity holders (omitting dt) is

$$NC_t = \phi V_t - (1-\tau_c) C + \sum_{i=1}^{2} [d_i (V_t, m_i) - p_i] . \tag{5}$$

The first term is the firm's dividend payout. The second term is the after-tax coupon payment, where τ_c denotes the marginal corporate tax rate. The third term captures the rollover gain/loss when the firm pays off maturing bonds by issuing new bonds at market prices. In this transaction, dt units of both class-1 and class-2 bonds mature. The maturing class-i bonds have a principal value of $p_i dt$. We denote the market value of the newly issued bonds with identical principal value and maturity m_i as $d_i (V_t, m_i) dt$, which depends on the firm fundamental V_t and bond maturity m_i. When the bond value $d_i (V_t, m_i) dt$ drops, the equity holders have to absorb the rollover loss $\sum_i [d_i (V_t, m_i) - p_i] dt$ to prevent the firm from bankruptcy.[3]

As in the Leland framework, we assume that the equity market is functional and liquid, i.e., the firm can freely raise more equity to pay for the rollover loss and the current coupon payments, as long as the equity value remains positive. In other words, equity holders have the option to keep servicing the debt (coupons and principals) in order to maintain the right to collect the future cash flows generated by the firm. Bankruptcy occurs endogenously when the firm fundamental drops to a certain threshold V_B so that the equity value becomes zero. At this point, equity holders are no longer willing to inject more capital to meet the coupon and principal payments, and the firm is bankrupt. When the firm is bankrupt, equity holders walk away, while the short-term and long-term bond holders divide the firm liquidation value αV_B on a pro rata basis.

Under the stationary debt structure specified earlier, the firm's bankruptcy boundary V_B is constant. We derive V_B in the next section based on a smooth pasting condition regarding

[3]Following the Leland framework, we assume that the firm cannot reduce the rollover loss by increasing the coupon payments of the newly issued bonds. Such an increase raises the firm's leverage, and hurts the existing debt holders. We also ignore renegotiation between debt and equity holders, which is analyzed in Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997).

the firm's equity value at the boundary. As in any trade-off theory, bankruptcy involves a dead-weight loss. The endogenous bankruptcy is a reflection of the debt-overhang problem originated from the conflict of interest between the debt and equity holders: when the bond prices are low, the equity holders are not willing to bear all of the rollover loss to avoid the deadweight loss in bankruptcy. This situation resembles the debt-overhang problem suggested by Myers (1977).

2.4 The Secondary Bond Markets

We assume that each bond holder is subject to a random liquidity shock, which arrives according to a Poisson process with intensity ξ. Upon the arrival of the liquidity shock, the bond holder has to exit by selling his bond holding in the secondary market. The liquidity shocks are independent across investors.

As documented by a series of empirical papers, e.g., Bessembinder, Maxwell, and Venkataraman (2006), Edwards, Harris, and Piwowar (2007), Mahanti et al (2008), and Bao, Pan, and Wang (2009), the secondary markets for corporate bonds are highly illiquid. The illiquidity is reflected by a large bid-ask spread that bond investors have to pay in trading with dealers, as well as a potential price impact of the trade. Edwards, Harris, and Piwowar (2007) show that the average effective bid-ask spread on corporate bonds ranges from 8 basis points for large trades to 150 basis points for small trades. Bao, Pan, and Wang (2009) estimate that the average effective trading cost, which incorporates bid-ask spread, price impact and other factors, ranges from 74 to 221 basis points depending on the trade size. There is also large variation across different bonds with the same trade size. In particular, Mahanti et al (2008) and Bao, Pan, and Wang (2009) document an increasing pattern of the net trading cost with respect to bond maturity (the sum of bond age and time-to-maturity). This result suggests that short-term debt is more liquid than long-term debt.[4] Furthermore, the bonds analyzed in their sample have a turnover rate of about once a year.

Motivated by these observations, we assume that when an investor sells a class-i bond in the secondary market, he only recovers a fraction $(1-\beta_i)$ of the bond value. The other fraction β_i represents the net trading cost. We shall broadly interpret this cost either as market impact of trade, e.g., Kyle (1985), or as bid-ask spread, e.g., Amihud and Mendelson (1986). Since short-term debt is more liquid, we impose that $\beta_1 < \beta_2$.

[4] Intuitively, the default probabilities of long-term bonds are usually higher than those of short-term bonds. As a result, there is a more uncertainty in valuing long-term bonds, which in turn makes long-term bonds less liquid.

The bond issuance cost in the primary markets tends to be much lower than the trading cost in the secondary markets. Issuing commercial papers through dealers usually costs about 5 basis points.[5] While the average cost of raising capital through long-term debt is about 220 basis points, e.g., Lee, Lochhead, and Ritter (1996), the effective cost when spread out across the debt maturity, which is typically 5-10 years, is still low relative to the secondary market trading cost. Thus, we ignore the issuance cost in the model.

In summary, our model captures the liquidity of the bond markets using three model parameters: ξ, which represents the demand of bond investors for market liquidity, and β_1 and β_2, which represent the illiquidity of the short-term and long-term bonds. In our later analysis, we will use an unexpected rise in the value of ξ to proxy for deteriorating market liquidity as it causes the liquidity premium to surge. We will focus on the impact of this surge on the firm's bankruptcy boundary and credit spreads. We will also use unexpected rises in the values of β_1 and β_2 to proxy for liquidity shocks to sepecific segments of the bond markets and analyze the spillover effects to other segments.

3 Valuation and Bankruptcy Boundary

In this section, we derive the debt and equity valuation and the firm's endogenous bankruptcy boundary.

3.1 Debt Value

We first derive the debt valuation by taking the firm's bankruptcy boundary V_B as given. Similar derivations can be found in Leland and Toft (1996). Denote $d_i\left(V,\tau;V_B\right)$ as the value of one unit of class-i bond with time-to-maturity $\tau < m_i$. We have the following partial differential equation for the bond value $d_i\left(V,\tau;V_B\right)$:

$$rd_i = \frac{\partial d_i}{\partial \tau} + \left(r-\phi\right)V\frac{\partial d_i}{\partial V} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 d_i}{\partial V^2} + c_i - \xi\beta_i d_i. \tag{6}$$

The left-hand side is the required return for the bond. This term should be equal to the expected increment in the bond value, which is the sum of the terms on the right-hand side.

- The first three terms $\frac{\partial d_i}{\partial \tau} + \left(r-\phi\right)V\frac{\partial d_i}{\partial V} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 d_i}{\partial V^2}$ capture the expected change in the bond value caused by the change in the time-to-maturity, and the fluctuation in the firm's asset value V_t.

[5] See the Wikipedia website at http://en.wikipedia.org/wiki/Commercial_paper for more background information on commercial paper.

- The fourth term c_i is the coupon payment per unit of time.
- The fifth term gives the bond holders' value change due to the liquidity shocks. During a time interval $(t, t+dt)$, with probability ξdt, a bond holder suffers a liquidity shock and has to sell the bond in the market for a cost of $d_i - (1-\beta_i)\, d_i = \beta_i d_i$.

We have two boundary conditions. At the bankruptcy boundary V_B, the bond holders share the firm's liquidation value proportionally. Thus, each unit of class-i bond gets

$$d_i\left(V_B, \tau; V_B\right) = \frac{P_i/m_i}{P}\alpha V_B = \frac{1}{m_i}\lambda_i \alpha V_B, \text{ for all } \tau \in [0, m_i]\,. \tag{7}$$

When $\tau = 0$, the bond matures and the bond holder gets the principal value p_i:

$$d_i\left(V, 0; V_B\right) = p_i, \text{ for all } V > V_B. \tag{8}$$

Individual Bond Value Similar to Leland and Toft (1996), we solve the individual bond value $d_i\left(V, \tau; V_B\right)$ based on equation (6) and the boundary conditions (7) and (8). Define the effective discount rate for this bond as

$$r_i \equiv r + \xi \beta_i,$$

and let

$$v \equiv \ln\left(\frac{V}{V_B}\right).$$

We have

$$d_i\left(V, \tau; V_B\right) = \frac{c_i}{r_i} + e^{-r_i \tau}\left[p_i - \frac{c_i}{r_i}\right]\left(1 - F\left(\tau\right)\right) + \left[\frac{1}{m_i}\lambda_i \alpha V_B - \frac{c_i}{r_i}\right] G_i\left(\tau\right), \tag{9}$$

where

$$\begin{aligned}
F\left(\tau\right) &= N\left(h_1\left(\tau\right)\right) + \left(\frac{V}{V_B}\right)^{-2a} N\left(h_2\left(\tau\right)\right); \qquad (10)\\
G_i\left(\tau\right) &= \left(\frac{V}{V_B}\right)^{-a+z_i} N\left(q_{1i}\left(\tau\right)\right) + \left(\frac{V}{V_B}\right)^{-a-z_i} N\left(q_{2i}\left(\tau\right)\right);\\
h_1\left(\tau\right) &= \frac{\left(-v - a\sigma^2\tau\right)}{\sigma\sqrt{\tau}};\, h_2\left(\tau\right) = \frac{\left(-v + a\sigma^2\tau\right)}{\sigma\sqrt{\tau}};\\
q_{1i}\left(\tau\right) &= \frac{\left(-v - z_i\sigma^2\tau\right)}{\sigma\sqrt{\tau}};\, q_{2i}\left(\tau\right) = \frac{\left(-v + z_i\sigma^2\tau\right)}{\sigma\sqrt{\tau}};\\
a &= \frac{r - \phi - \sigma^2/2}{\sigma^2};\, z_i = \frac{\left[a^2\sigma^4 + 2r_i\sigma^2\right]^{1/2}}{\sigma^2};
\end{aligned}$$

and $N\left(x\right) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$ is the cumulative standard normal distribution.

Bond Yield The bond yield is typically computed as the equivalent return on a bond conditional on it being held to maturity without default and trading. Given the bond price derived in equation (9), the bond yield y_i is given by the following equation:

$$d_i\left(V_t, m_i\right) = \frac{c_i}{y_i}\left(1 - e^{-y_i m_i}\right) + p_i e^{-y_i m_i} \tag{11}$$

where the right hand side is the price of a bond with a constant cash flows c_i over time $t \in [0, m_i]$ and a principal payment p_i at maturity $t = m_i$, conditional on that there are no default and liquidity shocks. The spread between y_i and the risk-free rate is often called the credit spread of the bond. Since the bond price in equation (9) contains the effects of trading cost and bankruptcy cost, the credit yield contains a liquidity premium and a default premium. The focus of our analysis is to uncover the intricate interaction between the liquidity and default premia.

It is easier to see the liquidity premium when there is no default risk. Consider a risk-free bond with coupon rate c_i, principal value p_i, and maturity of m_i. Anticipating their own future liquidity shocks and the cost in trading the bond, investors would value this risk-free bond at[6]

$$d_i^{riskfree}\left(m_i\right) = \frac{c_i}{r_i}\left(1 - e^{-(r+\xi\beta_i)m_i}\right) + p_i e^{-(r+\xi\beta_i)m_i}.$$

Based on equation (11), the yield of this risk-free bond is

$$y_i^{riskfree} = r_i = r + \xi\beta_i.$$

Consistent with Amihud and Mendelson (1986), this bond yield contains a liquidity premium determined by the arrival rate of investors' future liquidity shocks ξ multiplied by the trading cost β_i. This liquidity premium is consistent with the empirical findings of Longstaff, Mithal, and Neis (2005) and Chen, Lesmond, and Wei (2007) that less liquid bonds tend to have higher credit spreads. This also suggests that the liquidity premium in long-term debt is higher than that in short-term debt because it is more costly to trade long-term debt ($\beta_2 > \beta_1$). The cheaper cost of short-term debt is a key factor in our analysis of the firm's optimal debt maturity structure in Section 5.

[6] Specifically, the value of a risk-free bond with a time-to-maturity τ satisfies

$$r d_i^{riskfree}(\tau) = \frac{\partial d_i^{riskfree}(\tau)}{\partial \tau} + c_i - \xi\beta_i d_i^{riskfree}(\tau)$$

with boundary condition $d_i^{riskfree}(0) = p_i$. Therefore (as $r_i = r + \xi\beta_i$) $d_i^{riskfree}(\tau) = \frac{c_i}{r_i}\left(1 - e^{-(r+\xi\beta_i)\tau}\right) + p_i e^{-(r+\xi\beta_i)\tau}$.

Total Debt Value With the value of individual bonds, we can calculate the total value of all outstanding bonds in class i as

$$
\begin{aligned}
D_i(V;V_B) &= \int_0^{m_i} d_i(V,\tau;V_B)\,d\tau \\
&= \frac{C_i}{r_i} + \left[P_i - \frac{C_i}{r_i}\right]\left[\frac{1-e^{-r_i m_i}}{r_i m_i} - I_i(m_i)\right] + \left[\lambda_i \alpha V_B - \frac{C_i}{r_i}\right] J_i(m_i),
\end{aligned}
$$

where

$$
\begin{aligned}
I_i(m_i) &= \frac{1}{r_i m_i}\left[G_i(m_i) - e^{-r_i m_i}F(m_i)\right]; \\
J_i(m_i) &= \frac{1}{z_i \sigma \sqrt{m_i}}\left[-\left(\frac{V}{V_B}\right)^{-a+z_i} N(q_{1i}(m_i))\, q_{1i}(m_i) + \left(\frac{V}{V_B}\right)^{-a-z_i} N(q_{2i}(m_i))\, q_{2i}(m_i)\right].
\end{aligned}
$$

3.2 Equity Value and Endogenous Bankruptcy Boundary V_B

Leland (1994, 1998) and Leland and Toft (1996) indirectly derive the equity value as the difference between the total firm value and the debt value. The total firm value is the unlevered firm value V_t, plus the total tax-benefit, minus the bankruptcy cost. This approach does not apply to our model because part of the total firm value is consumed by future trading costs. Thus, we directly compute the equity value $E(V_t)$ through the following differential equation:

$$
rE = (r-\phi)VE_V + \frac{1}{2}\sigma^2 V^2 E_{VV} + \phi V - (1-\tau_c)C + \sum\nolimits_i \left[d_i(V,m_i) - p_i\right]. \tag{12}
$$

The left-hand side is the required return for the equity. This term should be equal to the expected increment in the equity value, which is the sum of the terms on the right-hand side.

- The first two terms $(r-\phi)VE_V + \frac{1}{2}\sigma^2 V^2 E_{VV}$ capture the expected change in the equity value caused by the fluctuation in the firm's asset value V_t.
- The third term ϕV is the dividend flow generated by the firm per unit of time.
- The fourth term $(1-\tau_c)C$ is the after-tax coupon payment per unit of time.
- The fifth term $\sum_i [d_i(V,m_i) - p_i]$ gives the equity holders' rollover gain/loss from paying off the maturing bonds by issuing new bonds at the market values.

Limited liability of equity holders provides the following boundary condition at V_B:

$$E(V_B) = 0. \tag{13}$$

Solving the differential equation in (12) is challenging because it contains the complicated bond's valuation function $d_i(V, m_i)$ given in (9). We manage to solve this equation using the Laplace transformation technique detailed in the Appendix. Based on the equity value, we then derive the equity holders' endogenous bankruptcy boundary V_B based on the smooth pasting condition that

$$E'(V_B) = 0.$$

The equity value and the endogenous bankruptcy boundary are given in the following proposition.

Proposition 1 *Let $v \equiv \ln(V/V_B)$. The equity value $E(V)$ is*

$$\begin{aligned} & E(V) \qquad (14) \\ = \; & V - \frac{\phi V_B}{z\sigma^2}\frac{e^{-\gamma v}}{\gamma+1} + \frac{(1-\tau_c)C + \sum_{i=1}^{2}\left(1-e^{-r_i m_i}\right)\left[\frac{P_i}{m_i} - \frac{C_i}{r_i m_i}\right]}{z\sigma^2}\left[-\frac{1}{\eta} - \frac{1}{\gamma}\left(1-e^{-\gamma v}\right)\right] \\ & + \sum_{i=1}^{2}\left\{ \begin{array}{c} \frac{e^{-r_i m_i}\left(\frac{P_i}{m_i} - \frac{C_i}{r_i m_i}\right)}{z\sigma^2}\left[\frac{K_i(v;a,a,\gamma)+k_i(v;a,-a,-\eta)}{\eta} + \frac{K_i(v;a,-a,\gamma)+k_i(v;a,a,-\eta)}{\gamma}\right] \\ -\frac{\frac{1}{m_i}\lambda_i\alpha V_B - \frac{C_i}{r_i m_i}}{z\sigma^2}\left[\frac{k_i(v;a,-z_i,-\eta)}{a-z_i+\eta} + \frac{K_i(v;a,-z_i,\gamma)}{\gamma-a+z_i} + \frac{k_i(v;a,z_i,-\eta)}{a+z_i+\eta} + \frac{K_i(v;a,z_i,\gamma)}{\gamma-a-z_i}\right] \end{array} \right\}, \end{aligned}$$

where $a \equiv \frac{r-\phi-\sigma^2/2}{\sigma^2}$, $z \equiv \frac{\left[a^2\sigma^4+2r\sigma^2\right]^{1/2}}{\sigma^2}$, $\gamma \equiv a + z > 0$, $\eta \equiv -a + z > 1$, *and*

$$\begin{aligned} & K_i(v; x, w, \gamma) \\ \equiv \; & \left\{N\left(w\sigma\sqrt{m_i}\right) - e^{\frac{1}{2}\left[(\gamma-x)^2-w^2\right]\sigma^2 m_i} N\left((\gamma-x)\sigma\sqrt{m_i}\right)\right\} e^{-\gamma v} \\ & + e^{\frac{1}{2}\left[(\gamma-x)^2-w^2\right]\sigma^2 m_i} e^{-\gamma v} N\left(\frac{-v+(\gamma-x)\sigma^2 m_i}{\sigma\sqrt{m_i}}\right) - e^{-(x+w)v} N\left(\frac{-v+w\sigma^2 m_i}{\sigma\sqrt{m_i}}\right), \end{aligned}$$

and

$$\begin{aligned} & k_i(v; x, w, -\eta) \\ \equiv \; & e^{\frac{1}{2}\left[(-\eta-x)^2-w^2\right]\sigma^2 m_i} e^{\eta v} N\left(\frac{-v+(-\eta-x)\sigma^2 m_i}{\sigma\sqrt{m_i}}\right) - e^{-(x+w)v} N\left(\frac{-v+w\sigma^2 m_i}{\sigma\sqrt{m_i}}\right). \end{aligned}$$

The endogenous bankruptcy boundary V_B is given by

$$V_B = \frac{\frac{(1-\tau_c)C + \sum_{i=1}^{2}\left(1-e^{-r_i m_i}\right)\left[\frac{P_i}{m_i} - \frac{C_i}{r_i m_i}\right]}{\eta} + \sum_{i=1}^{2}\left\{ \begin{array}{c} \left(\frac{P_i}{m_i} - \frac{C_i}{r_i m_i}\right)\left[\frac{1}{\eta}b_i(-a) + \frac{1}{\gamma}b_i(a)\right] \\ + \frac{C_i}{r_i m_i}\left[B_i(-z_i) + B_i(z_i)\right] \end{array} \right\}}{\frac{\phi}{\eta-1} + \sum_{i=1}^{2}\frac{1}{m_i}\lambda_i\alpha\left[B_i(-z_i) + B_i(z_i)\right].} \tag{15}$$

where

$$b_i(x) = e^{-r_i m_i}\left[N\left(x\sigma\sqrt{m_i}\right) - e^{rm_i}N\left(-z\sigma\sqrt{m_i}\right)\right],$$

$$B_i(x) = \frac{1}{a+x+\eta}\left[N\left(x\sigma\sqrt{m_i}\right) - e^{\frac{1}{2}\left[z^2-x^2\right]\sigma^2 m_i}N\left(-z\sigma\sqrt{m_i}\right)\right].$$

4 Liquidity Shocks and Debt Crises

We analyze the effects of deteriorating bond market liquidity on debt crises based on the model derived in the previous section. We first show that deteriorating liquidity can exacerbate the conflict of interest between debt and equity holders and lead to a debt crisis. We then discuss whether short-term debt can further amplify liquidity effects. Next, we discuss the flight-to-quality phenomenon caused by the different impacts of a market liquidity shock on different firms. Finally, we analyze the spillover of liquidity shocks across different market segments.

To facilitate our discussion, we use a set of baseline parameters:

$$\begin{aligned} r &= 10\%,\ \phi = 3\%,\ \alpha = 0.5,\ \tau_c = 35\%,\ \sigma = 7\%,\ V_0 = 100, \\ P &= 90,\ C = 9,\ m_1 = 0.25,\ m_2 = 5,\ \beta_1 = 0.1\%,\ \beta_2 = 2\%,\ \xi = 1. \end{aligned} \tag{16}$$

We choose the risk-free rate r to be 10%, the dividend payout rate ϕ to be 3%, the firm's liquidation recovery rate α to be 0.5, the tax rate τ_c to be 35%, and the asset volatility σ to be 7%. These values are fairly standard, except that σ is on the low end relative to the value used by Leland. We choose a small volatility because we intend to analyze a financial firm which uses high leverage to finance a relatively safe asset position. We let the initial value of the firm asset to be 100, the total principal value of the firm's debts P to be 90, and the total annual coupon payment C to be 9. These values imply that the firm's initial leverage is 73%. We choose the short-term debt maturity m_1 to be 3 months (commercial paper) and the long-term debt maturity m_2 to be 5 years (long-term corporate bond). We set their trading costs β_1 and β_2 to be 0.1% and 2%. These values are consistent with the empirical estimates of Bao, Pan, and Wang (2009). Finally, we let the arrival rate of the bond holders' liquidity shocks ξ to be 1, which is consistent with the average turnover rate of the corporate bonds in the data sample of Bao, Pan, and Wang (2009). The firm can choose its short-term debt fraction λ_1, which we will discuss in Section 5. For the analysis in this section, we treat λ_1 as exogenously given and with a value of 44.5%, which is the optimal value as we will show later. This value is close to the short-term debt fraction of a typical financial firm in the Compustat data base.[7] Under this set of parameters, the firm's optimal default boundary V_B is 88.27.

4.1 Market Liquidity and Endogenous Defaults

Deteriorating bond market liquidity can exacerbate the conflict of interest between debt and equity holders when the firm is in a financial distress. This is because as we see in Eq. (5) when deteriorating liquidity pushes the market prices of the firm's newly issued bonds to be below their principal values, equity holders have to absorb the rollover short-fall in paying off the maturing debts:

$$\sum_{i=1}^{2} \left(d_i \left(V_t, m_i; \xi \right) - p_i \right);$$

where we write d_i as a function of ξ to emphasize the dependence of rollover losses on the bond market liquidity. As a result, when the rollover short-fall becomes sufficiently large, the equity holders will choose to stop servicing the debts even if the falling bond prices are caused by liquidity reasons.

To illustrate the effects of a liquidity shock, we conduct the following thought experiment. Suppose that the arrival rate of liquidity shocks to the bond holders ξ has an unexpected change from its baseline value 1. One can broadly interpret the unexpected increase in ξ as a surge in the demand for liquidity after a major market disruption, such as the recent failure of Lehman Brothers or the crisis of LTCM. After the shock, investors will demand a higher liquidity premium and drive down the bond prices. To analyze the effects of the shock, we hold the firm's short-term debt fraction at the initial value. For example, bond covenants and other operational restrictions prevent firms in real life from swiftly modifying their debt structures in response to sudden market fluctuations.

Figure 1 illustrates the effects of a change in ξ on the equity holders' rollover loss and bankruptcy boundary. Panel A plots the rollover loss against ξ when $V = 97$. The line shows that the rollover loss increases with ξ. That is, as the arrival rate of bond holders' liquidity shocks increases, the increased liquidity premium in bond prices makes it more costly for the equity holders to roll over the maturing bonds. Consequently, Panel B shows that the firm's endogenous bankruptcy boundary V_B also increases with ξ. In other words, after a liquidity shock, the equity holders will choose to default at a higher fundamental threshold. We formally prove these results in the following proposition.

[7]For financial firms with CDS data in ***, the average fraction of short-term debt was 44% right before the failure of Bear Stearns in March 2009.

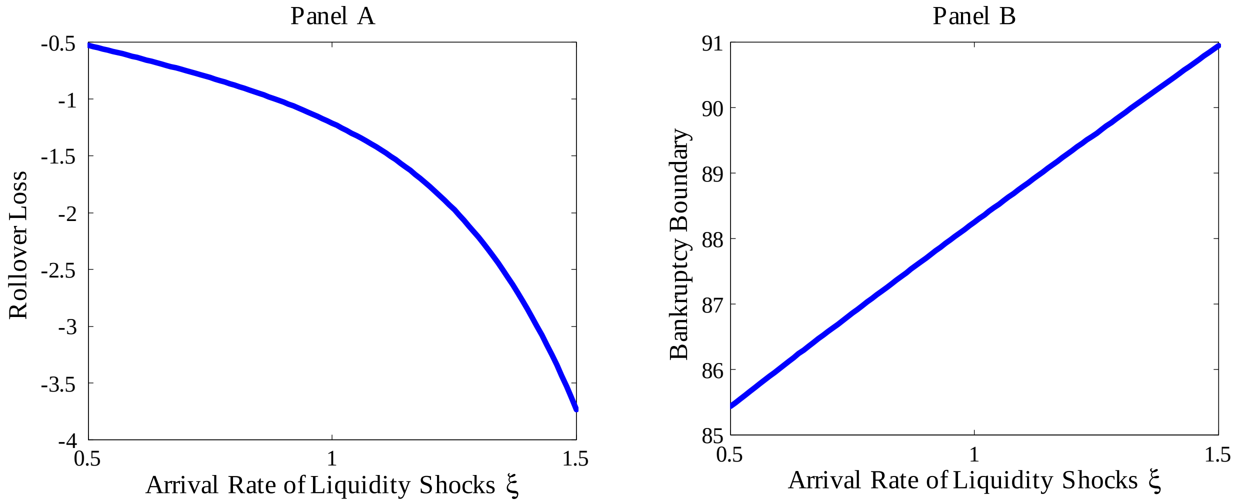

Figure 1: This figure shows the effects of a change in the arrival rate of liquidity shocks ξ on the firm's rollover loss and endogenous bankruptcy boundary, based on the baseline parameters given in (16) and $\lambda_1 = 44.5\%$. Panel A plots the firm's rollover loss against ξ when $V = 97$. Panel B plots the firm's endogenous bankruptcy boundary V_B against ξ.

Proposition 2 *All else equal, the bond prices d_i's decrease with the arrival rate of bond holders' liquidity shocks ξ. Consequently, equity holders' endogenous default boundary V_B increases with ξ.*

The firm's endogenous bankruptcy decision is rooted in the conflict of interest between the debt and equity holders. When the firm's bond values fall (even for liquidity reasons as we illustrated here), the equity holders have to bear all of the rollover losses to avoid bankruptcy, while the maturing debt holders get paid in full. This unequal sharing of losses causes the equity value to drop down to zero at V_B, at which point the equity holders stop servicing the debts. Could the debt and equity holders share the firm's losses, they would have avoided the social loss induced by bankruptcy.

The implication of Proposition 2 is consistent with the empirical findings of Collin-Dufresne, Goldstein, and Martin (2001). They find that proxies for both changes in the probability of future default based on standard fundamental-driven credit risk models and for changes in the recovery rate can only explain about 25% of the observed credit spread changes. On the other hand, they find that the residuals from these regressions are highly cross-correlated, and that over 75% of the variation in the residuals is due to the first principal component. While they cannot explain this systematic component, they attribute it to liquidity shocks. Our model explains their findings by suggesting that shocks to the ag-

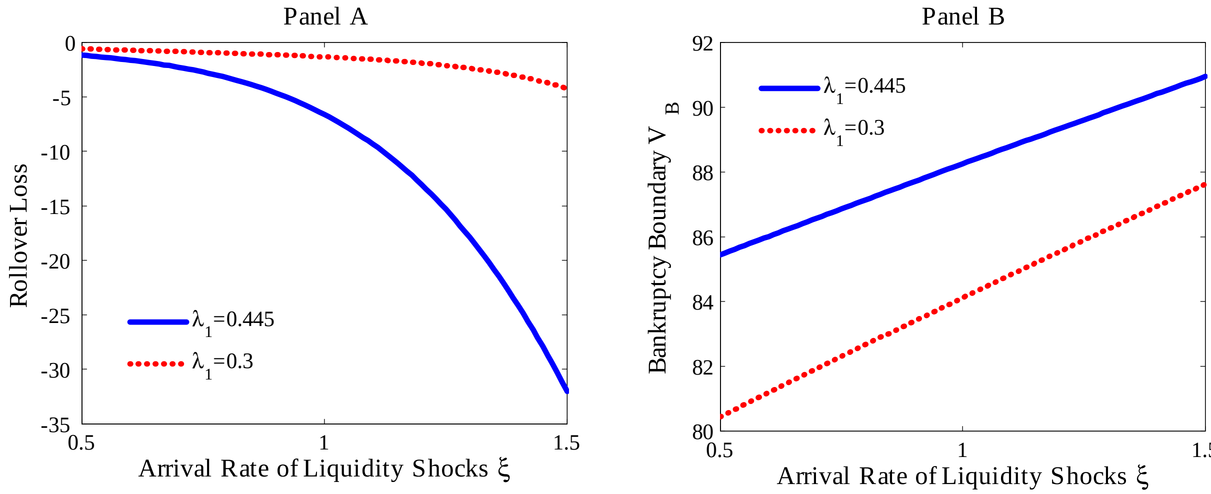

Figure 2: This figure plots the rollover loss and bankruptcy boundary against the arrival rate of liquidity shocks ξ for two otherwise identical firms, except one with a short-term debt fraction λ_1 of 44.5% and the other with 30%. This figure is based on the baseline parameters given in (16). The rollover loss is measured when the firm fundamental is at $V = 93$.

gregate demand for bond market liquidity can act as a common factor in individual bonds' credit spreads. Furthermore, our model shows that this liquidity factor affects not only the liquidity premium, but also their future default probabilities. This amplification mechanism through firms' endogenous defaults helps explain the large impact of the liquidity factor observed in the data.

4.2 Further Amplification by Short-term Debt

The ongoing financial crisis reveals that many financial firms heavily rely on short-term debt such as commercial paper and overnight repos to finance their investment positions. Commercial paper typically has a maturity of less than 270 days, while overnight repos have an extremely short maturity of one day. What is the effect of short-term debt on the firm's exposure to the liquidity shocks?

To examine this question, we compare two otherwise identical firms, one with a short-term debt fraction λ_1 of 44.5% and the other with 30%. Figure 2 plots both firms' rollover loss and endogenous bankruptcy boundary against the arrival rate of the bond holders' liquidity shocks ξ. Panel A of the figure shows that the rollover loss of the firm with higher λ_1 rises much faster with the increase in ξ. This is because short-term debt needs to be rolled over at a higher frequency. As a result, when bond prices drop below their principal values, the equity holders have to pay off the losses accumulated in the debt at a faster speed. The heavier financial burden could in turn cause the equity value to fall down to zero and the equity holders to quit servicing the debts at a higher fundamental threshold. Indeed, Panel B shows that the firm with the higher short-term debt fraction has higher bankruptcy boundary V_B. Taken together, this figure illustrates that short-term debt can further exacerbate the conflict of interest between the debt and equity holders in financial distresses, and thus amplify a debt crisis.

Why does short-term debt exacerbate the debt crisis? To see the intuition, we introduce $\overline{d}_i$ to normalize the market value of the newly issued short-term and long-term bonds so that

$$\overline{d}_i\left(V_t, m_i\right) = \frac{m_i}{\lambda_i} d_i\left(V_t, m_i\right) \tag{17}$$

correspond to the value of the i-th bond with a coupon rate C and a principal P (recall equations (3) and (4)). The two normalized bonds differ only in their maturities. This normalization allows us to rewrite the firm net rollover gain/loss in $(t, t+dt)$ as

$$\left[\sum_{i=1}^{2} \lambda_i \frac{\overline{d}_i\left(V_t, m_i\right) - P}{m_i}\right] dt. \tag{18}$$

For each class of debt, the rollover loss is proportional to the normalized loss $\overline{d}_i\left(V_t, m_i\right) - P$ and the rollver frequency $\frac{1}{m}$.

To understand the role of maturity in rollover losses in equation (18), let us first discuss the normalized loss $\overline{d}_i\left(V_t, m_i\right) - P$. Since short-term debt is more liquid than long-term debt ($\beta_1 < \beta_2$), this implies that if default is not a concern, i.e., when the firm's fundamental is strong, then we have $\overline{d}_1 > \overline{d}_2$. As a result, short-term debt has a *smaller* rollover loss for each unit of normailized bond. This then makes the dramatic effect of short-term debt on the firm's bankruptcy boundary more surprising.

The key to this effect lies in the rollover frequency $\frac{1}{m}$, i.e., a shorter maturity m_i means a higher rollover frequency and therefore amplifies the rollover loss. This effect is at the heart of the mounting financial burden caused by short-term debt exactly when the firm's fundamental is weak. For illustration, Figure 3 plots the firm's loss from rolling over its short-term and long-term debts with respect to the firm fundamental. The figure shows that when the firm's fundamental is strong, short-term debt does provide a smaller rollover loss than long-term debt.[8] However, when the firm's fundamental is weak and thus close

[8] Because of the low coupon rate specified in this illustration (i.e., the bonds are not issued at par when $C = P/r$), the firm always has a rollover short-fall, which serves in this situation as part of the interest payment for its debts. This amounts to a level shift in rollover losses in Figure 3 and will not affect the relative comparison between long-term and short-term debt.

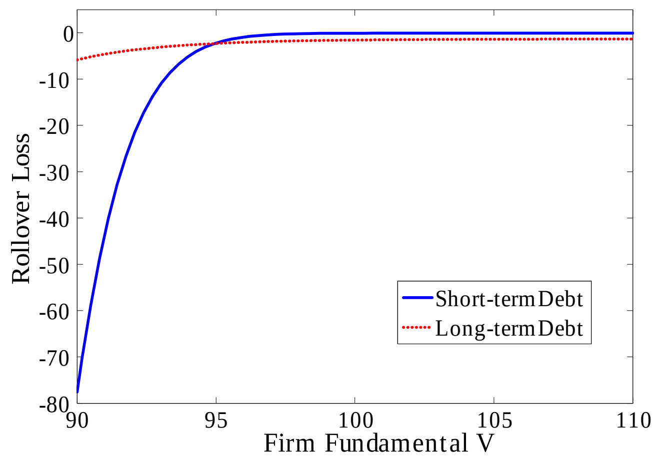

Figure 3: This figure plots the firm's rollover loss at different firm fundamentals for each \$100 face value of short-term and short-term debt. This figure is based on the parameters given in (16) and $\lambda_1 = 44.5\%$.

to bankruptcy, short-term debt generates much larger rollover loss than long-term debt. In other words, the rollover gain/loss from short-term debt is highly skewed on the downside, while that from long-term debt is relatively flat.

These different rollover gain/loss profiles have a direct impact on the value of the equity holders' embedded option of keeping the firm alive. Even if the current fundamental is weak, equity holders could choose to absorb the rollover losses in hope for a future fundamental comeback. The negatively skewed payoff from short-term debt makes keeping the firm alive costly and the option less valuable, while the flat payoff from long-term debt makes the option more valuable. In this sense, long-term debt gives the firm more flexibility, and, as we will discuss in Section 5, should be used as part of the firm's liquidity management strategy.

We can formally prove a set of results related to the discussion above. First, we can show that between two firms, one purely financed by the short-term debt and the other purely financed by the long-term debt, the short-term financed firm has a higher default boundary under some sufficient conditions.

Proposition 3 *Suppose that $\xi\beta_i = 0$ for $\forall i$, and $P = \frac{C}{r}$. Then $V_B(1) > V_B(0)$, i.e., the endogenous bankruptcy boundary of a 100% short-term financed firm is higher than that of a 100% long-term financed firm.*

Proposition 3 imposes two sufficient conditions. First, either the arrival rate ξ of bond

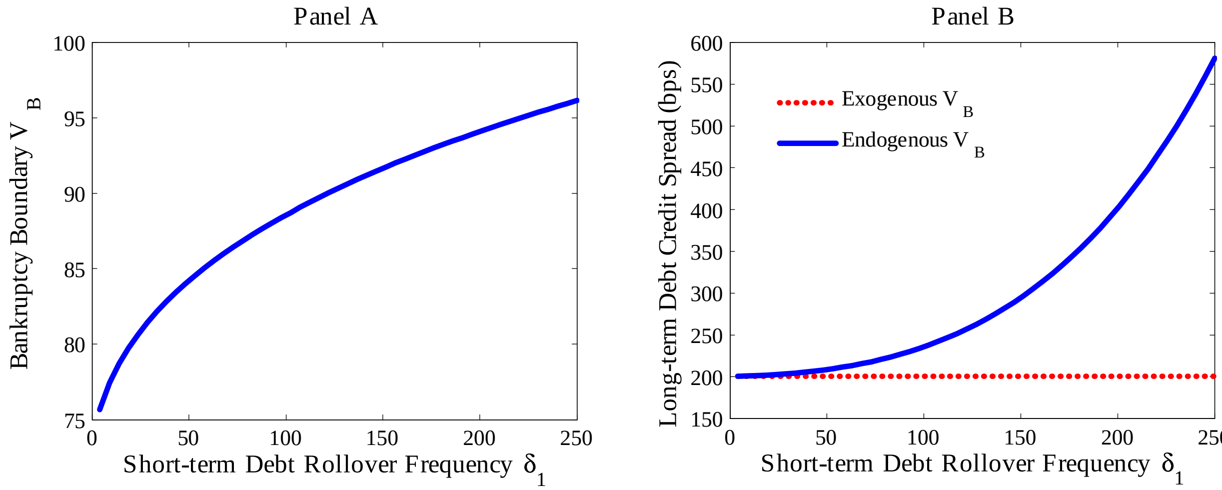

Figure 4: This figure shows the effect of shortening the maturity of the short-term debt, based on the model parameters given in (16) and fixing the short-term debt fraction λ_1 at 5%. Panel A plots the firm's bankruptcy boundary V_B against the short-term debt rollover frequency δ_1, Panel B plots the credit spread of newly issued long-term bond against δ_1.

holders' liquidity shocks or the trading cost β_i's is zero. Under this condition the bond valuation is not affected by market liquidity. Second, the bond's principal value is identical to the discounted value of the perpetual stream of its coupons, i.e., the firm faces no rollover short-fall when there is no default risk. These conditions are somewhat strong, as the complex expression of V_B in equation (15) prevents us from deriving the result of Proposition 3 under more relaxed conditions. However, by continuity arguments, the result must hold when the model parameters are close to the specified conditions (i.e., either ξ or β_i is small and the principal P is close to C/r). Furthermore, our numerical exercises show that the result holds in a wide range of parameter values.

We can further prove that if the result of Proposition 3 holds, the firm's bankruptcy boundary is monotonically increasing with the fraction of its short-term financing.

Proposition 4 *Suppose that $V_B(1) > V_B(0)$, i.e., the endogenous bankruptcy boundary of a 100% short-term financed firm is higher than that of a 100% long-term financed firm. Then $V_B(\lambda_1)$ is monotonically increasing with λ_1, i.e., the greater the fraction of the firm's short-term debt, the higher its endogenous bankruptcy boundary.*

This proposition provides another key factor in our analysis of the firm's optimal debt maturity structure in Section 5.

Repo Financing Right before the bankruptcy of Lehman Brothers, it had to roll over 25% of its debt every day through overnight repos. Repos are a type of collateralized lending agreement, in which a borrower finances the purchase of a financial security using the security as collateral. Overnight repos have an extremely short maturity of one day. What is the effect of overnight repos on the bankruptcy risk of a firm? To illustrate the impact of repo financing, we consider a hypothetical firm with the baseline parameters given in (16). We reduce the maturity of the short-term debt m_1 from 3 months to 1 day (overnight repo). We denote $\delta_1 = \frac{1}{m_1}$ as the rollover frequency of the short-term debt, which is 4 for commercial paper with a 3-month maturity and 250 for overnight repos. We also fix the short-term debt fraction at 5% to focus on the effect of shortening the maturity.[9] Figure 4 shows that even with a small fraction of short-term debt, shortening its maturity to 1 day generates a large impact on the firm's default probability. Panel A shows that as the rollover frequency increases from 4 to 250, the endogenous bankruptcy boundary V_B increases from slightly above 75 to 96. As a result of the substantial increase in V_B, the credit spread of newly issued long-term debt rises from 200 basis points to 575. This figures shows that simply shortening the maturity of a small fraction of the firm's debt from 3 months to 1 day could have a dramatic impact on the firm's financial stability.

There are several studies on the role played by repos in the ongoing financial crisis, e.g., Brunnermeier and Pedersen (2009), Geanakoplos (2009), and Acharya, Gale, and Yorulmzer (2009). These studies all focus on the destabilizing effect of the "haircut" (or margin) of the repos, i.e., creditors will increase the haircut when the market liquidity deteriorates or when the price volatility spikes. The increased margin requirement forces cash-constrained borrowers to liquidate their positions at firesale prices, resulting in a margin spiral. Our model shows that even in the absence of any cash constraint on borrowers, the fast rollover frequency of overnight repos can already lead to a debt crisis. Essentially, the repo rollover acts as a mark-to-market mechanism to force the borrowers to absorb the losses accumulated in their positions every day through margin calls. The heavy financial burden on the borrowers can in turn motivate them to default at a higher fundamental threshold.[10]

[9] We reduce the short-term debt fraction from 44.5% in the previous illustrations to 5% here because a higher fraction, when combined with the fast rollover frequency of overnight repos, would have caused the firm's bankruptcy boundary to be higher than the initial firm fundamental $V_0 = 100$.

[10] Since bankruptcy leads to a social loss, it is tempting to argue that debt restructuring, such as swapping debt into equity or lengthening the debt maturities, would be pareto improving. However, such modifications of the debt agreements are already defined as a credit event, which would trigger many credit default swap contracts to pay out. As a result, even if such debt modifications avoid the social loss, they would hurt some parties and thus run into resistance in practice.

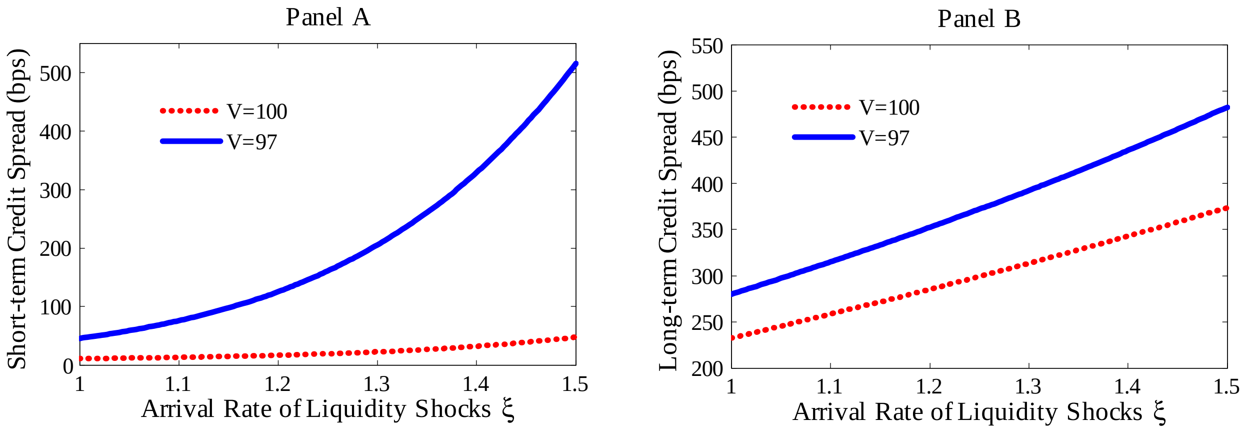

Figure 5: This figure plots the credit spreads of the short-term and long-term bonds of two firms with different fundamentals against the arrival rate of bond holders' liquidity shocks ξ, based on the baseline parameters given in (16) and by fixing the firms' short-term debt fraction at $\lambda_1 = 44.5\%$. One of the firm has a fundamental of $V = 100$, while the other firm has $V = 97$.

4.3 Flight to Quality

It is common to observe the so called flight-to-quality phenomenon after major liquidity disruptions in the financial markets—prices (credit spreads) of low quality bonds drop (rise) much more than those of high quality bonds. The recent episodes include the stock market crash of 1987, the events surrounding the Russian default and the crisis of LTCM in 1998, the events after the attacks of 9/11 in 2001, and the ongoing financial crisis. The CGFS (1999) report documents that during the 1998 LTCM crisis, which is widely regarded as a market-wide liquidity shock, the yields of speculative-grade corporate bonds and emerging market bonds increased much more than investment-grade bonds. A recent BIS report by Fender, Ho, and Hordahl (2009) shows that in a two-month period around the bankruptcy of Lehman Brothers in September 2008, the US five-year CDX high yield index spread shot up from around 700 basis points to over 1500, while the corresponding investment grade index spread widened from 150 basis points to a little above 250.

Can our model explain the flight-to-quality phenomenon? To address this question, we examine two otherwise identical firms, except one with a fundamental of $V = 100$ and the other with $V = 97$. We compare the changes in the credit spreads of these two firms' newly issued short-term and long-term bonds as the arrival rate of the bond holders' liquidity shocks ξ increases from 1 to 1.5. Figure 5 shows that the credit spreads of the weaker firm are significantly more sensitive to the increase in ξ than those of the stronger firm. The intuition is simple. As the increase in the arrival rate of the bond holders' liquidity shocks pushes up the firms' endogenous bankruptcy boundary, the weaker firm is now closer to bankruptcy. As a result, its credit spreads shoot up more than those of the stronger firm.

Our explanation of the flight-to-quality phenomenon is different from the existing ones. The CGFS (1999) report attributes them to suddenly increased risk aversion of market participants. Vayanos (2004) provides an explanation based on professional fund managers' career concerns, and Caballero and Krishnamurthy (2008) argue for Knightian uncertainty. These explanations are all based on considerations from the investor side. Our model focuses on the financing issues from the firm side and shows that deterioration of market liquidity would increase firms' refinancing cost of their maturing debt, exacerbate conflicts of interest inside the firms, and eventually cause the weaker firms to fail.

Corroborating to our theory, Fender, Ho, and Hordahl (2009) show that soon after the market liquidity breakdown caused by the failure of Lehman Brothers in September 2008, the default rates of speculative-grade bonds increased significantly from the very low levels (around 1%) observed in early 2008 to near 5% in March 2009, and were expected to rise further. The recent bankruptcy of General Growth Properties, one of the largest mall operators in the US, in April 2009 nicely illustrates how the deteriorating credit market liquidity put pressure on lower-quality firms. According to the New York Times (April 16, 2009), "*Despite bargaining for months with its creditors, General Growth faced dwindling options for handling its more than $25 billion in debt, largely in the form of short-term mortgages that will come due by next year. The company has been severely wounded by the trouble in the financial markets, which has wreaked havoc on its ability to refinance that debt.*"

4.4 Liquidity Spillover Effect

As is well known, bond markets are highly segmented. For example, the market for commercial paper (short-term debt with maturities less than 9 months) operates on different quote conventions from the market for long-term corporate bonds. These markets also have separate sets of institutional investors. Despite the segmentation between these markets, our model shows that liquidity shocks to one market could still affect bonds in the other market through the firms' endogenous default channel.

To illustrate this spillover effect, we use unexpected changes in the trading cost of short-term and long-term debts, β_1 and β_2, to proxy for liquidity shocks to these two different market segments. Specifically, we use the model parameters given in (16) and fix the firm's

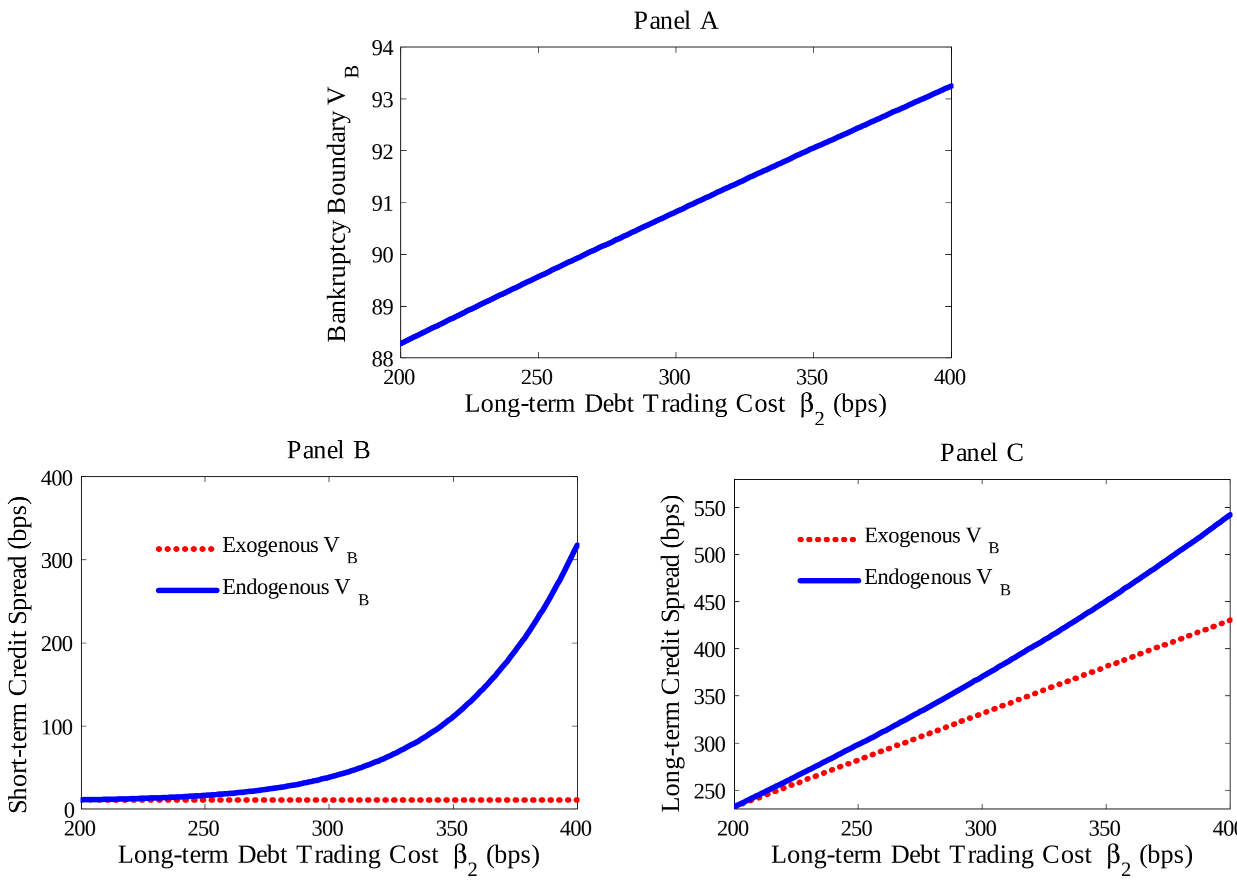

Figure 6: This figure shows the effects of the long-term debt trading cost β_2 on the firm's credit risk, based on the baseline parameters given in (16) and by fixing the firm's short-term debt fraction at $\lambda_1 = 44.5\%$. Panel A plots the firm's endogenous bankruptcy boundary V_B against β_2, Panels B and C plot the credit spreads of the newly issued short-term and long-term bonds against β_2.

long-term debt fraction at $\lambda_1 = 44.5\%$. Figure 6 shows the effects of an unexpected increase in the long-term bond trading cost β_2 on the firm's endogenous bankruptcy boundary V_B and the short-term and long-term credit spreads (credit spreads of the newly issued short-term and long-term bonds). Panel A shows that as β_2 increases from 200 basis points to 400, V_B increases from 88.2 to 93.2. This is because a higher trading cost reduces the long-term bond prices and increases the equity holders' rollover loss, thus causing the firm to default at a higher fundamental threshold. If the firm's bankruptcy boundary is fixed at 88.2, the short-term credit spread is not affected by the change in the long-term debt trading cost. However, Panel B shows that the short-term credit spread increases from below 10 basis points to above 300, as β_2 increases from 200 basis points to 400. This dramatic increase is exactly generated by the firm's higher bankruptcy boundary. This plot thus demonstrates

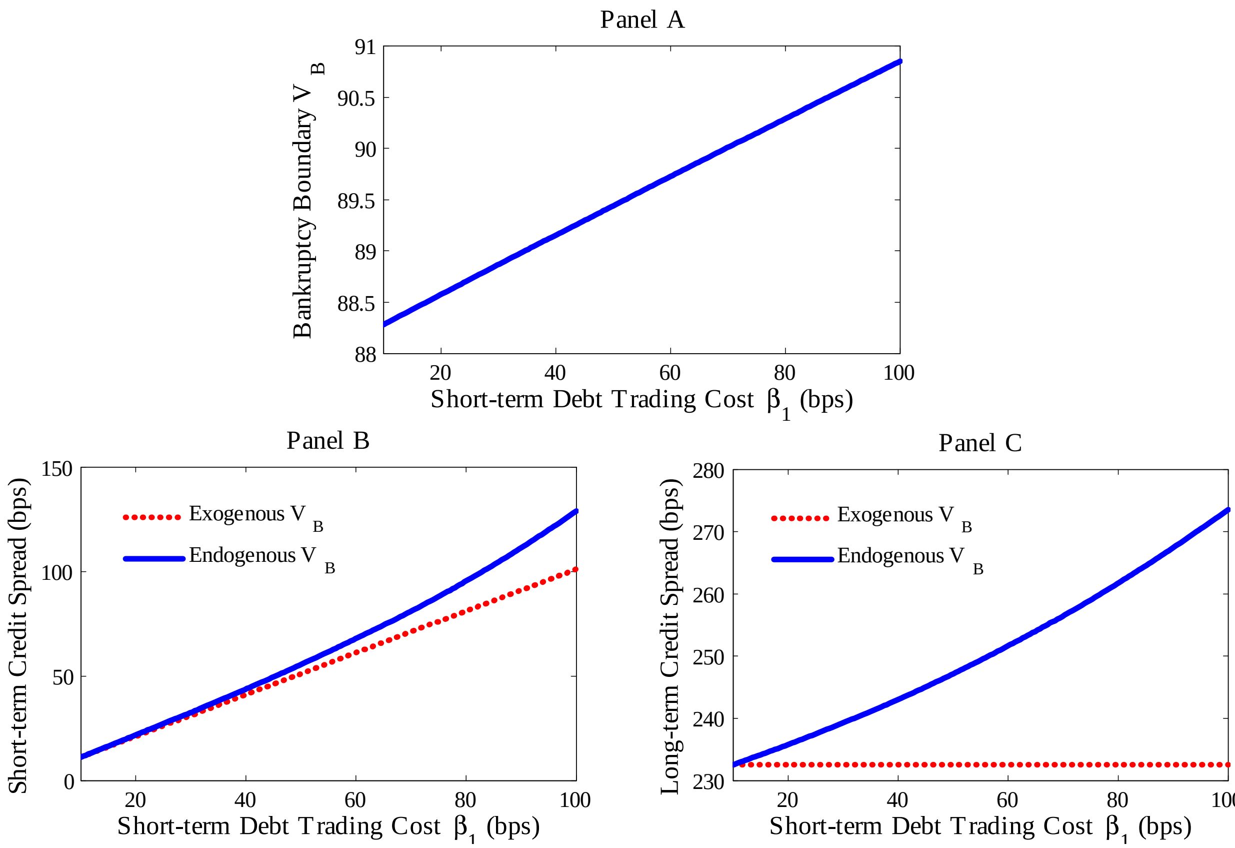

Figure 7: This figure shows the effects of the short-term debt trading cost β_1 on the firm's credit risk, based on the baseline parameters given in (16) and by fixing the short-term debt fraction at $\lambda_1^* = 44.5\%$, the optimal level under the baseline $\beta_1 = 0.1\%$. Panel A plots the firm's endogenous bankruptcy boundary V_B against β_1, Panels B and C plot the credit spreads of the short-term and long-term bonds against β_1.

the liquidity spillover effect from the long-term debt market to the short-term debt market.

Panel C also shows that as β_2 increases from 200 basis points to 400, the long-term credit spread increases from 230 basis points to near 550. However, when the firm commits to fix V_B at the initial level 88.2, the increase in the long-term credit spread is smaller, only from 230 to 430. This plot suggests that the firm's endogenous default also amplifies the effect of the increase in the long-term debt trading cost on the long-term credit spread.

Figure 7 confirms similar effects by an unexpected increase in the short-term debt trading cost β_1. Panel A shows that the bankruptcy boundary V_B increases with β_1. Panel C shows that the long-term credit spread increases significantly with β_1, a clear liquidity spillover effect from the short-term debt market to the long-term debt market. Panel B also shows that the firm's endogenous default significantly amplifies the effect of the increase in the short-term debt trading cost on the short-term credit spread.

We can formally prove the following proposition:

Proposition 5 *The firm's endogenous bankruptcy boundary V_B increases with both β_1 and β_2, the trading cost of the short-term and long-term debts.*

This proposition confirms that the firm's endogenous bankruptcy can serve as a channel for liquidity shocks to one segment of the bond markets to affect credit spreads in other segments.

5 Optimal Debt Maturity Structure

In our model, there are two opposing forces working together to determine the firm's ex ante optimal debt maturity structure. On one hand, short-term debts are more liquid and therefore are cheaper for the firm. On the other, short-term debts exacerbate the conflict of interest between the debt and equity holders and therefore increase the firm's future default probability. In this section, we examine this tradeoff between the short-term debt's cheaper financing cost and higher expected bankruptcy cost in determining the firm's optimal maturity structure.

Consider the firm's optimal maturity structure choice at time 0. The firm's objective is to maximize the total firm value, the sum of equity, short-term and long-term bonds:

$$\max_{\lambda_1 \in [0,1]} E(V) + D_1(V) + D_2(V).$$

This objective is also consistent with that of the equity holders at time 0 before the bonds are issued. Since the objective is a continuous function of λ_1 and λ_1 takes values in a closed set $[0, 1]$, there must exist an optimum.

Figure 8 plots the firm's endogenous bankruptcy boundary V_B and the total firm value against the firm's short-term debt fraction λ_1. Panel A shows that V_B increases with λ_1, consistent with our discussion before. Panel B shows that the total firm value is maximized when $\lambda_1^* = 44.5\%$. This interior optimum reflects the tradeoff between the short-term debt's cheaper financing cost and higher expected bankruptcy cost.

Figure 9 illustrates how firm characteristics affect its optimal short-term debt fraction λ_1^*, based on the baseline parameters given in (16). Panel A shows that λ_1^* decreases with the firm's asset volatility σ. As the volatility increases, it raises the firm's future default probability and therefore expected bankruptcy cost. As a result, it is desirable for the firm to use less short-term debt to reduce the bankruptcy cost. The figure also shows that, in the

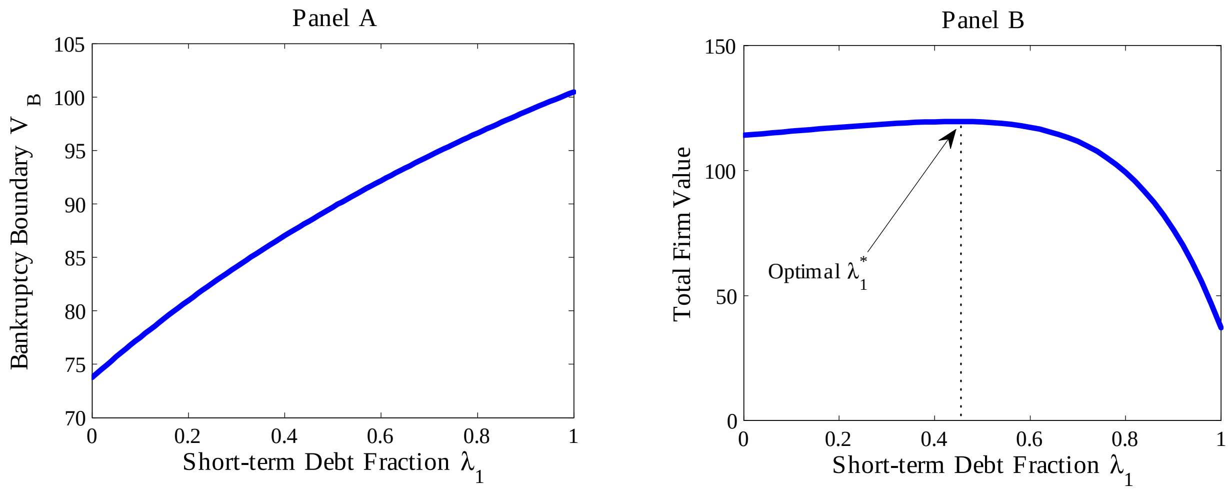

Figure 8: This figure plots the firm's endogenous bankruptcy boundary V_B and the total firm value against the firm's short-term debt fraction λ_1, based on the baseline parameters given in (16).

region where the asset volatility is low (lower than 5.5%), the cheaper cost effect dominates and induces the firm to use 100% short-term debt. Panel B shows that λ_1^* increases with the firm's bankruptcy recovery rate α. As α increases, the expected bankruptcy cost becomes smaller. As a result, the firm could take advantage of the cheaper financing cost of short-term debt more aggressively.

Panel C shows that there is a non-monotonic relationship between λ_1^* and the long-term debt trading cost β_2: λ_1^* first increases with β_2 when it is relatively low and decreases with β_2 when it becomes high. This plot again reflects the tradeoff between the financing cost and expected bankruptcy cost. As β_2 increases, the direct effect is that the long-term debt becomes more expensive. This effect makes the short-term debt more attractive, and thus explains the increasing part of the plot. When β_2 increases, an indirect effect is that it induces the firm to use a higher bankruptcy threshold, resulting in a higher future default probability. This indirect effect makes the short-term debt less attractive on the margin, and explains the decreasing part of the plot.

If the trading cost of both the short-term and long-term debts, β_1 and β_2, increase together, then the substitution effect between the types of bonds is void and only the second (indirect) effect through the firm's endogenous default is in operation. Panel D of Figure 9 shows that as the common component in β_1 and β_2 increases, the optimal short-term debt fraction λ_1^* decreases. This is because the increased market illiquidity makes the firm more likely to bankrupt in the future. As a result, the bankruptcy cost effect becomes more

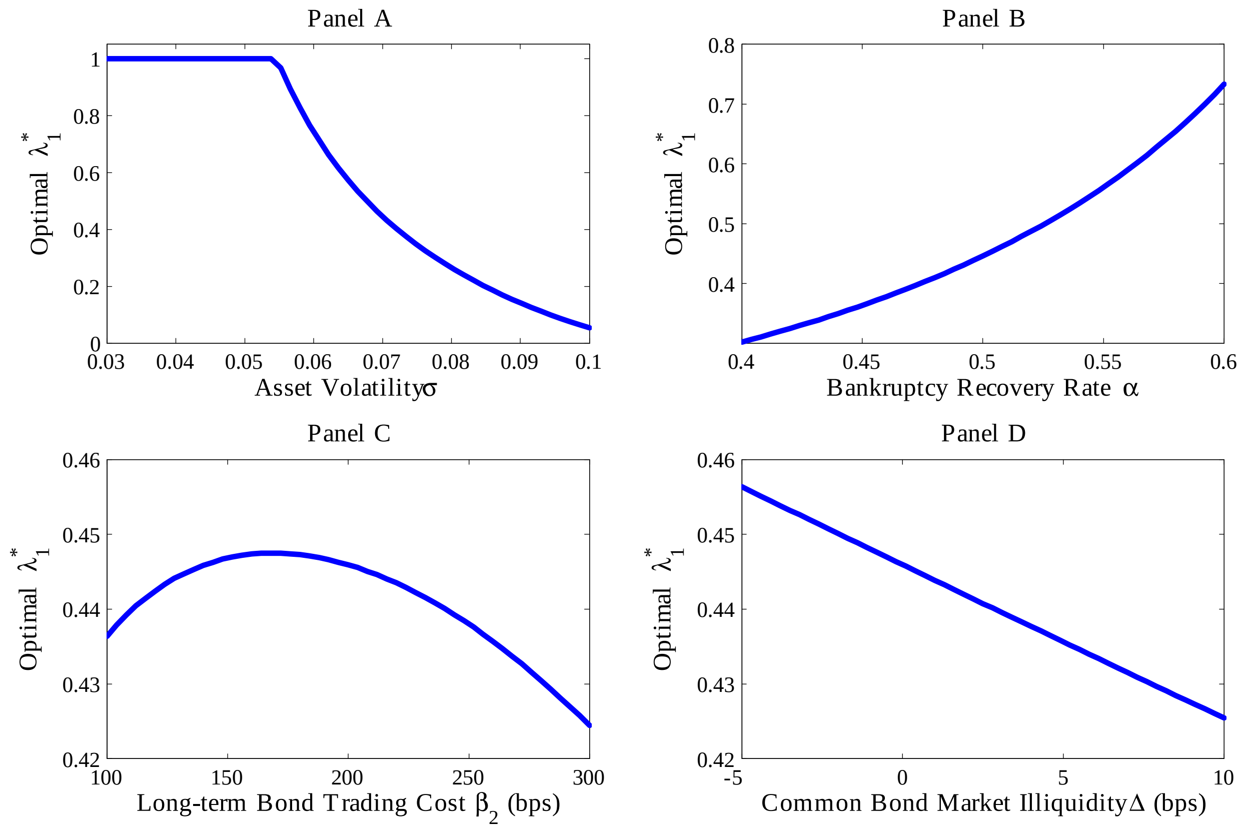

Figure 9: This figure shows how firm characteristics affect the firm's optimal short-term debt fraction λ_1^*, based on the baseline parameters given in (16). Panel A plots λ_1^* against the firm's asset volatility σ. Panel B plots λ_1^* against the bankruptcy recovery rate α. Panel C plots λ_1^* against the long-term debt trading cost β_2. Finally, Panel D plots λ_1^* against the common component Δ in the trading cost β_1 and β_2 of the firm's short-term and long-term bonds.

important.

The extant theories on firms' debt maturity choice have mostly focused on the disciplinary role of short-term debt in preventing managers' asset substitution, e.g., Flannery (1994) and Leland (1998), and private information of borrowers about their future credit ratings, e.g., Flannery (1986) and Diamond (1991). These theories have had some success in explaining the data, as shown by Barclay and Smith (1995) and Guedes and Opler (1996). Our model provides a new hypothesis, which relates firms' debt maturity structure to market illiquidity considerations.

Maturity Structure as Liquidity Management In fact, our analysis shows that debt maturity structure should be used as part of a firm's liquidity management strategy. As discussed in Section 4.2, despite its higher cost, long-term debt gives the firm more flexibility to delay realizing financial losses in adverse states, either when the firm's fundamental or bond market liquidity deteriorates. This benefit is analogous to the role of cash reserves, the standard tool for risk management, e.g., Holmstrom and Tirole (2001) and Bolton, Chen, and Wang (2009). Keeping cash is costly for a firm, but it allows the firm to avoid future financial constraints and to take advantage of future investment opportunities.

Brunnermeier and Yogo (2009) argue that firms should shift to long-term debt as their fundamentals deteriorate. This argument is consistent with the basic result of our model that the bankruptcy cost (or the loss of flexibility) from using short-term debt becomes higher as the firm's fundamental or bond market liquidity deteriorates. This effect motivates the firm to use less short-term debt in these states. This argument is appealing, but it is also countered by other conflicts between debt and equity holders. As pointed out by Leland (1994), adjusting debt policy in his model by issuing or retiring debt ex post is infeasible to the extent that it will hurt either equity or debt holders. This argument also applies to our setting.[11] A systematic analysis of these arguments is important, but is a challenge beyond our current framework. We will leave it for future research.

6 Conclusion

We examine the role played by deteriorating market liquidity in debt crises. We extend Leland's structural credit risk model with two realistic features: illiquid secondary bond markets and a mix of short-term and long-term bonds in a firm's debt structure. As liquidity shocks push down bond prices, they amplify the conflict of interest between the debt and equity holders because, to avoid bankruptcy, the equity holders have to absorb all of the short-fall from rolling over maturing bonds at the reduced market values. As a result, the equity holders choose to default at a higher fundamental threshold even if firms can freely raise more equity. A greater fraction of short-term debt further exacerbates the debt crisis

[11] It is clear that increasing λ_1 will lead to a higher bankruptcy boundary V_B, and therefore hurts long-term debt holders. Now suppose that the firm adjusts λ_1 downward. One realistic policy is to issue more long-term debt to replace the maturing short-term debt, until the desired maturity structure is achieved. In the interim period, the replaced short-term debt has coupon (principal) $\frac{\lambda_1}{m_1}C$ ($\frac{\lambda_1}{m_1}P$). Therefore, to maintain the firm's net coupon and debt principal, the firm needs to issue $\frac{\lambda_1}{m_1}\frac{m_2}{\lambda_2}$ units of long-term debt. The net financing effect, relative to the base case without the maturity adjustment, is

$$-\frac{\lambda_1}{m_1}\overline{d}_1\left(V_t,m_1\right)+\frac{\lambda_1}{m_1}\frac{m_2}{\lambda_2}\frac{\lambda_2}{m_2}\overline{d}_2\left(V_t,m_2\right)\propto -\overline{d}_1\left(V_t,m_1\right)+\overline{d}_2\left(V_t,m_2\right)$$

Since the short-term debt is safer than the long-term debt, the above term is negative. This heuristic argument implies that the financial burden on the equity holders is increased during the adjustment process, therefore it is not in the interest of the equity holders to increase the long-term debt fraction.

by forcing the equity holders to realize the rollover loss at a higher frequency. Our model illustrates the financial instability brought by overnight repos, an extreme form of short-term financing, to many financial firms, and provides a new explanation to the widely observed flight-to-quality phenomenon. We also examine a tradeoff between short-term debt's cheaper financing cost and higher future bankruptcy cost in determining firms' optimal debt maturity structure and liquidity management strategy.

A Appendix

A.1 Proof for Proposition 1

The equity satisfies the following ODE:

$$rE = (r-\phi)VE_V + \frac{1}{2}\sigma^2 V^2 E_{VV} + d_1(V, m_1) + d_2(V, m_2) + \phi V - \left[(1-\tau_c)C + \frac{P_1}{m_1} + \frac{P_2}{m_2}\right].$$

Define

$$v \equiv \ln\left(\frac{V}{V_B}\right).$$

Then,

$$rE = \left(r-\phi-\frac{1}{2}\sigma^2\right)E_v + \frac{1}{2}\sigma^2 E_{vv} + d_1(v, m_1) + d_2(v, m_2) + \phi V_B e^v - \left[(1-\tau_c)C + \frac{P_1}{m_1} + \frac{P_2}{m_2}\right].$$

with the boundary condition that

$$E(0) = 0 \text{ and } E_v(0) = l,$$

where the free parameter l is to be determined by the boundary condition when $v \to \infty$.

Define the Laplace transformation of $E(v)$ as

$$F(s) = L[E(v)] = \int_0^\infty e^{-sv} E(v)\, dv.$$

Then, apply the Laplace transformation to both sides of the ODE, we have:

$$\begin{aligned} rF(s) &= \left(r-\phi-\frac{1}{2}\sigma^2\right)L[E_v] + \frac{1}{2}\sigma^2 L[E_{vv}] + L[d_1(v, m_1) + d_2(v, m_2)] \\ &\quad + \frac{\phi V_B}{s-1} - \frac{(1-\tau_c)C + \frac{P_1}{m_1} + \frac{P_2}{m_2}}{s}. \end{aligned}$$

Note that

$$L[E_v] = sF(s) - E(0) = sF(s)$$

and

$$L\left[E_{vv}\right]=s^2F\left(s\right)-sE\left(0\right)-E_v\left(0\right)=s^2F\left(s\right)-l.$$

Thus,

$$\begin{aligned}\left[r-\left(r-\phi-\frac{1}{2}\sigma^2\right)s-\frac{1}{2}\sigma^2s^2\right]F\left(s\right) &= L\left[d_1\left(v,m_1\right)\right]+L\left[d_2\left(v,m_2\right)\right]-\frac{1}{2}\sigma^2 l\\ &\quad+\frac{\phi V_B}{s-1}-\frac{(1-\tau_c)C+\frac{P_1}{m_1}+\frac{P_2}{m_2}}{s}.\end{aligned}$$

Let

$$r-\left(r-\phi-\frac{1}{2}\sigma^2\right)s-\frac{1}{2}\sigma^2s^2=-\frac{1}{2}\sigma^2\left(s-\eta\right)\left(s+\gamma\right)$$

where $\eta>1$ and $\gamma>0$. Then,

$$\begin{aligned}&\frac{1}{2}\sigma^2F\left(s\right) \qquad (19)\\ =\ & -\frac{1}{\left(s-\eta\right)\left(s+\gamma\right)}\left\{L\left[d_1\left(v,m_1\right)\right]+L\left[d_2\left(v,m_2\right)\right]+\frac{\phi V_B}{s-1}-\frac{(1-\tau_c)C+\delta_1P_1+\delta_2P_2}{s}-\frac{1}{2}\sigma^2 l\right\}\\ =\ & -\frac{1}{\eta+\gamma}\left(\frac{1}{s-\eta}-\frac{1}{s+\gamma}\right)\left\{\begin{array}{c}L\left[d_1\left(v,m_1\right)\right]+L\left[d_2\left(v,m_2\right)\right]\\ +\frac{\phi V_B}{s-1}-\frac{(1-\tau_c)C+\frac{P_1}{m_1}+\frac{P_2}{m_2}}{s}-\frac{1}{2}\sigma^2 l\end{array}\right\}\end{aligned}$$

Since

$$d_i\left(v,m_i\right)=\frac{C_i}{r_im_i}+e^{-r_im_i}\left(\frac{P_i}{m_i}-\frac{C_i}{r_im_i}\right)\left(1-F\left(m_i\right)\right)+\left(\frac{1}{m_i}\lambda_i\alpha V_B-\frac{C_i}{r_im_i}\right)G_i\left(m_i\right),$$

where $F\left(m_i\right)$ and $G_i\left(m_i\right)$ are given in Eq. (10), i.e.,

$$\begin{aligned}F\left(m_i\right) &= N\left(\frac{\left(-v-a\sigma^2m_i\right)}{\sigma\sqrt{m_i}}\right)+e^{-2av}N\left(\frac{\left(-v+a\sigma^2m_i\right)}{\sigma\sqrt{m_i}}\right);\\ G_i\left(m_i\right) &= e^{(-a+z_i)v}N\left(\frac{\left(-v-z_i\sigma^2m_i\right)}{\sigma\sqrt{m_i}}\right)+e^{-(a+z_i)v}N\left(\frac{\left(-v+z_i\sigma^2m_i\right)}{\sigma\sqrt{m_i}}\right);\end{aligned}$$

where

$$a=\frac{r-\phi-\sigma^2/2}{\sigma^2};z_i=\frac{\left[a^2\sigma^4+2r_i\sigma^2\right]^{1/2}}{\sigma^2};z=\frac{\left[a^2\sigma^4+2r\sigma^2\right]^{1/2}}{\sigma^2}.$$

Plugging $d_i\left(v,m_i\right)$ in (19), we have

$$\begin{aligned}\frac{1}{2}\sigma^2F\left(s\right) &= -\frac{\frac{1}{s-\eta}-\frac{1}{s+\gamma}}{\eta+\gamma}\left\{\frac{\phi V_B}{s-1}-\frac{(1-\tau_c)C+\sum_i\left(1-e^{-r_im_i}\right)\left(\frac{P_i}{m_i}-\frac{C_i}{r_im_i}\right)}{s}-\frac{1}{2}\sigma^2 l\right\} \qquad (20)\\ &\quad-\frac{\frac{1}{s-\eta}-\frac{1}{s+\gamma}}{\eta+\gamma}\sum_i\left\{-e^{-r_im_i}\left(\frac{P_i}{m_i}-\frac{C_i}{r_im_i}\right)L\left[F\left(m_i\right)\right]+\left(\frac{1}{m_i}\lambda_i\alpha V_B-\frac{C_i}{r_im_i}\right)L\left[G_i\left(m_i\right)\right]\right\}\end{aligned}$$

Call the first line in (20) as $\widehat{F}(s)$, and it is easy to work out its inverse as

$$\begin{aligned}
\widehat{E}(v) &= -\frac{\phi V_B}{\eta+\gamma}\left[\frac{1}{\eta-1}\left(e^{\eta v}-e^{v}\right)+\frac{1}{\gamma+1}\left(e^{-\gamma v}-e^{v}\right)\right] \\
&\quad+\frac{(1-\tau_c)C+\sum\left(1-e^{-r_i m_i}\right)\left[\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right]}{\eta+\gamma}\left[\frac{1}{\eta}\left(e^{\eta v}-1\right)-\frac{1}{\gamma}\left(1-e^{-\gamma v}\right)\right] \\
&\quad+\frac{1}{2}\sigma^2 l\frac{1}{\eta+\gamma}\left(e^{\eta v}-e^{-\gamma v}\right) \\
&= V-\frac{\phi V_B}{\eta+\gamma}\left[\frac{1}{\eta-1}e^{\eta v}+\frac{1}{\gamma+1}e^{-\gamma v}\right] \\
&\quad+\frac{(1-\tau_c)C+\sum\left(1-e^{-r_i m_i}\right)\left[\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right]}{\eta+\gamma}\left[\frac{1}{\eta}\left(e^{\eta v}-1\right)-\frac{1}{\gamma}\left(1-e^{-\gamma v}\right)\right] \\
&\quad+\frac{1}{2}\sigma^2 l\frac{1}{\eta+\gamma}\left(e^{\eta v}-e^{-\gamma v}\right).
\end{aligned}$$

Call the second line in (20) as $\sum_i \overline{F}(s)$. One can show that

$$\begin{aligned}
&(\eta+\gamma)\overline{F}_i = e^{-r_i m_i}\left(\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right)\frac{1}{\eta}\left(\frac{1}{s-\eta}-\frac{1}{s}\right)\left[N\left(-a\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-a^2\right)\sigma^2 m_i}\right] \\
&-e^{-r_i m_i}\left(\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right)\frac{1}{\gamma}\left(\frac{1}{s}-\frac{1}{s+\gamma}\right)\left[N\left(-a\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-a^2\right)\sigma^2 m_i}\right] \\
&+e^{-r_i m_i}\left(\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right)\frac{1}{2a+\eta}\left(\frac{1}{s-\eta}-\frac{1}{s+2a}\right)\left[N\left(a\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-a^2\right)\sigma^2 m_i}\right] \\
&-e^{-r_i m_i}\left(\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right)\frac{1}{\gamma-2a}\left(\frac{1}{s+2a}-\frac{1}{s+k_2}\right)\left[N\left(a\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-a^2\right)\sigma^2 m_i}\right] \\
&-\left(\frac{1}{m_i}\lambda_i\alpha V_B-\frac{C_i}{r_i m_i}\right)\frac{1}{a-z_i+\eta}\left(\frac{1}{s-\eta}-\frac{1}{s+a-z_i}\right)\left[N\left(-z_i\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-z_i^2\right)\sigma^2 m_i}\right] \\
&+\left(\frac{1}{m_i}\lambda_i\alpha V_B-\frac{C_i}{r_i m_i}\right)\frac{1}{k_2-a+z_i}\left(\frac{1}{s+a-z_i}-\frac{1}{s+\gamma}\right)\left[N\left(-z_i\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-z_i^2\right)\sigma^2 m_i}\right] \\
&-\left(\frac{1}{m_i}\lambda_i\alpha V_B-\frac{C_i}{r_i m_i}\right)\frac{1}{a+z_i+\eta}\left(\frac{1}{s-\eta}-\frac{1}{s+a+z_i}\right)\left[N\left(z_i\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-z_i^2\right)\sigma^2 m_i}\right] \\
&\left(\frac{1}{m_i}\lambda_i\alpha V_B-\frac{C_i}{r_i m_i}\right)\frac{1}{\gamma-a-z_i}\left(\frac{1}{s+a+z_i}-\frac{1}{s+\gamma}\right)\left[N\left(z_i\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+a)^2-z_i^2\right)\sigma^2 m_i}\right].
\end{aligned}$$

Define

$$\begin{aligned}
M_i\left(v;x,w,p,q\right) &\equiv L^{-1}\left\{\left(\frac{1}{s+p}-\frac{1}{s+q}\right)\left[N\left(y\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left((s+x)^2-w^2\right)\sigma^2 m_i}\right]\right\} \\
&= \left\{N\left(w\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left[(p-x)^2-w^2\right]\sigma^2 m_i}N\left(\left(p-x\right)\sigma\sqrt{m_i}\right)\right\}e^{-pv} \\
&\quad +e^{\frac{1}{2}\left[(p-x)^2-w^2\right]\sigma^2 m_i}e^{-pv}N\left(\frac{-v+\left(p-x\right)\sigma^2 m_i}{\sigma\sqrt{m_i}}\right) \\
&\quad -\left\{N\left(w\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left[(q-x)^2-w^2\right]\sigma^2 m_i}N\left(\left(q-x\right)\sigma\sqrt{m_i}\right)\right\}e^{-qv} \\
&\quad -e^{\frac{1}{2}\left[(q-x)^2-w^2\right]\sigma^2 m_i}e^{-qv}N\left(\frac{-v+\left(q-x\right)\sigma^2 m_i}{\sigma\sqrt{m_i}}\right)
\end{aligned}$$

and

$$\begin{aligned}
K_i\left(x,w,p\right) &\equiv \left\{N\left(w\sigma\sqrt{m_i}\right)-e^{\frac{1}{2}\left[(p-x)^2-w^2\right]\sigma^2 m_i}N\left(\left(p-x\right)\sigma\sqrt{m_i}\right)\right\}e^{-pv} \\
&\quad +e^{\frac{1}{2}\left[(p-x)^2-w^2\right]\sigma^2 m_i}e^{-pv}N\left(\frac{-v+\left(p-x\right)\sigma^2 m_i}{\sigma\sqrt{m_i}}\right)-e^{-(x+y)v}N\left(\frac{-v+w\sigma^2 m_i}{\sigma\sqrt{m_i}}\right)
\end{aligned}$$

Then

$$M_i\left(v;x,w,x+w,q\right)=-K_i\left(x,w,q\right),\, M_i\left(v;x,w,p,x+w\right)=K_i\left(x,w,p\right).$$

Therefore (note that $\frac{2}{\sigma^2}\frac{1}{\eta+\gamma}=\frac{1}{z\sigma^2}$)

$$\begin{aligned}
E(v) &= \frac{2}{\sigma^2}\left(\widehat{E}(v)+\sum\overline{E}_i\right) \qquad (21)\\
&= V-\frac{\phi V_B}{z\sigma^2}\left[\frac{e^{\eta v}}{\eta-1}+\frac{e^{-\gamma v}}{\gamma+1}\right]+\frac{l}{2z}\left(e^{\eta v}-e^{-\gamma v}\right) \\
&\quad +\frac{(1-\tau_c)C+\sum\left(1-e^{-r_i m_i}\right)\left[\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right]}{z\sigma^2}\left[\frac{1}{\eta}\left(e^{\eta v}-1\right)-\frac{1}{\gamma}\left(1-e^{-\gamma v}\right)\right] \\
&\quad +\sum_i\left[\begin{array}{c}
\frac{e^{-r_i m_i}\left(\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right)}{z\sigma^2}\left[\begin{array}{c}\frac{1}{\eta}K_i\left(v;a,-a,-\eta\right)+\frac{1}{\gamma}K_i\left(v;a,-a,\gamma\right)\\ +\frac{1}{\gamma}K_i\left(v;a,a,-\eta\right)+\frac{1}{\gamma-2a}K_i\left(v;a,a,\gamma\right)\end{array}\right] \\
+\frac{\left(\frac{1}{m_i}\lambda_i\alpha V_B-\frac{C_i}{r_i m_i}\right)}{z\sigma^2}\left[\begin{array}{c}-\frac{1}{a-z_i+\eta}K_i\left(v;a,-z_i,-\eta\right)-\frac{1}{\gamma-a+z_i}K_i\left(v;a,-z_i,\gamma\right)\\ -\frac{1}{a+z_i+\eta}K_i\left(v;a,z_i,-\eta\right)-\frac{1}{\gamma-a-z_i}K_i\left(v;a,z_i,\gamma\right)\end{array}\right]
\end{array}\right]
\end{aligned}$$

There is one free parameter $l=E'\left(0\right)$ to be pinned down by the boundary condition at $v\to\infty$. Equity value has to grow linearly when $V\to\infty$. Since $e^{\eta v}=\left(\frac{V}{V_B}\right)^{\eta}$ and $\eta>1$, to avoid explosion we need the coefficient of $e^{\eta v}$ in $E(v)$ is zero. Collecting coefficients of $e^{\eta v}$, we need (note that $-\eta - a = -z$, $\gamma = 2a + \eta$, $\frac{1}{2}\left[z^2 - a^2\right]\sigma^2 m_i = rm_i$)

$$0 = -\frac{\phi V_B}{\eta - 1} + \left[(1-\tau_c)C + \sum\left(1 - e^{-r_i m_i}\right)\left[\frac{P_i}{m_i} - \frac{C_i}{r_i m_i}\right]\right]\frac{1}{\eta} + \frac{\sigma^2}{2} l \tag{22}$$

$$+\sum_i \left\{ \begin{array}{c} e^{-r_i m_i}\left(\frac{P_i}{m_i} - \frac{C_i}{r_i m_i}\right)\left[\begin{array}{c} \frac{\left\{N\left(-a\sigma\sqrt{m_i}\right) - e^{rm_i}N\left(-z\sigma\sqrt{m_i}\right)\right\}}{\eta} \\ +\frac{\left\{N\left(a\sigma\sqrt{m_i}\right) - e^{rm_i}N\left(-z\sigma\sqrt{m_i}\right)\right\}}{\gamma} \end{array}\right] \\ +\left(\frac{1}{m_i}\lambda_i \alpha V_B - \frac{C_i}{r_i m_i}\right)\left[\begin{array}{c} -\frac{\left\{N\left(-z_i\sigma\sqrt{m_i}\right) - e^{\frac{1}{2}\left[z^2 - z_i^2\right]\sigma^2 m_i}N\left(-z\sigma\sqrt{m_i}\right)\right\}}{a - z_i + \eta} \\ -\frac{\left\{N\left(z_i\sigma\sqrt{m_i}\right) - e^{\frac{1}{2}\left[z^2 - z_i^2\right]\sigma^2 m_i}N\left(-z\sigma\sqrt{m_i}\right)\right\}}{a + z_i + \eta} \end{array}\right] \end{array} \right\}$$

This condition gives l as an expression of primitive parameters and the bankruptcy boundary V_B. More important, this implies that the constant coefficient of $e^{\eta v}$ should be zero. This simplifies the expression of K_i that is involving $-\eta$ to be

$$\begin{aligned} K_i(x, w, -\eta) &= e^{\frac{1}{2}\left[(p-x)^2 - w^2\right]\sigma^2 m_i} e^{\eta v} N\left(\frac{-v + (-\eta - x)\sigma^2 m_i}{\sigma\sqrt{m_i}}\right) - e^{-(x+y)v} N\left(\frac{-v + w\sigma^2 m_i}{\sigma\sqrt{m_i}}\right) \\ &\equiv k_i(x, w, -\eta). \end{aligned}$$

The smooth pasting condition implies that $E'(V_B) = 0$, or $E_v'(0) = l = 0$. Then we can use condition (22) to obtain V_B. With these results, we have the closed-form expression for $E(v)$ and V_B stated in Proposition 1.

A.2 Proof of Proposition 2

We first fix the default boundary V_B. According to the Feynman-Kac formula, PDE (6) implies that at time 0, the price of a bond with time-to-maturity τ satisfies

$$d_i(V_0, \tau; V_B) = \mathbb{E}\left[\int_0^{\tau\wedge\tau_B} e^{-(r+\xi\beta_i)s} c_i ds + e^{-(r+\xi\beta_i)\cdot(\tau\wedge\tau_B)} d_i(\tau\wedge\tau_B)\right],$$

where $\tau_B = \inf\{t : V_t = V_B\}$ is the first hitting time of V_t to V_B. V_t follows (1), and $d_i(\tau\wedge\tau_B)$ is defined by the boundary conditions in (7) and (8):

$$d_i(\tau\wedge\tau_B) = \left\{ \begin{array}{cc} \frac{1}{m_i}\lambda_i\alpha V_B & \text{if } \tau\wedge\tau_B = \tau_B \\ p_i & \text{if } \tau\wedge\tau_B = \tau \end{array} \right. .$$

Because ξ enters d_i as raising the discount rate, a path-by-path argument implies that d_i decreases with ξ.

Similarly, the equity value can be written as

$$E(V_0, \tau; V_B) = \mathbb{E}\left\{\int_0^{\tau_B} e^{-rs}\left[\phi V_s - (1-\tau_c)C + \sum_{i=1}^2\left(d_i(V_s, m_i - s; \xi) - p_i\right)\right] ds\right\},$$

where we write the dependence of d_i on ξ explicitly. Again, a path-by-path argument implies that once fixing V_B, E decreases with ξ.

We now consider two different values of ξ: $\xi_1 < \xi_2$. Denote the corresponding default boundaries as $V_{B,1}$ and $V_{B,2}$. We need to show that $V_{B,1} < V_{B,2}$. Suppose that the opposite is true, i.e., $V_{B,1} \geq V_{B,2}$. Since the equity value is zero on the default boundary, we have

$$E\left(V_{B,1}; V_{B,1}, \xi_1\right) = E\left(V_{B,2}; V_{B,2}, \xi_2\right) = 0,$$

where we expand the notation to let the equity value $E(V_t; V_B, \xi)$ to explicitly depend on the default boundary V_B and the liquidity shock arrival rate ξ. Also, the optimality of default boundary implies that

$$0 = E\left(V_{B,1}; V_{B,1}, \xi_1\right) > E\left(V_{B,1}; V_{B,2}, \xi_1\right)$$

Since E decreases with ξ, $E\left(V_{B,1}; V_{B,2}, \xi_1\right) > E\left(V_{B,1}; V_{B,2}, \xi_2\right)$. Therefore $E\left(V_{B,1}; V_{B,2}, \xi_2\right) < 0$. This contradicts to limited liability which says that

$$E\left(V; V_{B,2}, \xi_2\right) \geq 0 \text{ for all } V \geq V_{B,2}.$$

A.3 Proof of Proposition 3

When the firm is only financed by one class of debt, and $\xi\beta_i = 0$, this setting is exactly Leland and Toft (1996). After translating to our notation, page 993 in Leland and Toft (1996) gives the endogenous default boundary as

$$V_B = \frac{\frac{C}{r}\left(\frac{A^{LT}}{rm} + B^{LT}\right) - P\frac{A^{LT}}{rm} - \frac{\tau C \gamma}{r}}{1 + (1-\alpha)\gamma + \alpha B^{LT}},$$

where

$$\begin{aligned} A^{LT} &= 2ae^{-rm} N\left(a\sigma\sqrt{m}\right) - 2zN\left(z\sigma\sqrt{m}\right) \\ &\quad - \frac{2}{\sigma\sqrt{m}} n\left(z\sigma\sqrt{m}\right) + \frac{2e^{-rm}}{\sigma\sqrt{m}} n\left(a\sigma\sqrt{m}\right) + z - a; \\ B^{LT} &= \left(2z + \frac{2}{z\sigma^2 m}\right) N\left(z\sigma\sqrt{m}\right) + \frac{2}{\sigma\sqrt{m}} n\left(z\sigma\sqrt{m}\right) - z + a - \frac{1}{z\sigma^2 m}. \end{aligned}$$

With potential abuse of notation, we denote V_B as a function of maturity m. If $P = \frac{C}{r}$,

$$V_B = \frac{\frac{C}{r}B^{LT} - \frac{\tau C \gamma}{r}}{1 + (1-\alpha)\gamma + \alpha B^{LT}} = \frac{\frac{C}{r} - \frac{\tau C \gamma}{r B^{LT}}}{\frac{1+(1-\alpha)\gamma}{B^{LT}} + \alpha}.$$

Note that $\frac{\partial V_B}{\partial m}$ has the same sign as $\frac{\partial B^{LT}}{\partial m}$. Then, simple algebra shows that (note that $z\sigma\sqrt{m} > 0$)

$$\frac{\partial B^{LT}}{\partial m} = \frac{1}{z\sigma^2 m^2}\left(1 - 2N\left(z\sigma\sqrt{m}\right)\right) < 0,$$

Therefore, $V_B(m_1) > V_B(m_2)$.

A.4 Proof of Proposition 4

We have

$$\begin{aligned} V_B(\lambda_1) &= \frac{\frac{(1-\tau_c)C+\sum\left(1-e^{-r_i m_i}\right)\left[\frac{P_i}{m_i}-\frac{C_i}{r_i m_i}\right]}{\eta} + \sum_i \left\{ \begin{array}{c} \left(\frac{P_i}{m_i} - \frac{C_i}{r_i m_i}\right)\left[\frac{1}{\eta}b(-a) + \frac{1}{\gamma}b(a)\right] \\ +\frac{C_i}{r_i m_i}\left[B_i(-z_i) + B_i(z_i)\right] \end{array} \right\}}{\frac{\phi}{\eta-1} + \sum_i \frac{1}{m_i}\lambda_i\alpha\left[B_i(-z_i) + B_i(z_i)\right]} \\ &= V_B(1)\frac{\lambda_1\left[\frac{\phi}{\eta-1} + \frac{\alpha[B_1(-z_1)+B_1(z_1)]}{m_1}\right]}{\frac{\phi}{\eta-1} + \sum_i \frac{1}{m_i}\lambda_i\alpha\left[B_i(-z_i) + B_i(z_i)\right]} \\ &\quad + V_B(0)\frac{\lambda_2\left[\frac{\phi}{\eta-1} + \frac{\alpha[B_2(-z_2)+B_2(z_2)]}{m_2}\right]}{\frac{\phi}{\eta-1} + \sum_i \frac{1}{m_i}\lambda_i\alpha\left[B_i(-z_i) + B_i(z_i)\right]}. \end{aligned}$$

Define

$$w(\lambda_1) \equiv \frac{\lambda_1\left[\frac{\phi}{\eta-1} + \frac{\alpha[B_1(-z_1)+B_1(z_1)]}{m_1}\right]}{\frac{\phi}{\eta-1} + \sum_i \frac{1}{m_i}\lambda_i\alpha\left[B_i(-z_i) + B_i(z_i)\right]},$$

then we can write $V_B(\lambda_1) = V_B(1)\,w(\lambda_1) + V_B(0)\left(1 - w(\lambda_1)\right)$. Because

$$\begin{aligned} \frac{1}{w(\lambda_1)} &= \frac{\frac{\phi}{\eta-1} + \sum_i \frac{1}{m_i}\lambda_i\alpha\left[B_i(-z_i) + B_i(z_i)\right]}{\lambda_1\left[\frac{\phi}{\eta-1} + \frac{\alpha[B_1(-z_1)+B_1(z_1)]}{m_1}\right]} \\ &= 1 + \frac{1-\lambda_1}{\lambda_1}\frac{\frac{\phi}{\eta-1} + \frac{\alpha[B_2(-z_2)+B_2(z_2)]}{m_2}}{\frac{\phi}{\eta-1} + \frac{\alpha[B_1(-z_1)+B_1(z_1)]}{m_1}} \end{aligned}$$

is decreasing in λ_1, $w(\lambda_1)$ is increasing in λ_1. Therefore our claim follows.

A.5 Proof of Proposition 5

The proof of this proposition follows directly from the proof of Proposition 2, as β_1 and β_2 play the same role as ξ in each step provided in Appendix A.2.

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