
Analyzing the Spectrum of Asset Returns: Jump and Volatility Components in High Frequency Data

Yacine Aït-Sahalia
Princeton University

Jean Jacod
Université de Paris VI

1. Introduction

- Basic framework: X , often the log of an asset price, is assumed to follow an **Itô semimartingale**.
- A semimartingale can be decomposed into the **sum** of a **drift**, a **continuous Brownian-driven part** and a discontinuous, or **jump**, part.
 - The jump part can in turn be decomposed into a sum of **small jumps** and **big jumps**.
 - Such a process will always generate a **finite number of big jumps**.
 - But it may give rise to **either a finite or infinite number of small jumps**.

- The model is

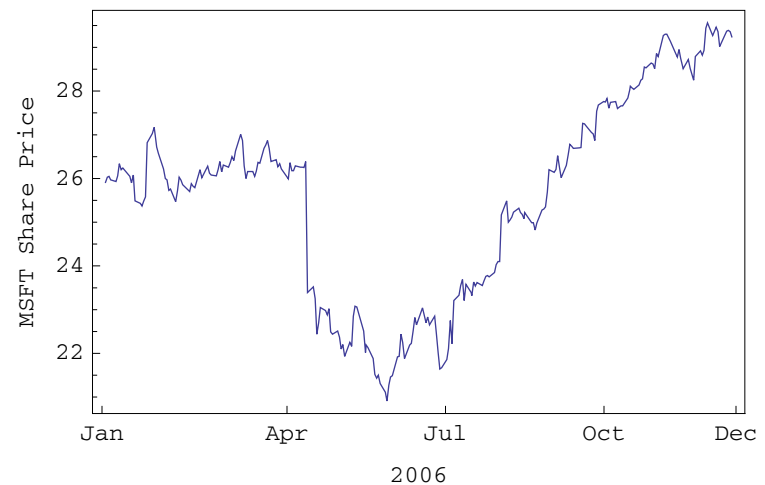
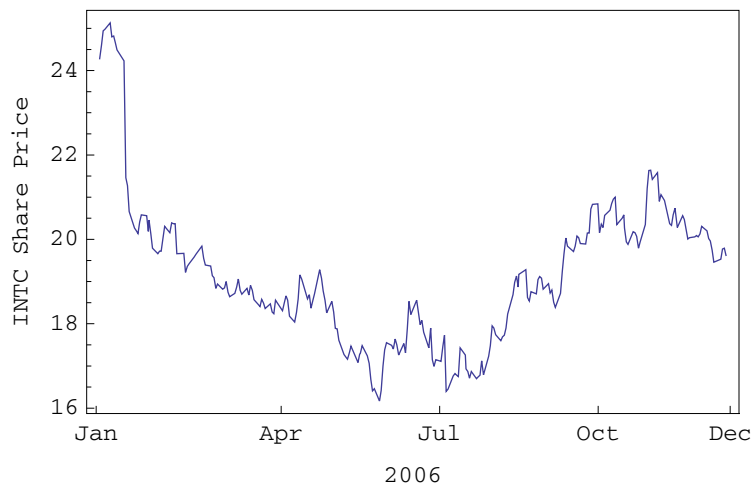
$$X_t = X_0 + \underbrace{\int_0^t b_s ds}_{\text{drift}} + \underbrace{\int_0^t \sigma_s dW_s}_{\text{continuous part}} + \text{JUMPS}$$

$$\text{JUMPS} = \underbrace{\int_0^t \int_{\{|x| \leq \varepsilon\}} x(\mu - \nu)(ds, dx)}_{\text{small jumps}} + \underbrace{\int_0^t \int_{\{|x| > \varepsilon\}} x\mu(ds, dx)}_{\text{big jumps}}$$

- μ is the jump measure of X , and its predictable compensator is the Lévy measure ν .
- The distinction between small and big jumps (ε) is arbitrary. What is important is that $\varepsilon > 0$ is fixed.

- In earlier work, we developed **tests** to determine on the basis of the observed sampled path on $[0, T]$:
 - whether a **jump part was present**
 - whether the jumps had **finite or infinite activity**
 - in the latter situation proposed a definition and an **estimator** of a **degree of jump activity** parameter
 - whether a Brownian **continuous component was needed** once infinite activity jumps are included
- In this talk, we show how these different results can be put in a common framework using a common methodology.

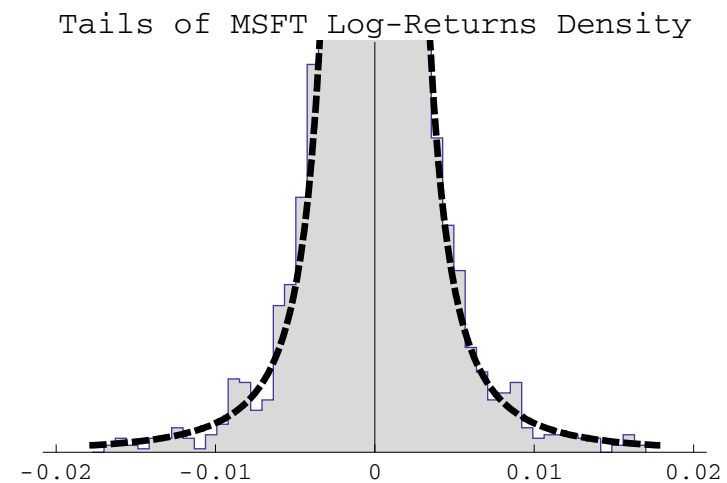
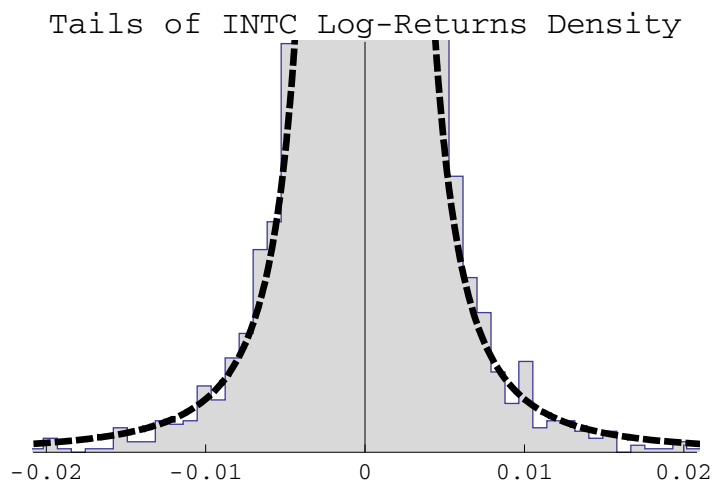
- We proceed by analogy with a **spectrographic analysis**
- We observe a **time series of high frequency returns** (a single path) over a finite length of time $[0, T]$
- For example, 2006 returns on MSFT and INTC



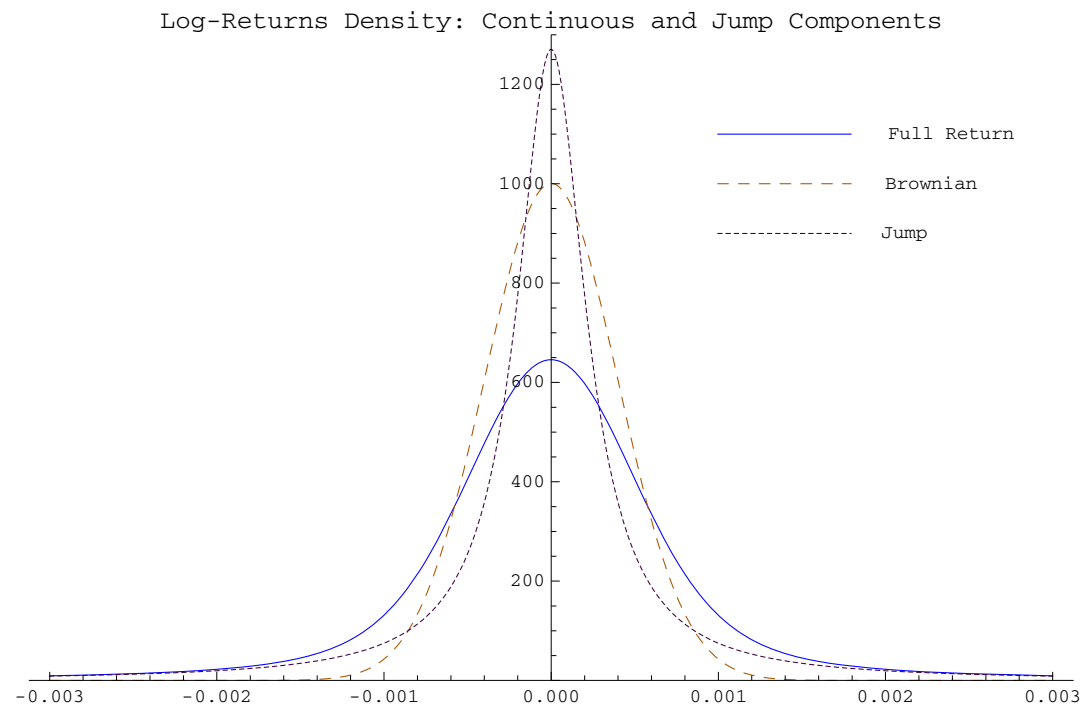
- And then design a set of statistical tools that can tell us something about **specific components of the process that produced the observations**
- These tools play the role of the **measurement devices** used in astrophysics to analyze the light emanating from a star, for instance
 - our observations are the **high frequency returns**; in astrophysics it's the light (visible or not)
 - here the data generating mechanism is assumed to be a **semimartingale**; in astrophysics it's whatever nuclear reactions inside the star are producing the light

- In astrophysics, one can look at a **specific range of the light spectrum** to learn something about specific chemical elements present in the star
- Here, we design statistics that focus on **specific parts of the distribution** of high frequency returns in order to **learn something about the different components of the semimartingale** that produced those returns
 - decide **which component(s) need to be included** in the model (jumps, finite or infinite activity, continuous component, etc.)
 - determine their **relative magnitude**
 - **magnify specific components** of the model if they are present, so we can **analyze their finer characteristics** (such as the degree of activity of jumps)

- From the time series of returns, we get the **distribution of returns** at time interval Δ_n
- 2006 returns on MSFT and INTC at 15 seconds



- From the previous plot, we would like to figure out which components should be included in the model
- And in what proportions



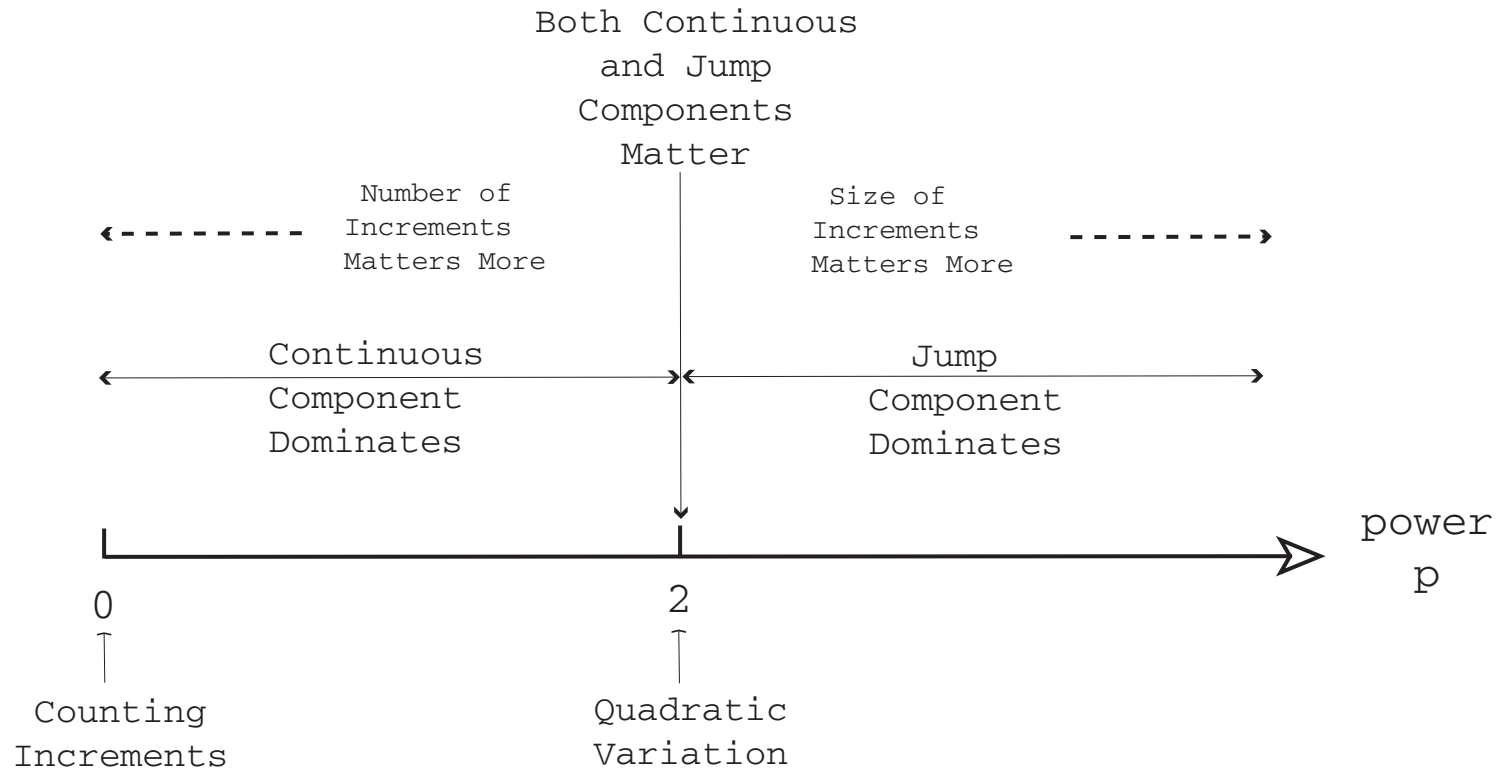
- Similarly to what is done in spectrographic analysis
 - we will emphasize **visual tools**
 - so we will only include the **LLN** here
 - and refer to the underlying papers for the formal derivations including regularity conditions and the **CLT**, as well as simulations.

2. The Measurement Device

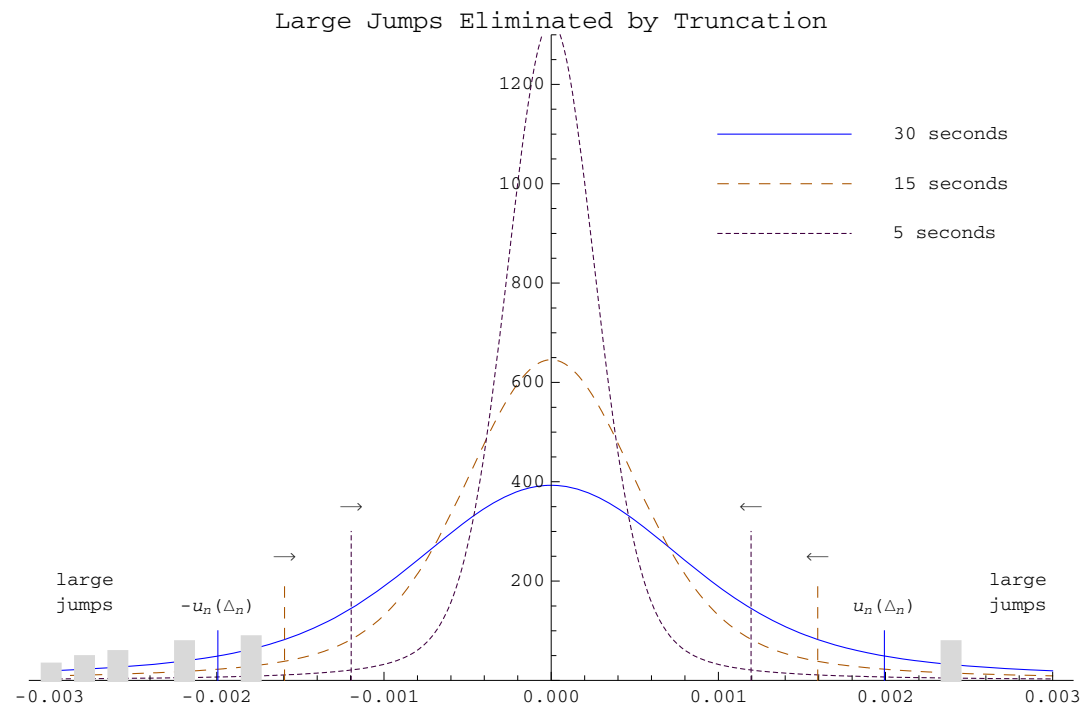
- We construct **power variations** of the increments, **suitably truncated** and/or **sampled at different frequencies**.
- We exploit the different asymptotic behavior of the variations as we vary:
 - the power p
 - the truncation level u
 - the sampling frequency Δ

- This gives us **three degrees of freedom**, or **tuning parameters**, with enough flexibility to isolate what we are looking for.
- Having these three parameters to play with, p , u and Δ , is like having three knobs to adjust in the measurement device.

- Varying the **power**
 - Powers $p < 2$ will emphasize the continuous component of the underlying sampled process.
 - Powers $p > 2$ will conversely accentuate its jump component.
 - The power $p = 2$ puts them on an equal footing.

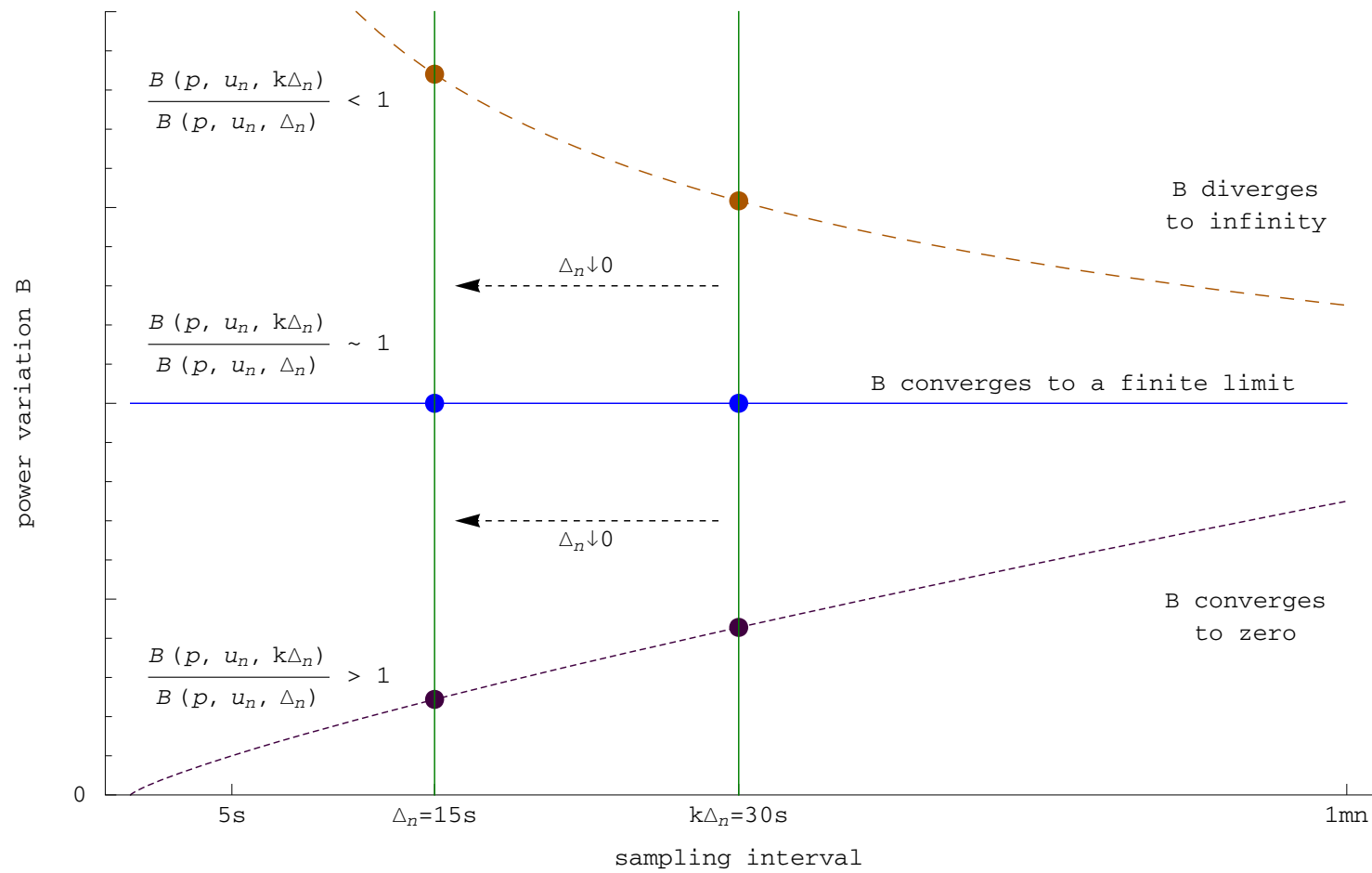


- **Truncating** the large increments at a suitably selected cutoff level can eliminate the big jumps when needed
- Early use of this device: Mancini (2001)



- **Sampling at different frequencies** can let us distinguish between situations where the variations:
 - converge to a finite limit;
 - converge to zero;
 - diverge to infinity.

Ratios of Power Variations at Two Frequencies
to Identify the Asymptotic Behavior of B



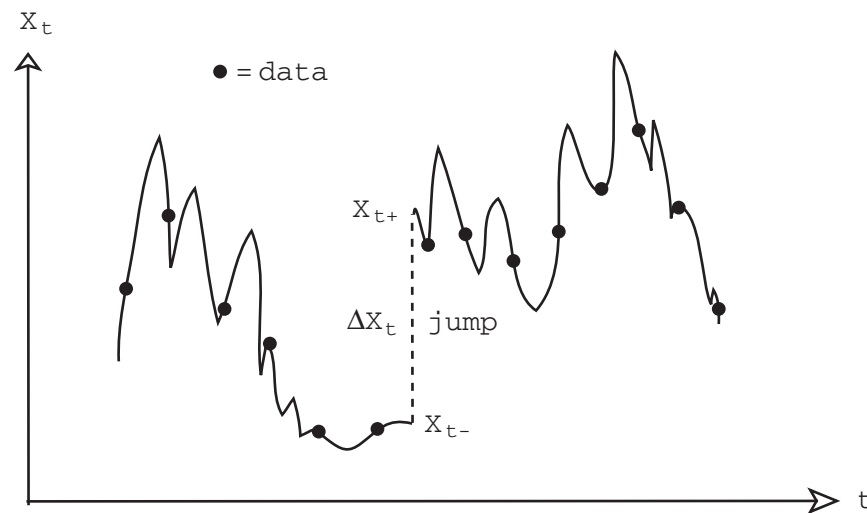
- These **various limiting behaviors** of the variations are indicative of **which component of the model dominates at a particular power** and **in a certain range of returns (by truncation)**
- Just like certain chemical elements have a very specific **spectrographic signature**.
- So they effectively allow us to distinguish between all manners of null and alternative hypotheses.

- There are n **observed increments** of X on $[0, T]$, which are

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n},$$

to be contrasted with the **actual (unobservable) jumps** of X :

$$\Delta X_s = X_s - X_{s-}$$



- For any real $p \geq 0$, the basic instruments are the sum of the p^{th} power of the increments of X , sampled at time interval Δ_n , and truncated at level u_n :

$$B(p, u_n, \Delta_n) = \sum_{i=1}^{[T/\Delta_n]} |\Delta_i^n X|^p \mathbf{1}_{\{|\Delta_i^n X| \leq u_n\}}$$

- The entire methodology relies only on the computation of B for various values of (p, u_n, Δ_n) :

$$B(p, u, \text{del}) = \text{sum}((\text{abs}(dX(\text{del})) . \hat{p}) . * (\text{abs}(dX(\text{del})) \leq u(\text{del})))$$

- T is fixed, asymptotics are all with respect to $\Delta_n \rightarrow 0$.
- u_n is the **cutoff** level for truncating the increments
- $u_n \rightarrow 0$ when $n \rightarrow \infty$: in the form $u_n = \alpha \Delta_n^\varpi$ for some $\varpi \in (0, 1/2)$.
- $\varpi < 1/2$ to keep all the increments which contain a Brownian contribution.
- There will be further restrictions on the rate at which $u_n \rightarrow 0$, expressed in the form of restrictions on the choice of ϖ .
- If we don't want to truncate, we write $B(p, \infty, \Delta_n)$.

- Sometimes we will truncate in the other direction, that is retain only the increments **larger than u** :

$$U(p, u_n, \Delta_n) = \sum_{i=1}^{[T/\Delta_n]} |\Delta_i^n X|^p \mathbf{1}_{\{|\Delta_i^n X| > u_n\}}.$$

- With $u_n = \alpha \Delta_n^\varpi$ and $\varpi < 1/2$, that can allow us to **eliminate** all the increments from the continuous part of the model.
- In terms of the power variations B :

$$U(p, u_n, \Delta_n) = B(p, \infty, \Delta_n) - B(p, u_n, \Delta_n).$$

- Sometimes, we will simply **count the number of increments** of X , that is, take the power $p = 0$

$$U(0, u_n, \Delta_n) = \sum_{i=1}^{\lceil T/\Delta_n \rceil} \mathbf{1}_{\{|\Delta_i^n X| > u_n\}}.$$

3. Which Component(s) Are Present

- Leaving aside the drift (effectively invisible at high frequency), the model has **three components**

$$X_t = X_0 + \underbrace{\int_0^t b_s ds}_{\text{drift}} + \underbrace{\int_0^t \sigma_s dW_s}_{\text{continuous part}} + \text{JUMPS}$$

$$\text{JUMPS} = \underbrace{\int_0^t \int_{\{|x| \leq 1\}} x(\mu - \nu)(ds, dx)}_{\text{small jumps}} + \underbrace{\int_0^t \int_{\{|x| > 1\}} x\mu(ds, dx)}_{\text{big jumps}}$$

- The analogy with spectrography would be that we are looking for three possible chemical elements (say, hydrogen, helium and everything else).

- Consider the sets

$$\begin{aligned}
 \Omega_T^c &= \{X \text{ is continuous in } [0, T]\} \\
 \Omega_T^j &= \{X \text{ has jumps in } [0, T]\} \\
 \Omega_T^f &= \{X \text{ has finitely many jumps in } [0, T]\} \\
 \Omega_T^i &= \{X \text{ has infinitely many jumps in } [0, T]\} \\
 \Omega_T^W &= \{X \text{ has a Wiener component in } [0, T]\} \\
 \Omega_T^{\text{no}W} &= \{X \text{ has no Wiener component in } [0, T]\}
 \end{aligned}$$

- Formally, $\Omega_T^W = \left\{ \int_0^T \sigma_s^2 ds > 0 \right\}$ and $\Omega_T^{\text{no}W} = \left\{ \int_0^T \sigma_s^2 ds = 0 \right\}$.

- We observe a time series and wish to determine in which set(s) the path was.
- There are theoretically many possible ways to do this, even if we restrict attention to power variations only.
- However, we wish to construct test statistics that are **model-free** in the sense that:
 - their implementation does **not require** that we estimate or calibrate the model, which can potentially be quite complicated (stochastic volatility, jumps, jumps in volatility, jumps in jump intensity, etc.)
 - so we want the distribution of the test statistics to be assessed using **only power variations** (of perhaps other powers, truncation levels and sampling frequencies)

3.1. Jumps: Present or Not

- Here are processes which measure **some kind of variability** of X and depend on the whole (unobserved) path of X :

$$A(p) = \int_0^T |\sigma_s|^p ds, \quad B(p) = \sum_{s \leq T} |\Delta X_s|^p$$

where $p > 0$ and $\Delta X_s = X_s - X_{s-}$ are the **jumps of X** .

- $A(p)$ is finite for all $p > 0$. $B(p)$ is finite if $p \geq 2$ but often not when $p < 2$.
- The quadratic variation of the process is $[X, X]_T = A(2) + B(2)$.

- We have

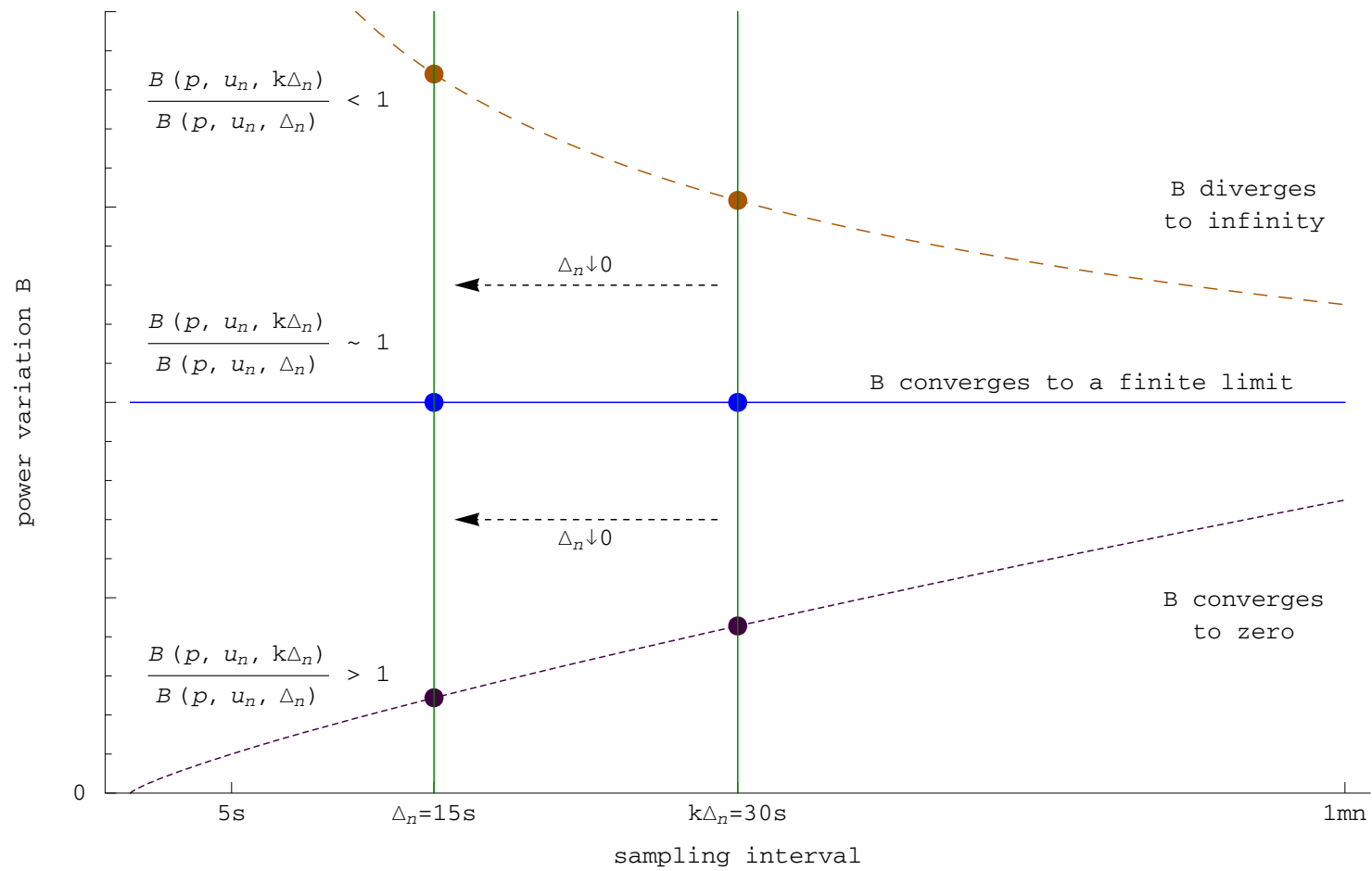
$$\begin{cases} p > 2, \text{ all } X & \Rightarrow B(p, \infty, \Delta_n) \xrightarrow{\mathbb{P}} B(p) \\ \text{all } p, X \text{ continuous} & \Rightarrow \frac{\Delta_n^{1-p/2}}{m_p} B(p, \infty, \Delta_n) \xrightarrow{\mathbb{P}} A(p). \end{cases}$$
- We see that, when $p > 2$, $B(p, \infty, \Delta_n)$ tends to $B(p)$: the jump component dominates.
- If there are jumps, the limit $B(p)_t > 0$ is finite.
- On the other hand when X is continuous, then the limit is $B(p) = 0$ and $B(p, \infty, \Delta_n)_t$ converges to 0 at rate $\Delta_n^{p/2-1}$.

- These considerations lead us to pick a value of $p > 2$ and compare $B(p, \infty, \Delta_n)_t$ on two different sampling frequencies.
- Specifically, for an integer k , consider the test statistic S_J :

$$S_J(p, k, \Delta_n) = \frac{B(p, \infty, k\Delta_n)_T}{B(p, \infty, \Delta_n)_T}.$$

- The ratio in S_J exhibits a markedly different behavior depending upon whether X has jumps or not.

Ratios of Power Variations at Two Frequencies
to Identify the Asymptotic Behavior of B



- Theorem

$$S_J(p, k, \Delta_n)_t \rightarrow \begin{cases} 1 & \text{on } \Omega_T^j \\ k^{p/2-1} & \text{on } \Omega_T^c \end{cases}$$

- This is valid on Ω_T^j whether the jump component include finite or infinite components, or both.
- We provide a CLT under Ω_T^c and one under Ω_T^j , so one can test either $H_0 : \Omega_T^c$ vs. $H_1 : \Omega_T^j$ or the reverse $H_0 : \Omega_T^j$ vs. $H_1 : \Omega_T^c$.

3.2. Jumps: Finite or Infinite Activity

- Many models in mathematical finance do **not** include jumps.
- But among those that do, the framework most often adopted consists of a **jump-diffusion**: these models include a drift term, a Brownian-driven continuous part, and a finite activity jump part (compound Poisson process): early examples include Merton (1976), Ball and Torous (1983) and Bates (1991).
- Other models are based on **infinite activity jumps**: see for example Madan and Seneta (1990), Eberlein and Keller (1995), Barndorff-Nielsen (1998), Carr, Geman, Madan and Yor (2002), Carr and Wu (2003), etc.

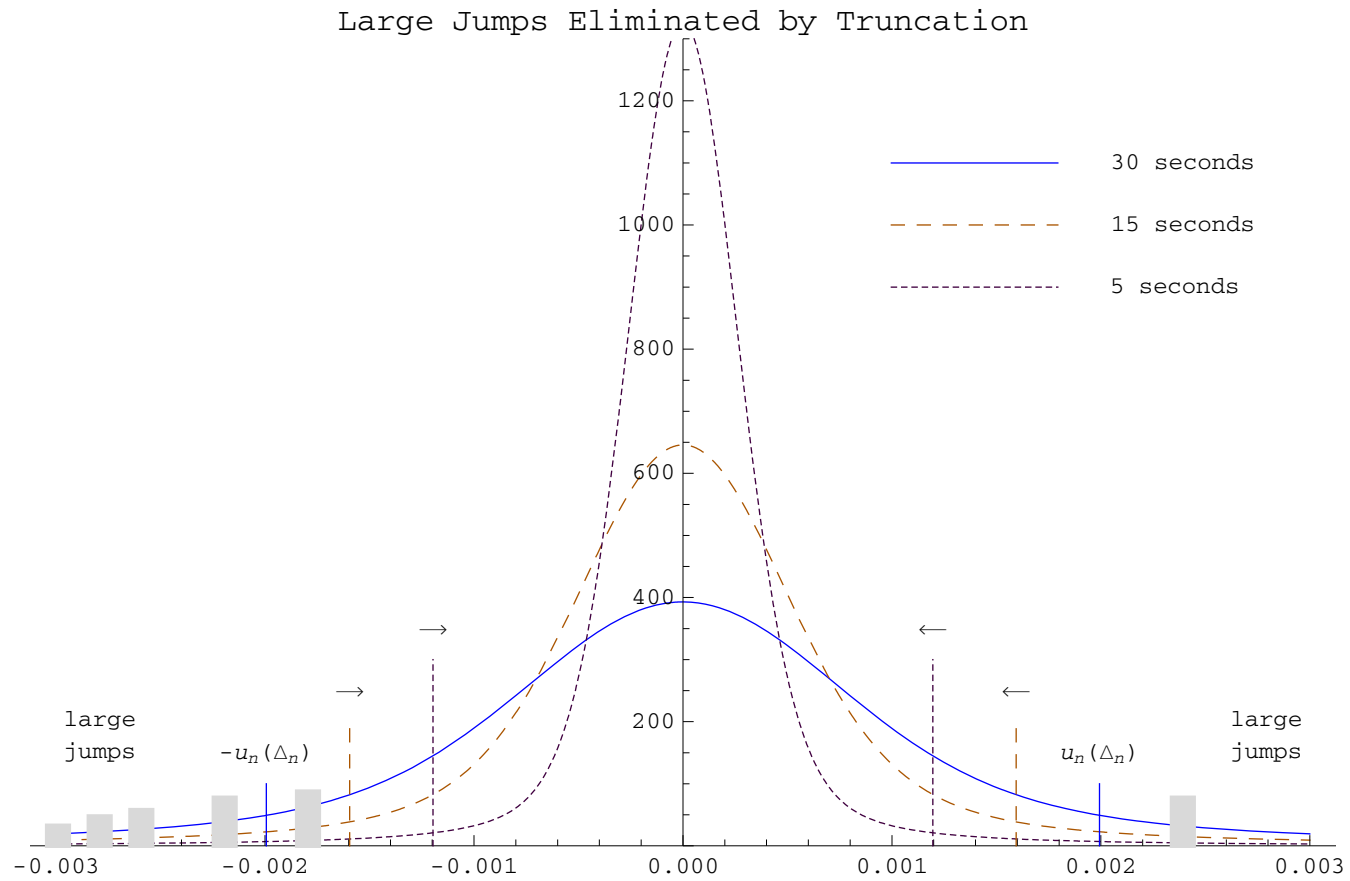
3.2.1. Null Hypothesis: Finite Activity

- We first set the null hypothesis to be finite activity, that is $H_0 : \Omega_T^f \cap \Omega_T^W$, whereas the alternative is $H_1 : \Omega_T^i$.
- We choose an integer $k \geq 2$ and a real $p > 2$.
- The only difference is that we now truncate

$$S_{FA}(p, u_n, k, \Delta_n) = \frac{B(p, u_n, k\Delta_n)}{B(p, u_n, \Delta_n)}.$$

- Without truncation, we could discriminate between jumps and no jumps, but not among **different types of jumps**.

- Like before, we set $p > 2$ to **magnify the jump component**.
- But since we want to separate big and small jumps, we now **truncate** as a means of **eliminating the large jumps**.
- Since the large jumps are of finite size (independent of Δ_n), at some point in the asymptotics $\Delta_n \downarrow 0$, the truncation level $u_n = O(\Delta_n^\rho)$ will have eliminated all the large jumps.



- Then if there are only big jumps and the Brownian component, the two power variations $B(p, u_n, k\Delta_n)$ and $B(p, u_n, \Delta_n)$ will behave as if there were no jumps and the limit of the ratio will be 2 as in the test for jumps.
- But if there are small jumps, then the truncation cannot eliminate them because their size is Δ_n -dependent then each B truncated tends to the small of remaining jumps and the ratio tends to 1.

- Theorem: Under regularity conditions on u_n ,

$$S_{FA}(p, u_n, k, \Delta_n) \xrightarrow{\mathbb{P}} \begin{cases} k^{p/2-1} & \text{on } \Omega_T^f \cap \Omega_T^W. \\ 1 & \text{on } \Omega_T^i. \end{cases}$$

3.2.2. Null Hypothesis: Infinite Activity

- We next set the null hypothesis to be infinite activity, that is $H_0 : \Omega_T^i$, whereas the alternative is $H_1 : \Omega_T^f \cap \Omega_T^W$.
- Why do we need different statistics? Because the distribution of S_{FA} is not model-free under Ω_T^i , and that of S_{IA} is not model-free under $\Omega_T^f \cap \Omega_T^W$.
- We choose three reals $\gamma > 1$ and $p' > p > 2$ and define a family of test statistics as follows:

$$S_{IA}(p, u_n, \gamma, \Delta_n) = \frac{B(p', \gamma u_n, \Delta_n) B(p, u_n, \Delta_n)}{B(p', u_n, \Delta_n) B(p, \gamma u_n, \Delta_n)}.$$

- Theorem: Under regularity conditions on u_n ,

$$S_{IA}(p, u_n, \gamma, \Delta_n) \xrightarrow{\mathbb{P}} \begin{cases} \gamma^{p'-p} & \text{on } \Omega_T^i \\ 1 & \text{on } \Omega_T^f \cap \Omega_T^W \end{cases}$$

3.3. Brownian Motion: Present or Not

- We would like to construct procedures which allow to:
 - decide whether the Brownian motion is really there
 - or if it can be forgone with in favor of a pure jump process with infinite activity.
- When infinitely many jumps are included, there are a number of models in the literature which dispense with the Brownian motion altogether. The log-price process is then a purely discontinuous Lévy process with infinite activity jumps, or more generally is driven by such a process: see for example Madan and Seneta (1990), Eberlein and Keller (1995), Carr, Geman, Madan and Yor (2002), Carr and Wu (2003), etc.

3.3.1. Null Hypothesis: Brownian Motion Present

- In order to construct a test, we seek a statistic with markedly different behavior under the null and alternative.
- The idea is now to consider **powers less than 2**
 - since in the presence of Brownian motion the power variation would be dominated by it
 - while in its absence it would behave quite differently.

- Specifically, the **large number of small increments** generated by a continuous component would cause a power variation of order less than 2 to diverge to infinity.
- Without the Brownian motion, however, and when $p > \beta$, the power variation converges to 0 at exactly the same rate for the two sampling frequencies Δ_n and $k\Delta_n$
- Whereas with a Brownian motion the choice of sampling frequency will influence the magnitude of the divergence.
- Taking a **ratio** will eliminate all unnecessary aspects of the problem and focus on that key aspect.

- We choose an integer $k \geq 2$ and a real $p < 2$.
- We propose the test statistic

$$S_W(p, u_n, k, \Delta_n) = \frac{B(p, u_n, \Delta_n)}{B(p, u_n, k\Delta_n)}.$$

- Theorem: Under regularity conditions on u_n ,

$$S_W(p, u_n, k, \Delta_n) \xrightarrow{\mathbb{P}} \begin{cases} k^{1-p/2} & \text{on } \Omega_T^W \\ 1 & \text{on } \Omega_T^{\text{no}W} \cap \Omega_T^i, \quad p > \beta \end{cases}$$

- An alternative: Tauchen and Todorov (2009)
- They employ the test statistic for jumps S_J , plot its logarithm for different values of the power argument and contrast the behavior of the plot above two and below two in order to identify the presence of a Brownian component.
- This works when there is a Brownian motion is present under the null.

3.3.2. Null Hypothesis: No Brownian Motion

- The null model is now **pure jump** (plus perhaps a drift) with jumps.
 - When there are no jumps, or finitely many jumps, and no Brownian motion, X reduces to a pure drift plus occasional jumps, and such a model is fairly unrealistic in the context of most financial data series.
 - But one can certainly consider models that consist **only of a jump component**, plus perhaps a drift, **if that jump component is allowed to be infinitely active**.

- Designing a test under this null is trickier
 - because we are aiming for a test that **remains model-free** even for this model.
 - that is, despite being driven by what is now a pure jump process, the behavior of the statistic should **not depend** on the characteristics of the pure jump process
 - such as for instance its **degree of activity β**
 - since those characteristics are a priori unknown.

- We choose a real $\gamma > 1$ to define two different truncation ratios
- And define a family of test statistics as follows:

$$S_{\text{no}W}(p, u_n, \gamma, \Delta_n) = \frac{B(2, \gamma u_n, \Delta_n) U(0, u_n, \Delta_n)}{B(2, u_n, \Delta_n) U(0, \gamma u_n, \Delta_n)}.$$

- To understand the construction of this test statistic, recall that in a **power variation of order 2** the contributions from the Brownian and jump components are of the **same order**.
- If the Brownian motion is present ($H_1 : \Omega_T^W$)
 - Once that power variation is properly **truncated**, the Brownian motion will dominate it if it is present.
 - And the truncation can be chosen to be sufficiently loose that it retains essentially all the increments of the Brownian motion at cutoff level u_n and a fortiori γu_n , thereby making the ratio of the two truncated quadratic variations converge to 1 under the alternative hypothesis.

- If the Brownian motion is not present ($H_0 : \Omega_T^{\text{no}W}$)
 - Then the nature of the tail of jump distributions is such that the **difference in cutoff levels** between u_n and γu_n **remains material** no matter how far we go in the tail
 - And the limit of the ratio $\frac{B(2, \gamma u_n, \Delta_n)}{B(2, u_n, \Delta_n)}$ in $S_{\text{no}W}$ will reflect it: it will now be $\gamma^{2-\beta}$.
 - Since absence of a Brownian motion is now the null hypothesis, the issue for constructing a test is that this limit depends on the unknown β .

- Canceling out that dependence is the role devoted to the ratio $\frac{U(0, u_n, \Delta_n)}{U(0, \gamma u_n, \Delta_n)}$ of the number of large increments, the U 's.
 - The U 's are always dominated by the jump components of the model whether the Brownian motion is present or not.
 - Their inclusion in the statistic is merely to ensure that the statistic is model-free, by effectively **canceling out the dependence on the jump characteristics** that emerges from the ratio of the truncated quadratic variations.
 - Indeed, the limit of the ratio of the U 's is γ^β under both the null and alternative hypotheses. As a result, the probability limit of $S_{\text{no}W}$ will be γ^2 under the null, independent of β .

- Theorem: Under regularity conditions on u_n ,

$$S_{\text{no}W}(p, u_n, \gamma, \Delta_n) \xrightarrow{\mathbb{P}} \begin{cases} \gamma^2 & \text{on } \Omega_T^{\text{no}W} \cap \Omega_T^i \\ \gamma^\beta & \text{on } \Omega_T^W \end{cases}$$

4. The Relative Magnitude of the Components

- A typical “main sequence” star might be made of 90% hydrogen, 10% helium and 0.1% everything else.
- Here, what is the relative magnitude of the two jump and the continuous components?
- We can answer this question using the same device.
- It makes sense to consider $p = 2$ since this is the power where **all the components are present together**.

- We can then truncate to **split the QV into its continuous and jump components**

- And not truncate to estimate the full QV:

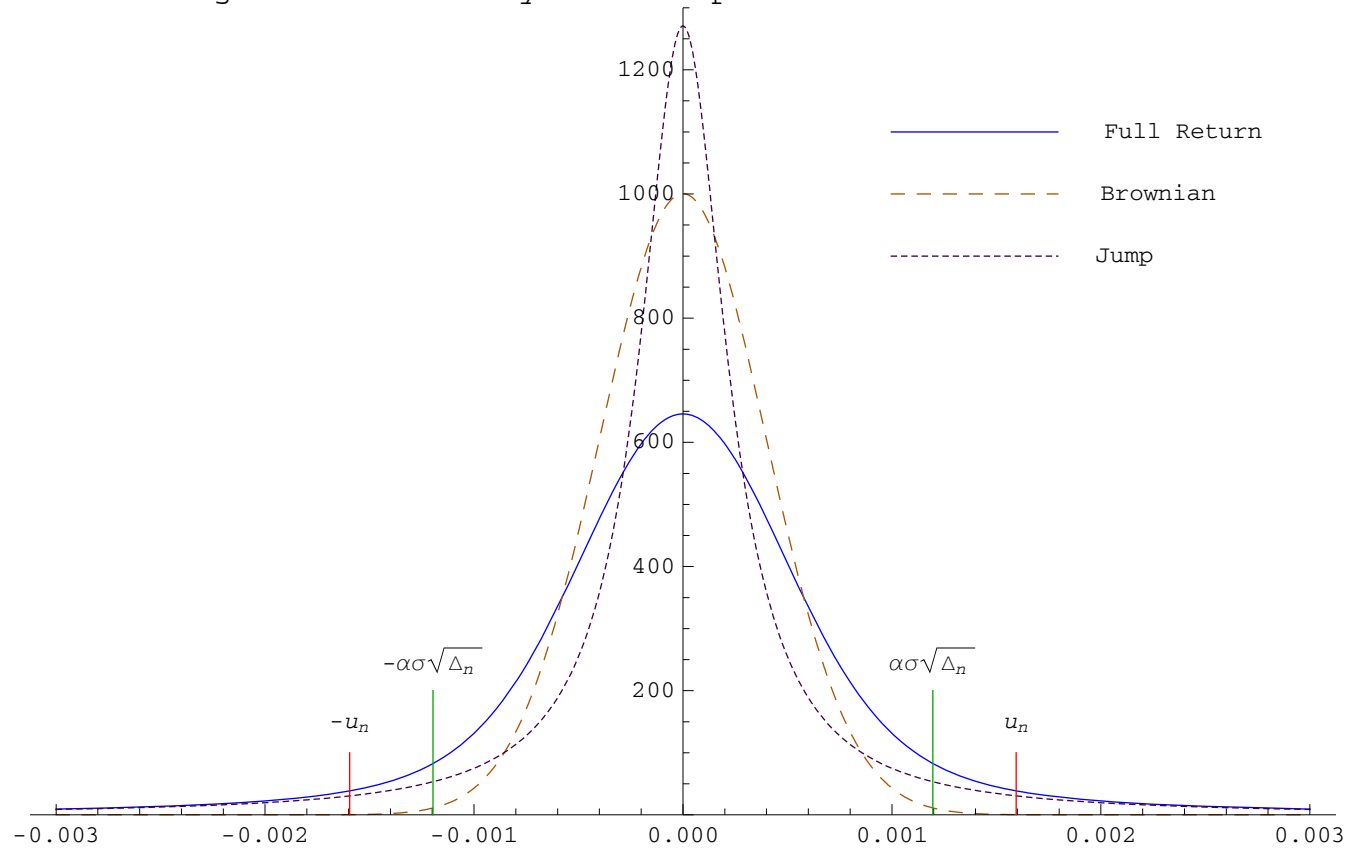
$$\frac{B(2, u_n, \Delta_n)}{B(2, \infty, \Delta_n)} = \% \text{ of QV due to the } \mathbf{continuous} \text{ component}$$

$$1 - \frac{B(2, u_n, \Delta_n)}{B(2, \infty, \Delta_n)} = \% \text{ of QV due to the } \mathbf{jump} \text{ component}$$

- Alternative splitting of the QV based on bipower variation instead of truncating: Barndorff-Nielsen and Shephard (2004), Huang and Tauchen (2005), Andersen, Bollerslev and Diebold (2007).

4 THE RELATIVE MAGNITUDE OF THE COMPONENTS

Log>Returns Density: Sub-Components and Cutoff Levels

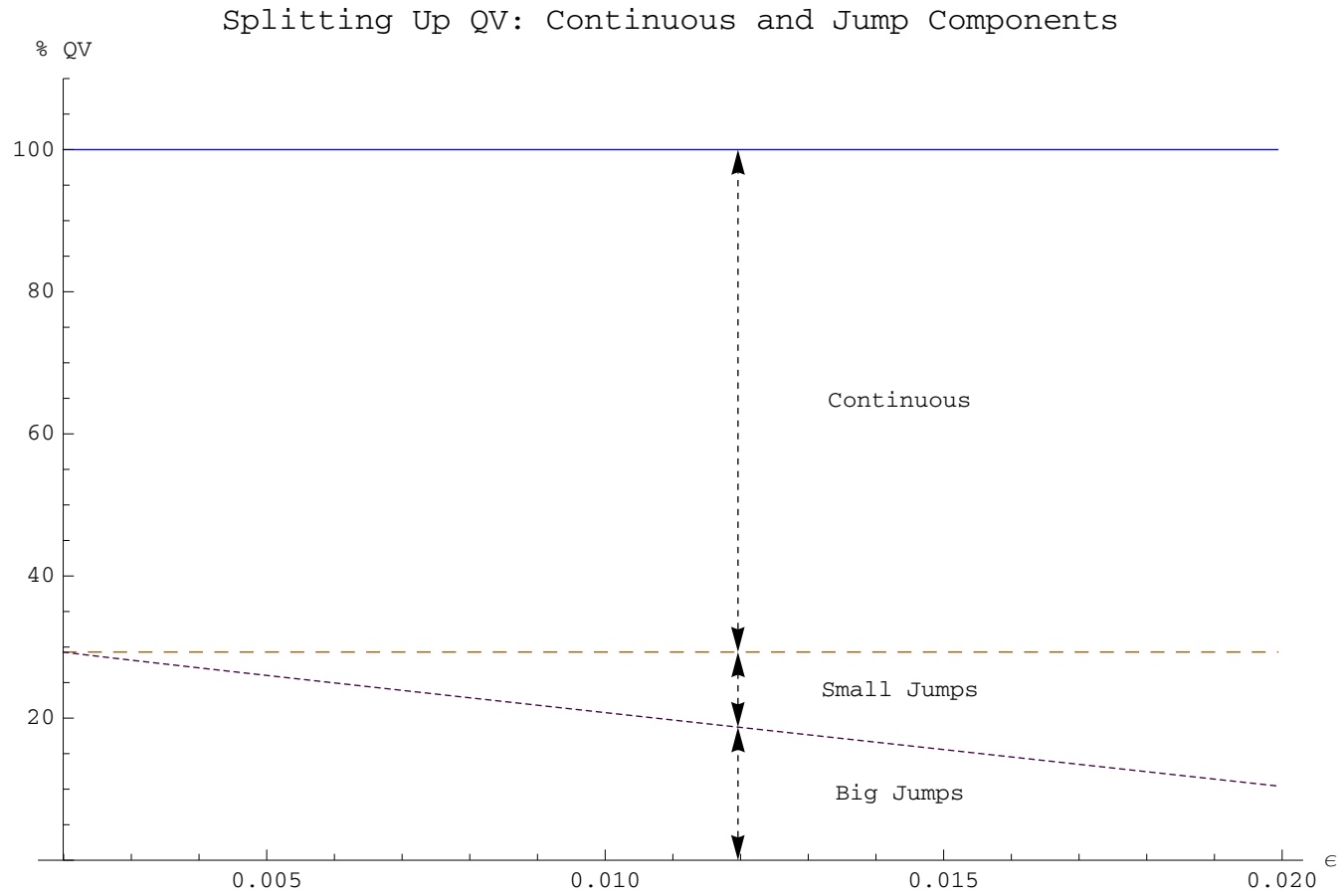


- We can then split the rest of the QV, which by construction is attributable to jumps, into a small jumps and a big jumps component.
- This depends on the cutoff level ε selected to distinguish big and small jumps:

$$\frac{U(2,\varepsilon,\Delta_n)}{B(2,\infty,\Delta_n)} = \% \text{ of QV due to } \mathbf{big \ jumps}$$

$$\frac{B(2,\infty,\Delta_n) - B(2,u_n,\Delta_n) - U(2,\varepsilon,\Delta_n)}{B(2,\infty,\Delta_n)} = \% \text{ of QV due to } \mathbf{small \ jumps}$$

4 THE RELATIVE MAGNITUDE OF THE COMPONENTS



5. The Finer Characteristics of the Components

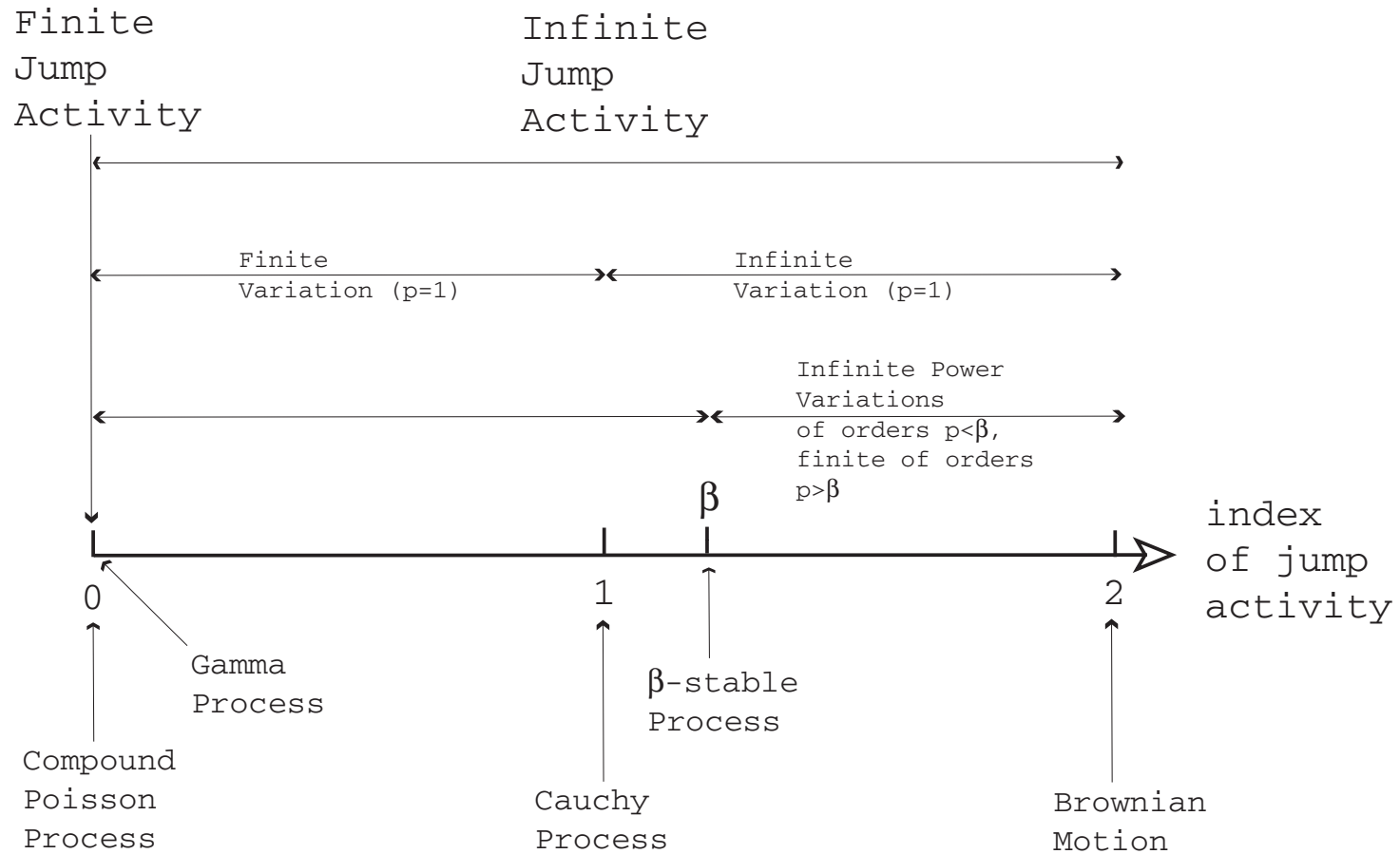
5.1. Defining an Index of Jump Activity

- Recall $B(p) = \sum_{s \leq T} |\Delta X_s|^p$.
- Define $I_T = \{p \geq 0 : B(p) < \infty\}$.
- Necessarily, the (random) set I_T is of the form $[\beta_T, \infty)$ or (β_T, ∞) for some $\beta_T(\omega) \leq 2$, and $2 \in I_T$ always.

- We call $\beta_T(\omega)$ the **jump activity index** for the path $t \mapsto X_t(\omega)$ at time T .
- We define this index in analogy with the special case where X is a Lévy process:
 - Then $\beta_T(\omega) = \beta$ does not depend on (ω, T) , and it is also the infimum of all $r \geq 0$ such that $\int_{\{|x| \leq 1\}} |x|^r \nu(dx) < \infty$, where ν is the Lévy measure
 - This property shows that, for a Lévy process, the jump activity index coincides with the **Blumenthal-Gettoor index** of the process.
 - In the further special case where X is a stable process, then β is also the **stable index** of the process.

- β captures an essential qualitative feature of ν , which is its **level of activity**: when β increases, the (small) jumps tend to become more and more frequent.
 - Processes with finite jump activity have $\beta = 0$.
 - Processes with infinite jump activity may also have $\beta = 0$ if the rate of divergence of the jump measure is sub-polynomial.
 - Processes with $\beta \in (0, 2)$ have infinite jump activity
 - And the higher β , the more active the jumps.
- Brownian motion has $\beta = 2$ in the limit.

5.1 Defining an Index of Jump Activity 5 THE FINER CHARACTERISTICS OF THE COMPONENTS

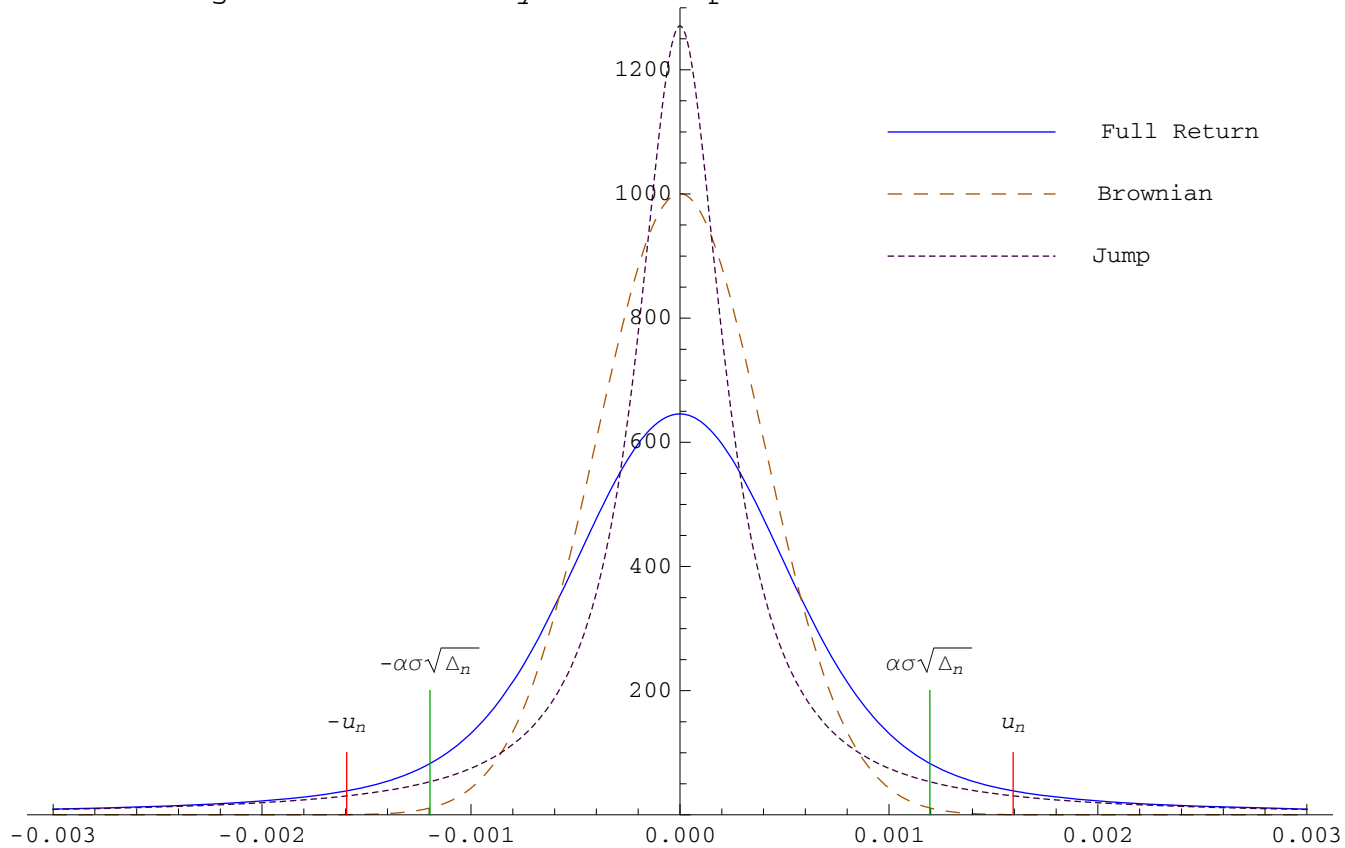


- The problem is made more challenging by the presence in X of a **continuous, or Brownian, martingale part**:
 - β characterizes the **behavior of ν near 0**.
 - Hence it is natural to expect that the **small increments** of the process are going to be the ones that are most informative about β .
 - But that is where the contribution from the **continuous martingale** part of the process is inexorably **mixed** with the contribution from the **small jumps**.
 - We need to **see through the continuous part** of the semimartingale in order to say something about the number and concentration of **small jumps**.

- So we are now looking in a **different range of the spectrum of returns**
- Considering only returns that are larger than the cutoff $u_n = \alpha \Delta_n^{\varpi}$ for some $\varpi \in (0, 1/2)$.
- This allows us to **eliminate** the increments due to the continuous component.
- We can then use all values of p , not just those $p > 2$.

5.1 Defining an Index of Jump Activity 5 THE FINER CHARACTERISTICS OF THE COMPONENTS

Log>Returns Density: Sub-Components and Cutoff Levels



5.2. Estimating Jump Activity

- We propose two estimators of β based on counting the number of increments greater than the cutoff u_n .

- The first one: fix $0 < \alpha < \alpha'$ and consider **two cutoffs** $u_n = \alpha \Delta_n^{\varpi}$ and $u'_n = \alpha' \Delta_n^{\varpi}$ with $\gamma = \alpha'/\alpha$:

$$\hat{\beta}_n(\varpi, \alpha, \alpha') = \frac{\log(U(0, u_n, \Delta_n)/U(0, \gamma u_n, \Delta_n))}{\log(\gamma)},$$

- The second one: sample on **two time scales**, Δ_n and $2\Delta_n$.

$$\hat{\beta}'_n(\varpi, \alpha, k) = \frac{\log(U(0, u_n, \Delta_n)/U(0, u_n, k\Delta_n))}{\varpi \log k}.$$

- Given consistent estimators and with a CLT
- We could test various hypotheses, for instance whether $\beta > 1$ or $\beta < 1$ which correspond to **finite or infinite variation** for X .
- Related methods: testing whether $\beta = 1$ (Cont and Mancini (2009)), testing whether $\beta = 2$ or $\beta < 2$ (Tauchen and Todorov (2009)).

6. Summary: (p, u, Δ)

		Jumps: Present or Not	
		Ω_T^c	Ω_T^j
H_1	H_0		
Ω_T^c		\dots	$S_J :$ $\left(\begin{array}{c} p > 2 \\ \infty \\ \Delta_n, k\Delta_n \end{array} \right)$
Ω_T^j		$S_J :$ $\left(\begin{array}{c} p > 2 \\ \infty \\ \Delta_n, k\Delta_n \end{array} \right)$	\dots

		Jumps: Finite or Infinite Activity	
		Ω_T^f	Ω_T^i
H_1	H_0		
Ω_T^f		\dots	$S_{IA} :$ $\left(\begin{array}{c} p > 2, p' > 2 \\ u_n, \gamma u_n \\ \Delta_n \end{array} \right)$
Ω_T^i		$S_{FA} :$ $\left(\begin{array}{c} p > 2 \\ u_n \\ \Delta_n, k\Delta_n \end{array} \right)$	\dots

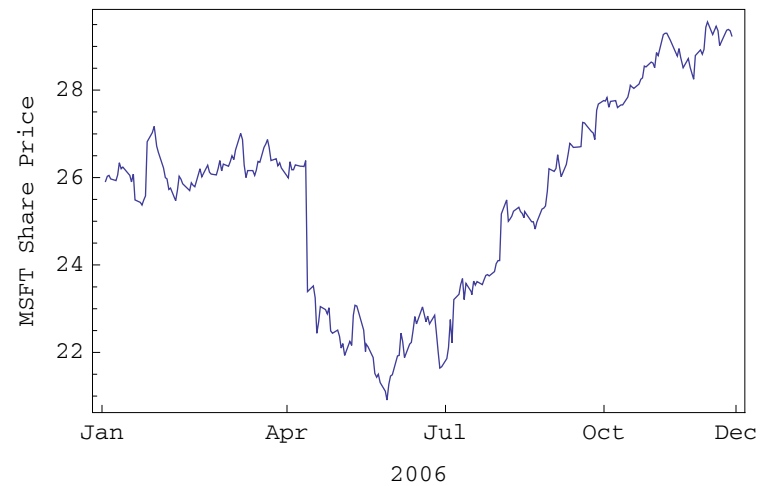
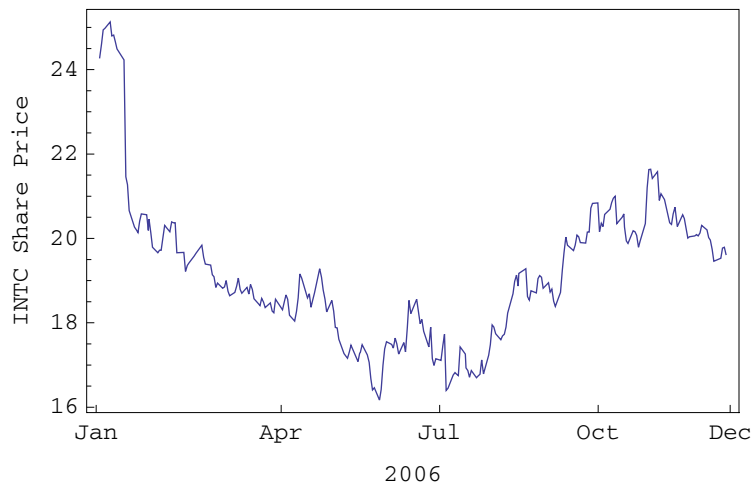
		Brownian Motion: Present or Not	
H_1	H_0	Ω_T^W	$\Omega_T^{\text{no}W}$
Ω_T^W		...	$S_{\text{no}W} :$ $\left(\begin{array}{c} p = 0, p' = 2 \\ u_n, \gamma u_n \\ \Delta_n \end{array} \right)$
$\Omega_T^{\text{no}W}$		$S_W :$ $\left(\begin{array}{c} p < 2 \\ u_n \\ \Delta_n, k\Delta_n \end{array} \right)$...

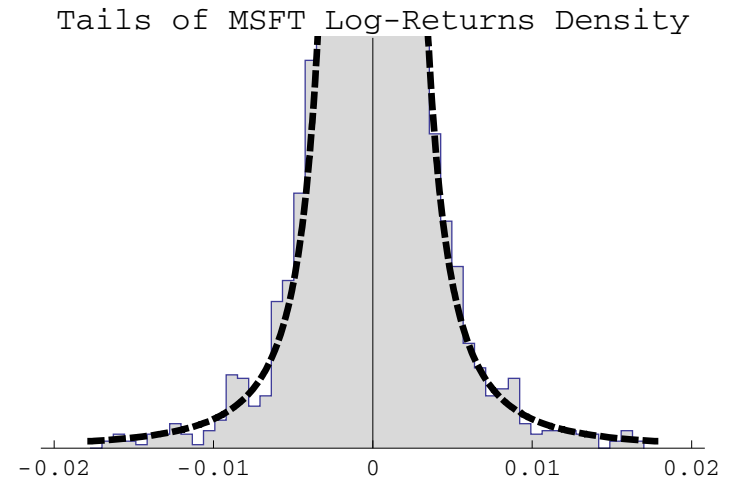
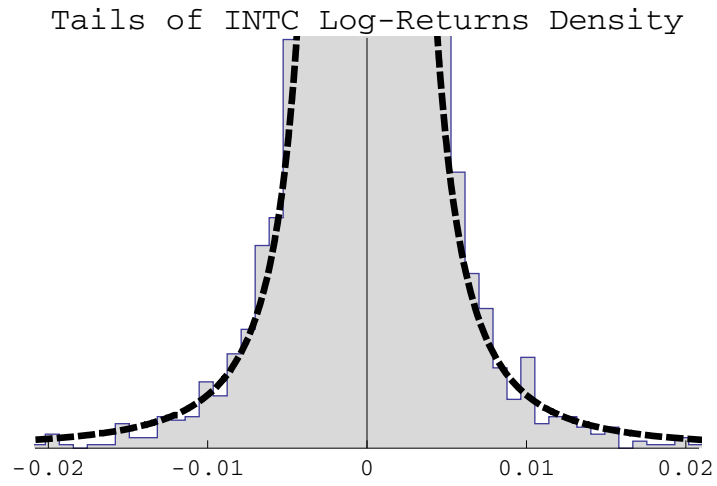
Relative Magnitude of the Components
$\left(\begin{array}{c} p = 2 \\ u_n \\ \Delta_n \end{array} \right)$

Estimating the Degree of Jump Activity β	
$\hat{\beta}$	$\left(\begin{array}{c} p = 0 \\ u_n, \gamma u_n \\ \Delta_n \end{array} \right)$
$\hat{\beta}'$	$\left(\begin{array}{c} p = 0 \\ u_n \\ \Delta_n, k\Delta_n \end{array} \right)$

7. Empirical Results: Intel & Microsoft 2006

7.1. The Data





- Whenever we need to truncate, we express the truncation cutoff level u_n in terms of a number of standard deviations of the continuous part of the semimartingale.
- We consider sampling frequencies up to 5 seconds.
- In real data, observations of the process X are blurred by **market microstructure noise**, which messes things up at **very high frequency**.

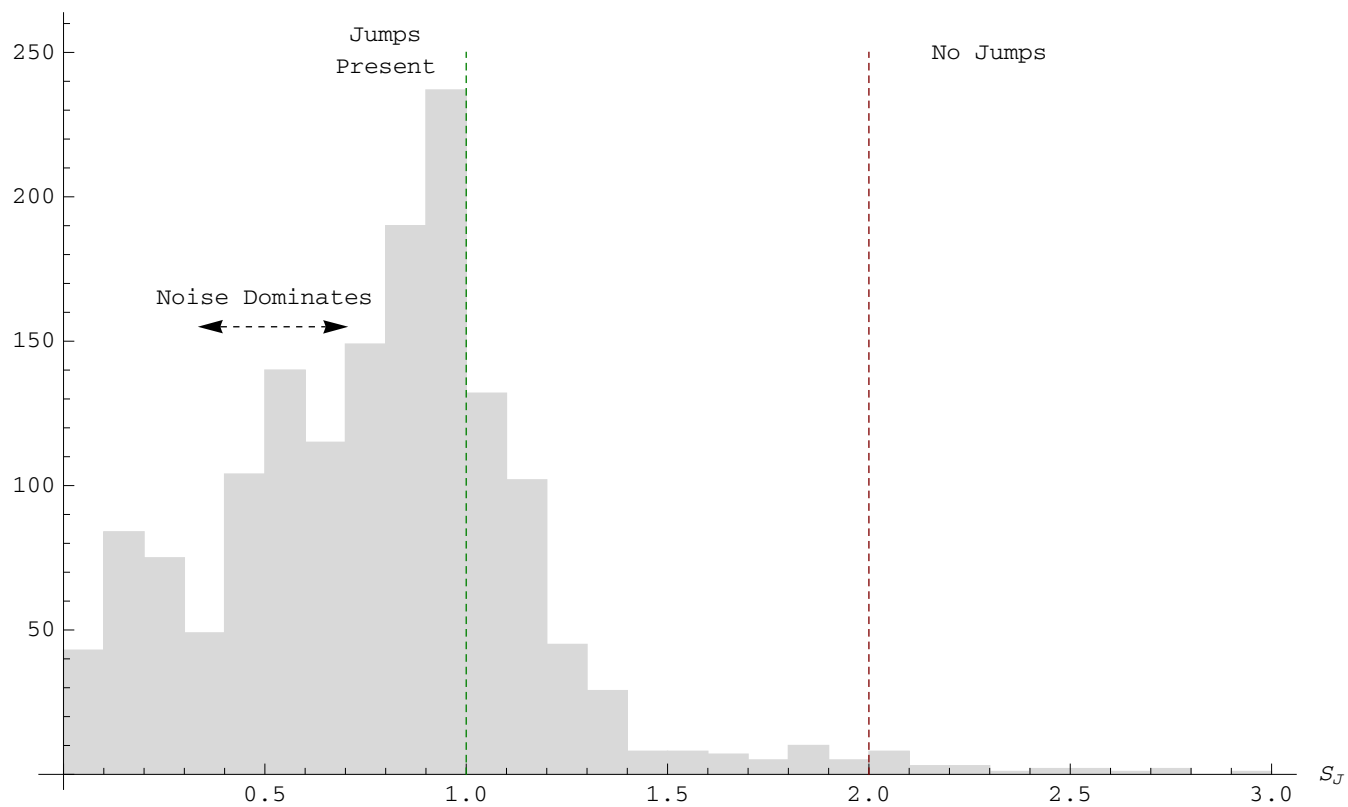
7.2. Jumps: Present or Not

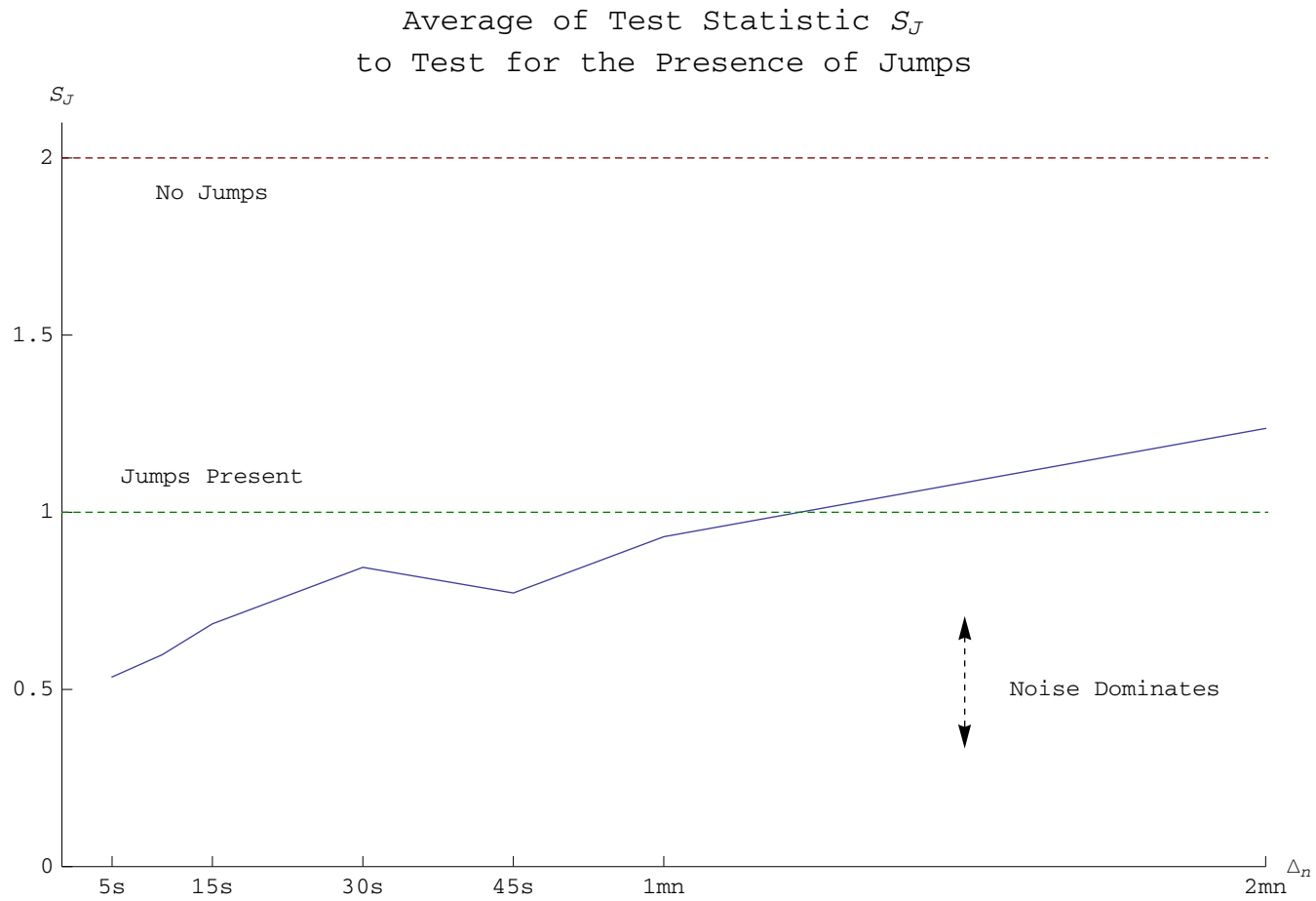
- Two polar cases: observations are blurred with either an **additive white noise** or with **noise due to rounding**
 - Observations are affected by an additive noise, that is instead of $X_{i\Delta_n}$ we really observe $Y_{i\Delta_n} = X_{i\Delta_n} + \varepsilon_i$, and the ε_i are i.i.d. with $E(\varepsilon_i^2)$ and $E(\varepsilon_i^4)$ finite.
 - Or we observe $Y_{i\Delta_n} = [X_{i\Delta_n}]_a$, that is X rounded to the nearest multiple of a , say 1 cent for a decimalized asset.
- We show that, **in the presence of additive noise**, $S_J(4, k, \Delta_n) \xrightarrow{\mathbb{P}} \frac{1}{k}$.
- In the presence of **rounding error noise**, the limit is $\frac{1}{k^{1/2}}$.

- So S_J has four possible limits: with $k = 2$ and $p = 4$,

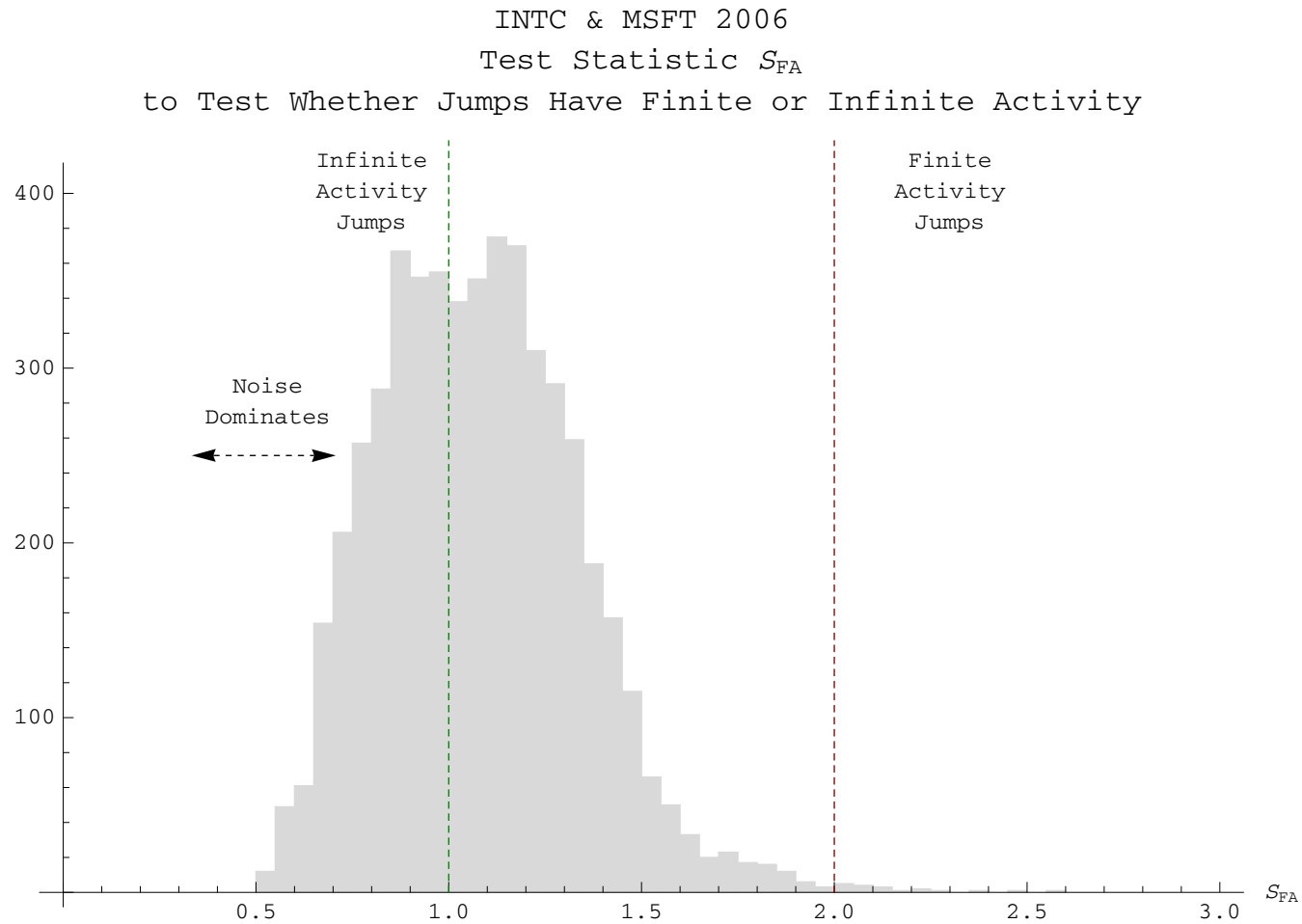
$1/2$:	additive noise dominates
$1/2^{1/2}$:	rounding error dominates
1	:	jumps present
2	:	no jumps present

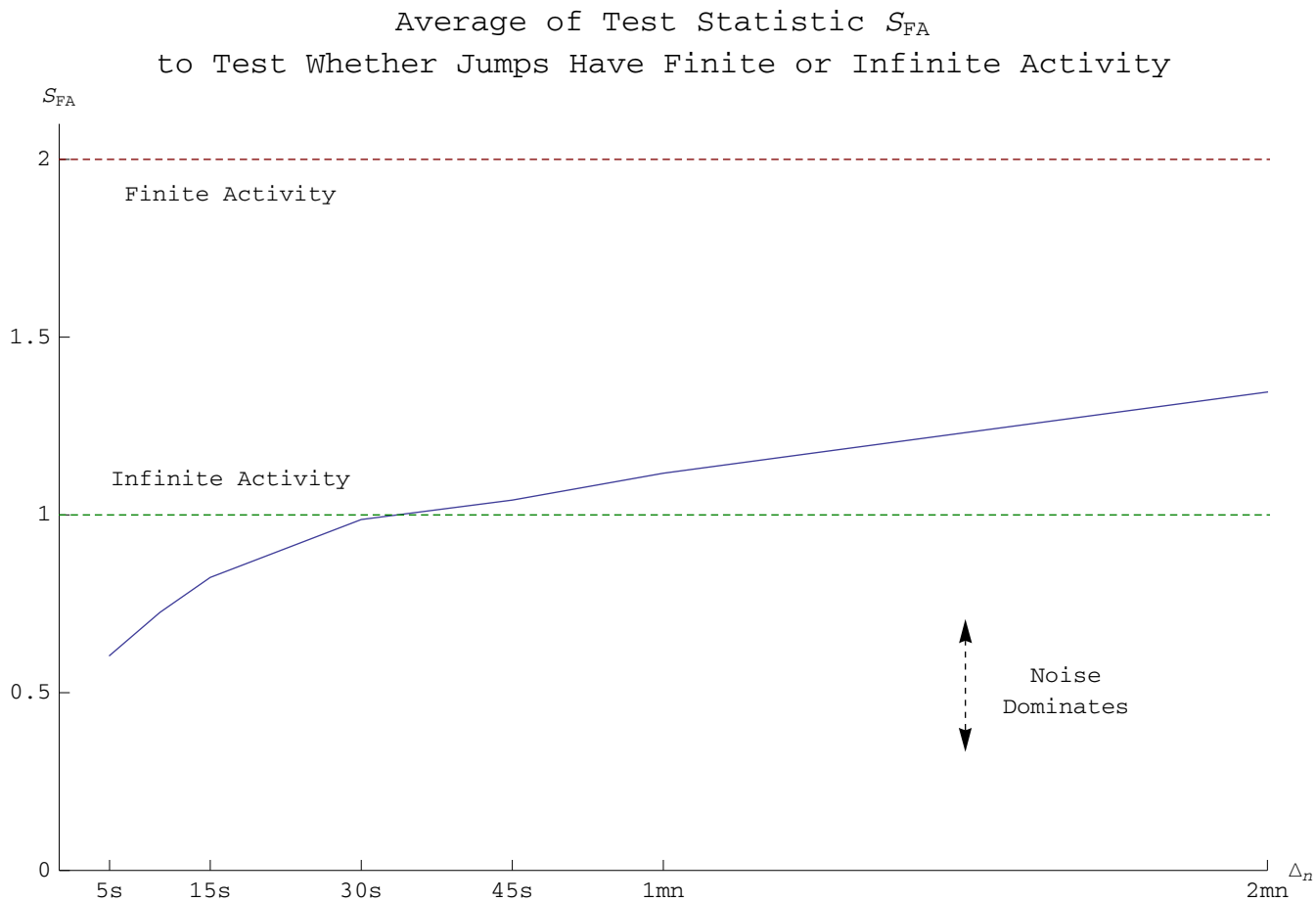
INTC & MSFT 2006
 Test Statistic S_J
 to Test for the Presence of Jumps





7.3. Jumps: Finite or Infinite Activity





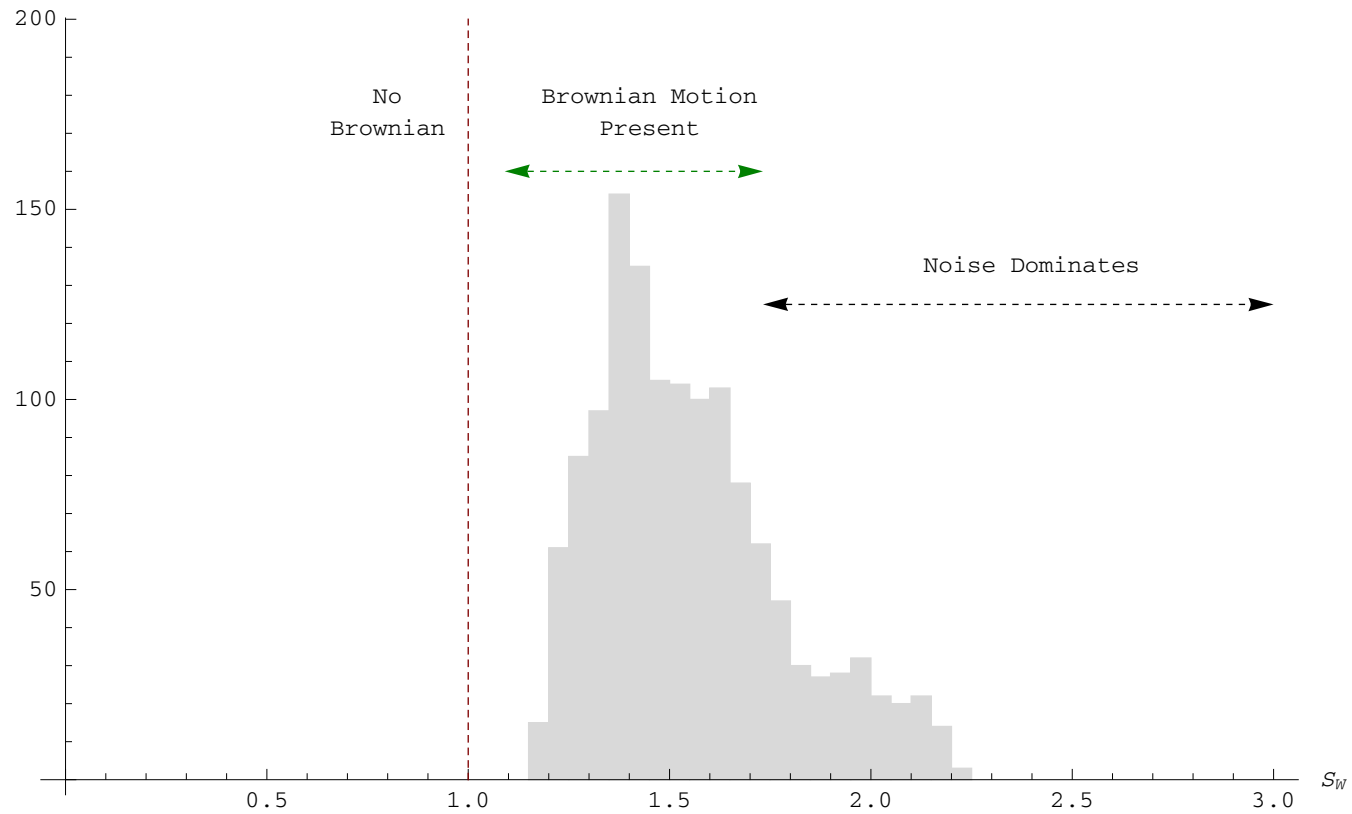
7.4. Brownian Motion: Present or Not

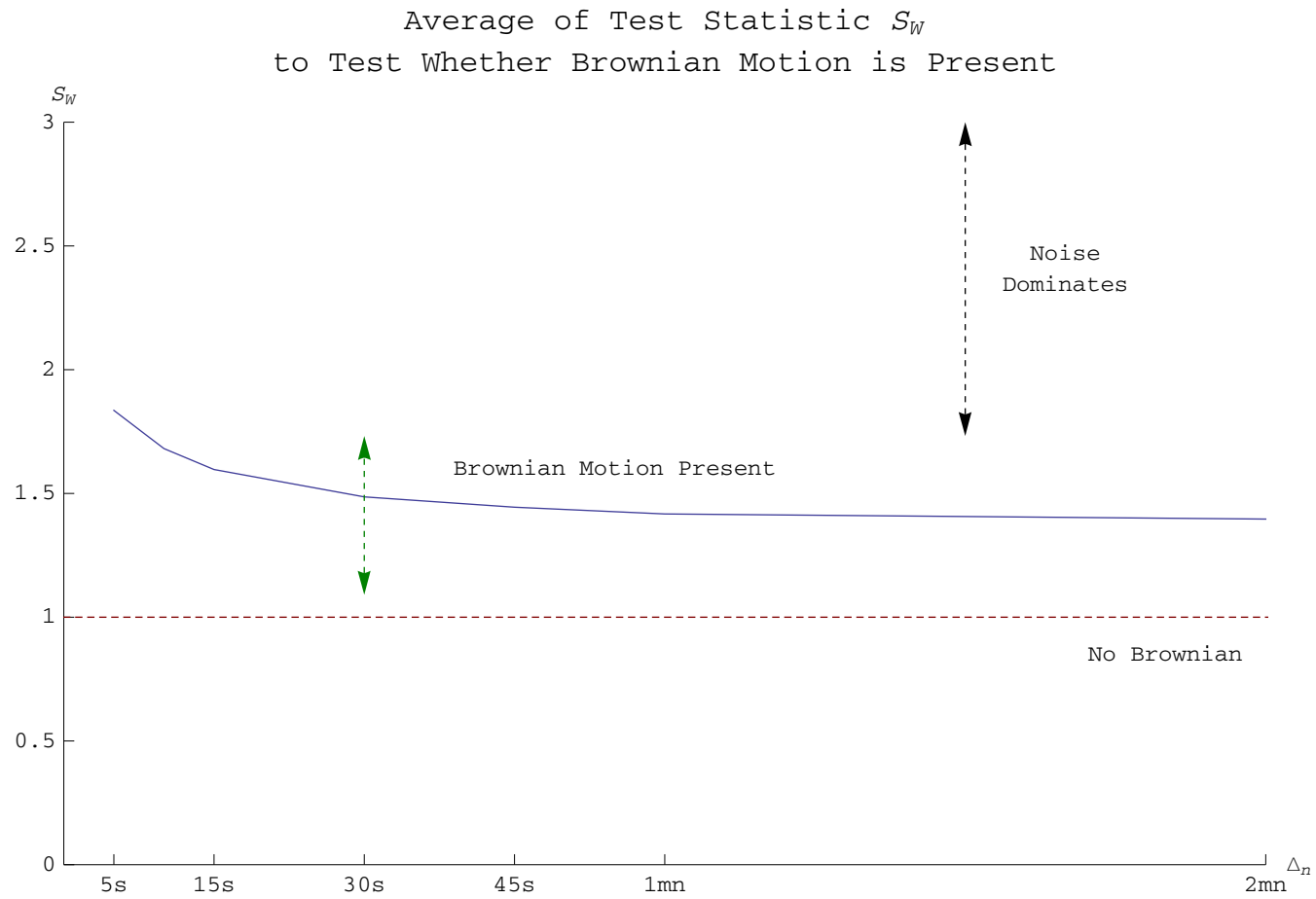
- Market microstructure noise with either an **additive white noise** or with **noise due to rounding**, the respective limits of S_W become 2 and $2^{1/2}$ with $k = 2$.

- S_W has four possible limits:

1	:	No Brownian motion
$k^{1-p/2}$:	Brownian motion present
$k^{1/2}$:	rounding error dominates
k	:	additive noise dominates

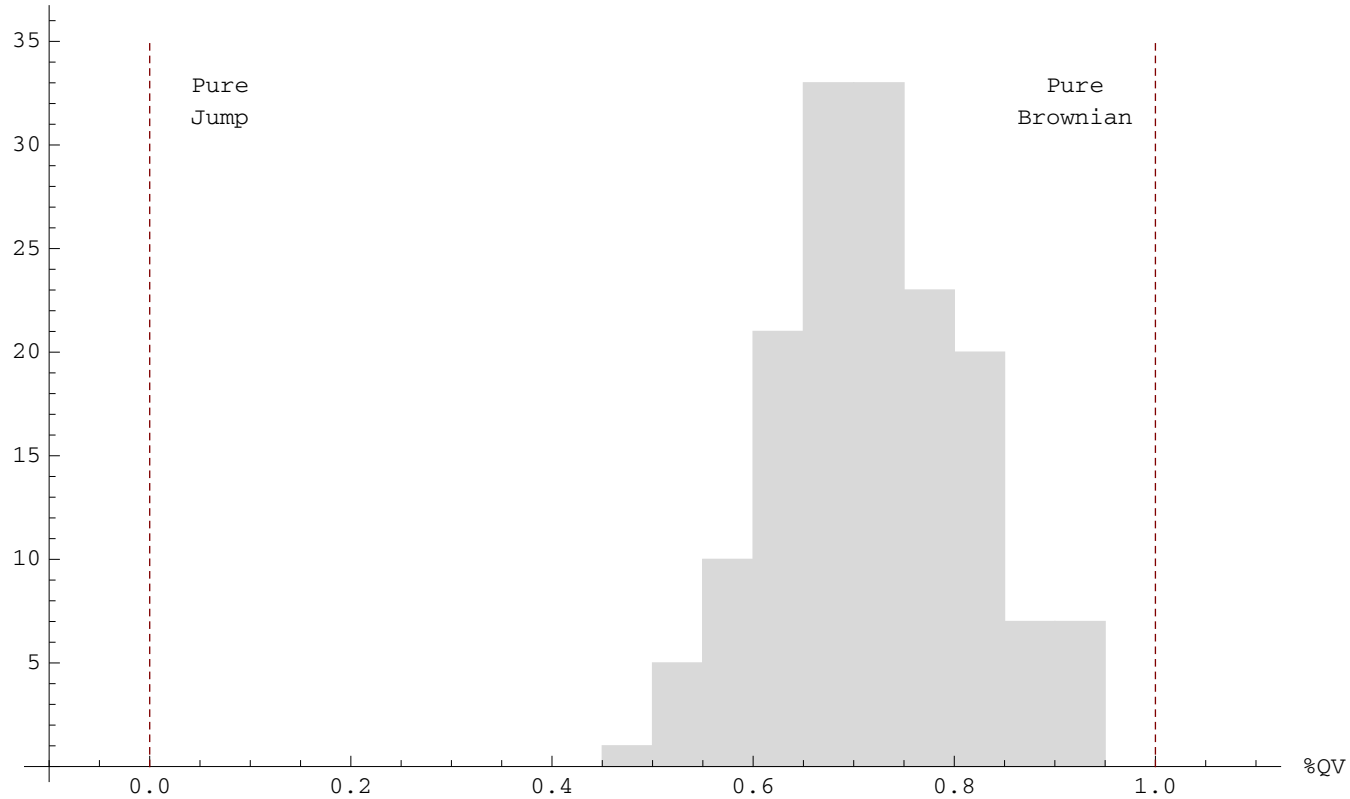
INTC & MSFT 2006
Test Statistic S_W
to Test Whether Brownian Motion is Present

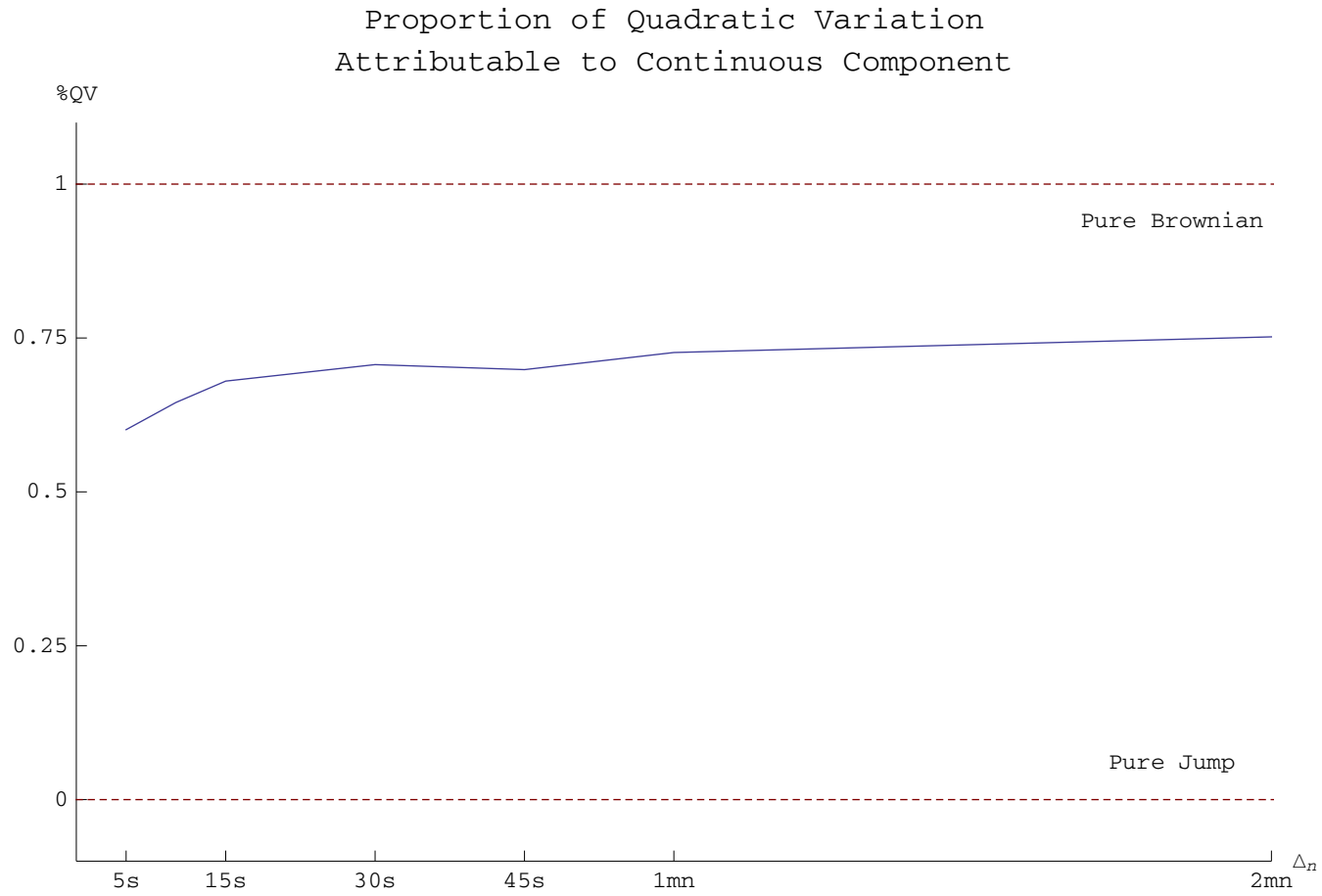




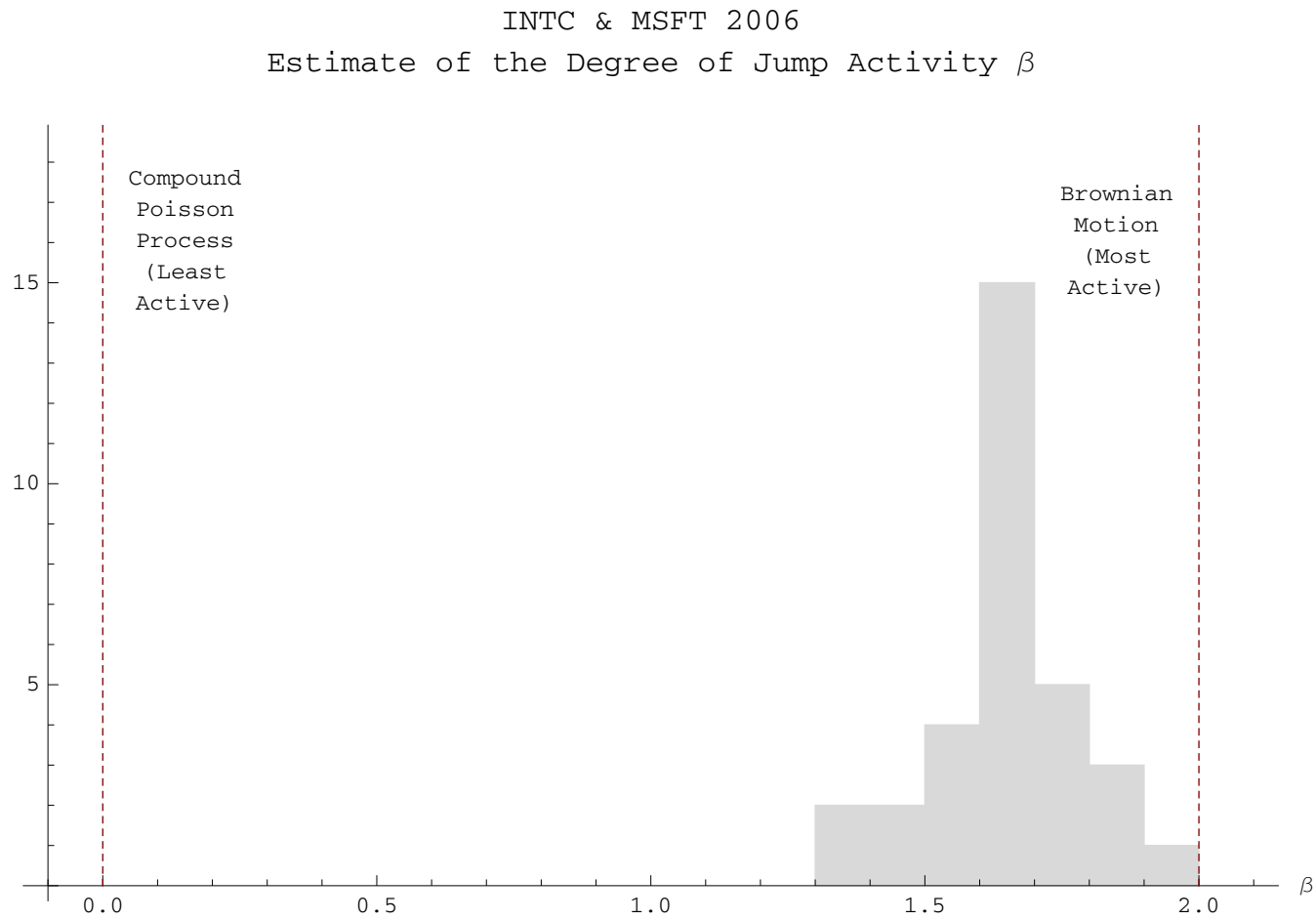
7.5. QV Relative Magnitude

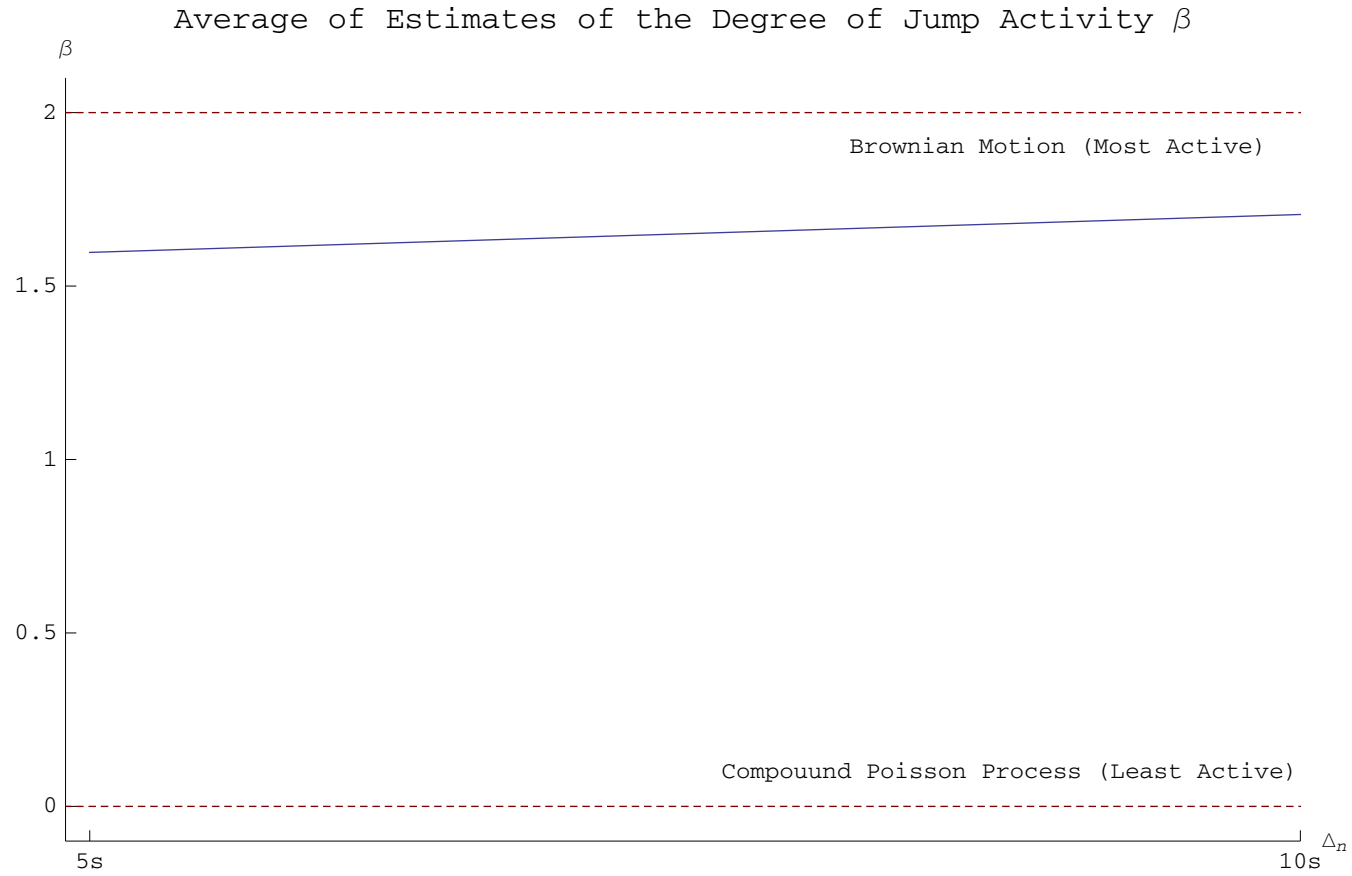
INTC & MSFT 2006
Proportion of Quadratic Variation
Attributable to Continuous Component





7.6. Estimating Jump Activity





8. Conclusions

The empirical results for these data appear to:

- Indicate that **jumps are present** in the data
- Point towards the presence of **infinite activity** jumps
- Of **degree of jump activity** that is somewhere **around 1.5 or higher**.
- Indicate that a **continuous component is present**.
- Representing about **3/4** of total QV.

- **Pros**
 - Unified methodology to address all these specification questions in a common framework
 - Symmetric treatment of null and alternative in each case, including distribution theory
 - Model-free
 - Extremely simple to implement
 - Impact of the noise on the statistics is characterized

- Cons
 - Not necessarily the optimal approach for each one of these questions taken individually.
 - Requires high frequency data (particularly the estimation of β)
 - Still to do: a full development of noise-robust statistics.