

User's Guide for the Matlab Library Implementing Closed Form MLE for Diffusions*

Yacine Aït-Sahalia

Department of Economics and Bendheim Center for Finance
Princeton University and NBER[†]

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Abstract

This document explains the use of the attached Matlab code for estimating the parameters of diffusions using closed-form maximum-likelihood.

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1 References

This code implements in Matlab the closed-form maximum-likelihood estimation method for diffusions developed in:

- Aït-Sahalia, Y., 1999, Transition Densities for Interest Rate and Other Nonlinear Diffusions, *Journal of Finance*, 54, 1361-1395.
- Aït-Sahalia, Y., 2002, Maximum-Likelihood Estimation of Discretely-Sampled Diffusions: A Closed-Form Approximation Approach, *Econometrica*, 70, 223-262.

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[†]Princeton, NJ 08544-1021. Phone: (609) 258-4015. E-mail: yacine@princeton.edu.

- Aït-Sahalia, Y., 2008, Closed-Form Likelihood Expansions for Multivariate Diffusions, *Annals of Statistics*, 36, 906-937.

Extensions to models with latent factors can be found in:

- Aït-Sahalia, Y. and Kimmel, R., 2007, Maximum Likelihood Estimation of Stochastic Volatility Models, *Journal of Financial Economics*, 83, 413-452.
- Aït-Sahalia, Y. and Kimmel, R., 2010, Estimating Affine Multifactor Term Structure Models Using Closed-Form Likelihood Expansions, *Journal of Financial Economics*, 98, 113-144.

2 Installation

This code uses “anonymous functions,” which require Matlab version 7 (R14, released in 2004), or later.

2.1 Installing the Package

The only thing you need to do is to download the file `ClosedFormMLE.zip` and unzip it somewhere in your computer. You have two options:

1. You can work under the folder `ClosedFormMLE\`, (you may rename this folder, this does not matter), keeping its subfolders in place, and save all your files in that folder.
2. Or you add this library to your Matlab’s search path.

We recommend the first option for casual users of Matlab.

This package contains the following folders and files:

Type	Name	Usage
Folder	<code>DERIVESTsuite</code>	Numerical differentiation
Folder	<code>Models</code>	A library of closed-form log-likelihood functions
Matlab function	<code>mymle.m</code>	Implementing MLE
Matlab function	<code>mymle_summary.m</code>	Post estimation analysis
Matlab function	<code>example_cev.m</code>	Demo: CEV model estimation
Matlab function	<code>logdensity2loglik.m</code>	A helper function

Note: the package `DERIVESTsuite` was written by John D’Errico and downloaded from: <http://www.mathworks.com/matlabcentral/fileexchange/13490>.

It is enclosed here as a convenience for endusers. Please refer to the package for its license.

The folder “`Models`” contains the formulae for the closed-form approximations of log likelihood functions for various stochastic differential equations (SDE) that have been implemented in this toolbox. Section 3 below gives the list of models that are currently implemented. Since this is an ongoing project, we will keep adding models into the library. Interested users can visit the author’s website for updates, and request additional models by email. The folder “`DERIVESTsuite`” contains routines for numerical differentiation (necessary to compute Fisher’s information so that we can obtain the asymptotic standard errors of the estimates). Neither one of these folders need to be edited or modified.

The two functions `mymle.m` and `mymle_summary.m` need not be modified either.

The simplest way to use this code is to first run `example_cev.m`, and then modify the file `example_cev.m` as needed: estimate a different model, replace the simulated data with real data, etc.

2.2 An Example: Estimating the CEV Model on Simulated Data

- Start Matlab and switch your “current directory” to `ClosedFormMLE\`.
- In Matlab, open the file `example_cev.m`, and run it by pressing the F5 key (in Matlab’s editor: Debug >> Run)
- Step 1 of the code simulates a time series according to the CEV model (i.e. Model U3, see 3.1.3) with parameters

$$a = 0.08, b = 0.5, c = 0.7, d = 1.5.$$

The sampling interval Δ is set to be $1/52$, which should be thought of as weekly data. There are $n = 500$ observations, saved in the variable `x`. Note: since the sample size is small, the estimation is fast but the estimates are inaccurate. You may also want to experiment with $n = 5,000$ to get statistically better results.

- Step 2 does the estimation with the function `mymle.m`. The code is

```
output=mymle(@ModelU3,x,delta,param0);
```

where “`@ModelU3`” is the function handle of the log likelihood function, “`x`” is our data, “`delta`” is our sampling interval $\Delta = 1/52$, and “`param0`” is the initial value of the optimization procedure underlying MLE. The user may need to experiment with different initial values to find the global maximum. In this example, we set the initial value as the true value of the parameters, that is,

$$\text{param0}=[0.08,0.5,0.7,1.5].$$

(See Section 3 for the ordering of parameters in each model.) If we wanted to estimate the other model, say Model U1 (see 3.1.1), the syntax would be

```
output=mymle(@ModelU1,x,delta,param0forU1);
```

where the initial value `param0forU1` should be 3-dimensional, since Model U1 has only 3 parameters. The “`output`” is a Matlab structure which packages the results of the estimation. You do not have to worry if you are not familiar with this data structure. We provide the output in step 3.

- As the code runs, it first maximize the log likelihood function and then prints the MLE estimates on the screen. It also tells you whether the maximization procedure converges under the default tolerance. After that, it moves onto computing the standard error and then prints the estimates, standard error (constructed from the inverse of Hessian), and the misspecification-robust standard error (i.e., the sandwich estimate).
- Step 3 does the post estimation analysis. The code is

```
mymle_summary(output);
```

where you take the “`output`” structure from step 2 directly as your input in this step. The code does two things. First, it reprints the table of estimates, standard errors, robust standard errors, and exit status of the maximization procedure as below:

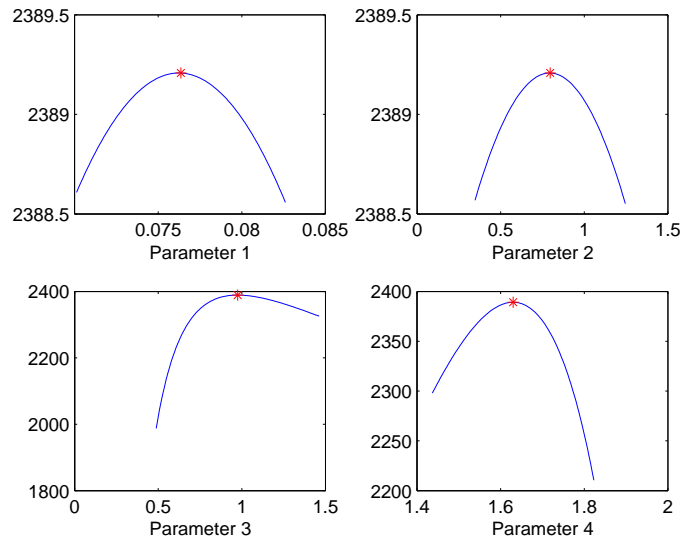


Figure 1: Example: Log-Likelihood Near the Solution

```
*****
COEF SE Robust SE
0.0764 0.0062 0.0062
0.7961 0.4397 0.4497
0.9755 0.4812 0.4875
1.6297 0.1917 0.1932
*****
fminsearch converged to a solution.
```

Secondly, it plots the marginal log likelihood for each parameter in a neighborhood of the estimates. See Figure 1. The blue curve is the likelihood function, while each red star is our estimate for the corresponding parameter. This gives us a way to visually check whether the maximization is successful (of course, this says nothing about global optimality!).

2.3 Syntax of the Two Functions (for reference only)

2.3.1 `mymle`

- Syntax:

```
output = mymle(logdensity, x, del, param0, options);
```

- Inputs:

- `logdensity` is the function handle of the model, for example, `@ModelU1`, `@ModelU3`, `@ModelB4...`
- `x`: is the data. This is a $n \times k$ matrix, where n is the number of observations, and k is the dimensionality of your model. For univariate models, $k = 1$, and for bivariate models $k = 2, \dots$

- `del`: is the sampling interval of the time series
 - `param0` : initial parameter value of the maximization procedure. The dimensionality and ordering of its entries should be consistent with the underlying model, listed in Section 3.
 - `options` : this input is optional. The input is a Matlab option structure for its optimization toolbox. See Matlab’s help document of “`optimset`” for more details.
- Output is structure consisting the following fields:
 - `exitmsg`: exit status of the optimization procedure
 - `param`: the MLE estimates, see Section 3 for the ordering of parameters.
 - `variance`: covariance estimates
 - `variance_robust`: robust covariance estimates
 - `se`: standard errors
 - `se_robust`: robust standard errors
 - `objfun`: the function handle of the negative log likelihood function used internally in the optimization procedure
 - `loglik`: the optimized value of log likelihood
 - `exitflag`: exit status reported by `fminsearch`.

2.3.2 `mymle_summary`

- Syntax:

`mymle_summary(mle_output)`

- The input of this function is the output structure from `mymle`.

3 List of Models Currently Implemented

The models below are in no particular order.

The 9 canonical affine models, and semi-affine models, are implemented separately (see below). There is of course some duplication since U4 and U7 are affine, for instance.

3.1 Univariate Models

Univariate models have the following canonical form:

$$dX_t = \mu(X_t; \text{parameters}) dt + \sigma(X_t; \text{parameters}) dW_t.$$

3.1.1 Model U1

$$\mu(x) = ax + bx^2,$$

$$\sigma(x) = \sigma x^{3/2},$$

Parameters: a, b, σ .

3.1.2 Model U2

$$\mu(x) = a + bx,$$

$$\sigma(x) = dx,$$

Parameters: a, b, d .

3.1.3 Model U3

$$\mu(x) = b(a - x),$$

$$\sigma(x) = cx^\rho,$$

Parameters: a, b, c, ρ .

Note: do not let the parameter ρ go below 1. Special cases for $\rho = 1$ and $3/2$.

3.1.4 Model U4

$$\mu(x) = \kappa(\alpha - x),$$

$$\sigma(x) = \sigma x^{1/2},$$

Parameters: κ, α, σ .

3.1.5 Model U5

$$\mu(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3,$$

$$\sigma(x) = \gamma x^\rho,$$

Parameters: $\gamma, \rho, \theta_0, \theta_1, \theta_2, \theta_3$.

Note: do not let the parameter ρ go below 1. Special cases for $\rho = 1$ and $3/2$.

3.1.6 Model U6

$$\mu(x) = a + bx + cx^2 + dx^3,$$

$$\sigma(x) = f,$$

Parameters: a, b, c, d, f .

3.1.7 Model U7

$$\mu(x) = \kappa(\alpha - x),$$

$$\sigma(x) = \sigma,$$

Parameters: κ, α, σ .

3.1.8 Model U8

$$\mu(x) = \frac{a_{-1}}{x} + a_0 + a_1x + a_2x^2,$$

$$\sigma(x) = \sigma x^\rho,$$

Parameters: $a_{-1}, a_0, a_1, a_2, \sigma, \rho$.

Note: do not let the parameter ρ go below 1. Special cases for $\rho = 1$ and $3/2$.

3.1.9 Model U9

$$\mu(x) = \frac{a_{-1}}{x} + a_0 + a_1x + a_2x^2,$$

$$\sigma(x) = (b_0 + b_1x + b_2x^{b_3})^{1/2},$$

Parameters: $a_{-1}, a_0, a_1, a_2, b_0, b_1, b_2, b_3$.

3.1.10 Model U10

$$\mu(x) = \frac{a_{-1}}{x} + a_0 + a_1x + a_2x^2,$$

$$\sigma(x) = b_0 + b_1x + b_2x^{b_3},$$

Parameters: $a_{-1}, a_0, a_1, a_2, b_0, b_1, b_2, b_3$.

Note: do not let the parameter ρ go below 1.

3.1.11 Model U11

$$\mu(x) = a + bx,$$

$$\sigma(x) = f + dx,$$

Parameters: a, b, f, d .

3.1.12 Model U12

$$\mu(x) = \frac{\beta}{x} - \alpha x^3,$$

$$\sigma(x) = \gamma x^{1/2},$$

Parameters: β, α, γ .

3.1.13 Model U13

$$\mu(x) = a_0 + \frac{a_{-1}}{x} + a_1x + a_2x^2 + a_3x^3$$

$$\sigma(x) = \sigma x^\rho$$

Parameters: $a_0, a_{-1}, a_1, a_2, a_3, \sigma, \rho$.

Note: do not let the parameter ρ go below 1. Special cases for $\rho = 1$ and $3/2$.

3.1.14 Model U14

$$\mu(x) = h + ax + bx^2,$$

$$\sigma(x) = vx^{3/2},$$

Parameters: h, a, b, v .

3.2 Bivariate Models

Consider a 2-dimensional process $X_t = (X_{1t}, X_{2t})^\top$ determined by

$$dX_t = \mu(X_t) dt + \Sigma(X_t) dW_t,$$

or written componentwise,

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \end{pmatrix} = \begin{pmatrix} \mu_1(X_{1t}, X_{2t}) \\ \mu_2(X_{1t}, X_{2t}) \end{pmatrix} dt + \begin{pmatrix} \Sigma_{11}(X_{1t}, X_{2t}) & \Sigma_{12}(X_{1t}, X_{2t}) \\ \Sigma_{21}(X_{1t}, X_{2t}) & \Sigma_{22}(X_{1t}, X_{2t}) \end{pmatrix} \begin{pmatrix} dW_{1t} \\ dW_{2t} \end{pmatrix}.$$

The Brownian motions are independent. Any correlation between the components is introduced by the off-diagonal terms in the Σ matrix.

3.2.1 Model B1

$$\mu(x_1, x_2) = \begin{pmatrix} a + bx_2 \\ c + dx_2 \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} r\sqrt{x_2} & 0 \\ h & \sqrt{(1-r^2)x_2} \end{pmatrix},$$

Parameters: r, h, a, b, c, d .

3.2.2 Model B2

$$\mu(x_1, x_2) = \begin{pmatrix} a_0 + a_1x_1 + a_2x_2 \\ b_0 + b_1x_1 + b_2x_2 \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} c_0 + c_1x_1 + c_2x_2 & 0 \\ 0 & d_0 + d_1x_1 + d_2x_2 \end{pmatrix},$$

Parameters: $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, d_0, d_1, d_2$.

3.2.3 Model B3

$$\mu(x_1, x_2) = \begin{pmatrix} \mu - x_2/2 \\ \alpha + \beta x_2 \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_2} & 0 \\ \sigma\rho x_2^\gamma & \sigma\sqrt{1-\rho^2}x_2^\gamma \end{pmatrix},$$

Parameters: $\mu, \alpha, \beta, \sigma, \rho, \gamma$.

3.2.4 Model B4

$$\mu(x_1, x_2) = \begin{pmatrix} a_0 + a_1 x_2 \\ b(a - x_2) + \lambda g x_2^\beta \sqrt{a + f(x_2 - a)} \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{1 - r^2} \sqrt{a + f(x_2 - a)} & r \sqrt{a + f(x_2 - a)} \\ 0 & g x_2^\beta \end{pmatrix},$$

Parameters: $a_0, a_1, a, b, \lambda, g, \beta, f, r$.

3.2.5 Model B5

$$\mu(x_1, x_2) = \begin{pmatrix} b x_1 \\ c - d x_2 \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} h x_1 \sqrt{x_2} & 0 \\ g r \sqrt{x_2} & g \sqrt{1 - r^2} \sqrt{x_2} \end{pmatrix},$$

Parameters: b, c, d, h, g, r .

3.2.6 Model B6

This is the Heston model with x_1 being the log of the asset price and x_2 its stochastic volatility.

$$\mu(x_1, x_2) = \begin{pmatrix} m - x_2/2 \\ a - b x_2 \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_2} & 0 \\ s r \sqrt{x_2} & s \sqrt{1 - r^2} \sqrt{x_2} \end{pmatrix},$$

Parameters: m, a, b, s, r .

Note that this model is observationally equivalent to the ones with

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{1 - r^2} \sqrt{x_2} & r \sqrt{x_2} \\ 0 & s \sqrt{x_2} \end{pmatrix}$$

or

$$\Sigma(x_1, x_2) = \begin{pmatrix} r \sqrt{x_2} & \sqrt{1 - r^2} \sqrt{x_2} \\ s \sqrt{x_2} & 0 \end{pmatrix}.$$

3.2.7 Model B7

$$\mu(x_1, x_2) = \begin{pmatrix} 0 \\ a_1 - a_2 x_2 \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \frac{2x_1}{\gamma \sqrt{x_2}} & \frac{2\eta x_1}{\gamma} \\ 2\sqrt{x_2} & 0 \end{pmatrix},$$

Parameters: a_1, a_2, γ, η .

3.2.8 Model B8

$$\mu(x_1, x_2) = \begin{pmatrix} a + bx_1 \\ cx_2 \end{pmatrix}$$
$$\Sigma(x_1, x_2) = \begin{pmatrix} dx_1^\gamma e^{x_2} & 0 \\ 0 & f \end{pmatrix},$$

Parameters: a, b, c, d, γ, f .

3.2.9 Model B9

$$\mu(x_1, x_2) = \begin{pmatrix} \theta_5 - \frac{1}{2}e^{2x_2} \\ \theta_2(\theta_1 - x_2) \end{pmatrix},$$
$$\Sigma(x_1, x_2) = \begin{pmatrix} e^{x_2} & 0 \\ \theta_3\theta_4 & \sqrt{1 - \theta_4^2\theta_3} \end{pmatrix},$$

Parameters: $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$.

3.2.10 Model B10

$$\mu(x_1, x_2) = \begin{pmatrix} b_1(a_1 - x_1) \\ b_2(a_2 - x_2) \end{pmatrix},$$
$$\Sigma(x_1, x_2) = \begin{pmatrix} g_1 & 0 \\ 0 & g_2\sqrt{x_2} \end{pmatrix},$$

Parameters: $a_1, a_2, b_1, b_2, g_1, g_2$.

3.2.11 Model B11

$$\mu(x_1, x_2) = \begin{pmatrix} k_1 + k_2x_2 \\ \kappa(\theta - x_2) \end{pmatrix},$$
$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{1 - \rho^2}\sqrt{x_2} & \rho\sqrt{x_2} \\ 0 & \sigma x_2 \end{pmatrix},$$

Parameters: $k_1, k_2, \rho, \kappa, \theta, \sigma$.

3.2.12 Model B12

$$\mu(x_1, x_2) = \begin{pmatrix} ax_1 \\ -bx_2 \end{pmatrix},$$
$$\Sigma(x_1, x_2) = \begin{pmatrix} cx_1e^{x_2} & 0 \\ dr & d\sqrt{1 - r^2} \end{pmatrix},$$

Parameters: a, b, c, d, r .

3.2.13 Model B13

$$\mu(x_1, x_2) = \begin{pmatrix} b_{11}(a_1 - x_1) + b_{12}(a_2 - x_2) \\ b_{21}(a_1 - x_1) + b_{22}(a_2 - x_2) \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Parameters: $a_1, a_2, b_{11}, b_{12}, b_{21}, b_{22}, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$.

3.2.14 Model B14

$$\mu(x_1, x_2) = \begin{pmatrix} k_1(x_2 - x_1) \\ k_2(\theta - x_2) \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sigma_1\sqrt{x_1} & 0 \\ 0 & \sigma_2\sqrt{x_2} \end{pmatrix},$$

Parameters: $k_1, k_2, \theta, \sigma_1, \sigma_2$.

3.2.15 Model B15

$$\mu(x_1, x_2) = \begin{pmatrix} a + bx_1 \\ fx_1 + dx_2 \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_1} & 0 \\ h & \sqrt{1 + gx_1} \end{pmatrix},$$

Parameters: g, h, a, b, f, d .

3.2.16 Model B16

$$\mu(x_1, x_2) = \begin{pmatrix} a + bx_1 + gx_2 \\ d + \eta x_1 + fx_2 \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_1} & 0 \\ 0 & \sqrt{x_2} \end{pmatrix},$$

Parameters: a, b, g, d, η, f .

3.2.17 Model B17

$$\mu(x_1, x_2) = \begin{pmatrix} a_{00} - (a_1 + a_2x_2)/2 + \left(n_0\sqrt{1 - g_1^2} + nu_1g_1 \right) (\sqrt{a_1 + a_2x_2})^{b+d} \\ a_{01} + a_{11}x_2 + (nu_1g_{11}) (\sqrt{a_1 + a_2x_2})^{b+d} \end{pmatrix},$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{1 - g_1^2}\sqrt{a_1 + a_2x_2} & g_1\sqrt{a_1 + a_2x_2} \\ 0 & g_{11}(\sqrt{a_1 + a_2x_2})^b \end{pmatrix},$$

Parameters: $a_{00}, a_{01}, a_{11}, A_1, A_2, n_0, nu_1, g_1, g_{11}, b, d$.

3.2.18 Model B18

$$\mu(x_1, x_2) = \begin{pmatrix} b_1 x_1 \\ a_2 + b_2 x_2 \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} g_{11} e^{x_1} & 0 \\ g_{22} r & g_{22} \sqrt{1-r^2} \end{pmatrix},$$

Parameters: $b_1, a_2, b_2, g_{11}, g_{22}, r$.

3.2.19 Model B19

$$\mu(x_1, x_2) = \begin{pmatrix} b_1 x_1 \\ a_2 + b_2 x_2 \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} e^{x_2} & 0 \\ g_{22} r & g_{22} \sqrt{1-r^2} \end{pmatrix},$$

Parameters: b_1, a_2, b_2, g_{22}, r .

3.2.20 Model B20

$$\mu(x_1, x_2) = \begin{pmatrix} a_1 + b_1 x_1 \\ a_2 + b_2 x_2 \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_2} & 0 \\ g r \sqrt{x_2} & g \sqrt{1-r^2} \sqrt{x_2} \end{pmatrix},$$

Parameters: a_1, b_1, a_2, b_2, g, r .

3.2.21 Model B21

$$\mu(x_1, x_2) = \begin{pmatrix} a_1(b_1 - x_1) \\ a_{21}(b_1 - x_1) + a_2(b_2 - x_2) \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_1} & 0 \\ g_{21} \sqrt{x_1} & g_{22} \sqrt{x_1} \end{pmatrix},$$

Parameters: $a_1, b_1, a_{21}, a_2, b_2, g_{21}, g_{22}$.

3.2.22 Model B22

$$\mu(x_1, x_2) = \begin{pmatrix} k_1 + k_2 x_2 \\ k_3(a - x_2) \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{1-r^2} \sqrt{x_2} & r \sqrt{x_2} \\ 0 & s x_2^b \end{pmatrix},$$

Parameters: $k_1, k_2, k_3, a, r, s, b$.

3.3 Trivariate Models

Again, the Brownian motions are independent. Any correlation between the components is introduced by the off-diagonal terms in the Σ matrix. Some models are specified by the local variance matrix $\Sigma \Sigma^\top$ directly ($\Sigma \Sigma^\top$ is what matters, not Σ).

3.3.1 Model T1

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} k_{11}(m_1 - x_1) + k_{12}(m_2 - x_2) \\ k_{21}(m_1 - x_1) + k_{22}(m_2 - x_2) \\ k_{33}(m_3 - x_3) \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} (a_1 + b_1 x_1)^{1/2} & 0 & 0 \\ 0 & (a_2 + b_2 x_2)^{1/2} & 0 \\ 0 & 0 & (a_3 + b_3 x_3) \end{pmatrix}$$

Parameters: $m_1, m_2, m_3, k_{11}, k_{12}, k_{21}, k_{22}, k_{33}, a_1, b_1, a_2, b_2, a_3, b_3$.

3.3.2 Model T2

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} (a_1 - x_2 + x_3)x_1 \\ a_2 + b_2 x_2 \\ a_3 + b_3 x_3 \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) \Sigma(x_1, x_2, x_3)^\top = \begin{pmatrix} s_1^2 x_1^2 & s_1 s_2 r_{12} x_1 & s_1 s_3 r_{13} x_1 \\ \cdot & s_2^2 & s_2 s_3 r_{23} \\ \cdot & \cdot & s_3^2 \end{pmatrix}$$

Parameters: $a_1, a_2, b_2, a_3, b_3, s_1, s_2, s_3, r_{12}, r_{13}, r_{23}$.

3.3.3 Model T3

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} k_{11}(g_1 - x_1) + k_{12}(g_2 - x_2) + k_{13}(g_3 - x_3) \\ k_{22}(g_2 - x_2) \\ k_{33}(g_3 - x_3) \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) \Sigma(x_1, x_2, x_3)^\top = \begin{pmatrix} s_{11} x_1 & s_{12} x_2 & s_{13} x_3 \\ 0 & s_{22} x_2 & 0 \\ 0 & 0 & s_{33} x_3 \end{pmatrix}$$

Parameters: $g_1, g_2, g_3, k_{11}, k_{12}, k_{13}, k_{22}, k_{33}, s_{11}, s_{12}, s_{13}, s_{22}, s_{33}$.

3.3.4 Model T4

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} b_1 x_1 \\ b_2 x_2 \\ b_3 x_3 \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} d_{11} x_1^2 & 0 & 0 \\ d_{21} x_2 x_1 & d_{22} x_2^2 & 0 \\ d_{31} x_3 x_1 & d_{32} x_3 x_2 & d_{33} x_3^2 \end{pmatrix}$$

Parameters: $b_1, b_2, b_3, d_{11}, d_{22}, d_{33}, d_{21}, d_{31}, d_{32}$

3.3.5 Model T5

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} x_3 - x_2/2 \\ v_1(v_2 - x_2) \\ r_1 - r_2 x_3 \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} \sqrt{x_2} & 0 & 0 \\ \rho_{12} v_3 \sqrt{x_2} & \sqrt{1 - \rho_{12}^2} v_3 \sqrt{x_2} & 0 \\ r_3 \rho_{13} & r_3 \frac{\rho_{23} - \rho_{13} \rho_{12}}{\sqrt{1 - \rho_{12}^2}} & r_3 \sqrt{1 - \rho_{13}^2 - \left(\frac{\rho_{23} - \rho_{13} \rho_{12}}{\sqrt{1 - \rho_{12}^2}} \right)^2} \end{pmatrix}$$

Parameters: $v_1, v_2, v_3, r_1, r_2, r_3, \rho_{12}, \rho_{13}, \rho_{23}$.

3.3.6 Model T6

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} a_1(b_1 - x_1) \\ a_{21}(b_1 - x_1) + a_2(b_2 - x_2) \\ a_{31}(b_1 - x_1) + a_{32}(b_2 - x_2) + a_3(b_3 - x_3) \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} \sqrt{x_1} & 0 & 0 \\ g_{21}\sqrt{x_1} & g_{22}\sqrt{x_1} & 0 \\ g_{31}\sqrt{x_1} & g_{32}\sqrt{x_1} & g_{33}\sqrt{x_1} \end{pmatrix}$$

Parameters: $a_1, b_1, a_{21}, a_2, b_2, a_{31}, a_{32}, a_3, b_3, g_{21}, g_{22}, g_{31}, g_{32}, g_{33}$.

3.4 Canonical Affine Models (Dimensions 1 to 3)

3.4.1 Model AY01

$$\mu(x_1) = -\kappa_{11} x_1$$

$$\sigma(x_1) = 1$$

Parameters: κ_{11} .

3.4.2 Model AY11

$$\mu(x_1) = \kappa_{11}(\theta_1 - x_1)$$

$$\sigma(x_1) = x_1^{1/2}$$

Parameters: θ_1, κ_{11} .

3.4.3 Model AY02

$$\mu(x_1, x_2) = \begin{pmatrix} -\kappa_{11} x_1 \\ -\kappa_{21} x_1 - \kappa_{22} x_2 \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Parameters: $\kappa_{11}, \kappa_{21}, \kappa_{22}$.

3.4.4 Model AY12

$$\mu(x_1, x_2) = \begin{pmatrix} \kappa_{11}(\theta_1 - x_1) \\ \kappa_{21}(\theta_1 - x_1) - \kappa_{22}x_2 \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} x_1^{1/2} & 0 \\ 0 & (1 + \beta_{21}x_1)^{1/2} \end{pmatrix}$$

Parameters: $\theta_1, \kappa_{11}, \kappa_{21}, \kappa_{22}, \beta_{21}$.

3.4.5 Model AY22

$$\mu(x_1, x_2) = \begin{pmatrix} \kappa_{11}(\theta_1 - x_1) + \kappa_{12}(\theta_2 - x_2) \\ \kappa_{21}(\theta_1 - x_1) + \kappa_{22}(\theta_2 - x_2) \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} x_1^{1/2} & 0 \\ 0 & x_2^{1/2} \end{pmatrix}$$

Parameters: $\theta_1, \theta_2, \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}$.

3.4.6 Model AY03

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} -\kappa_{11}x_1 \\ -\kappa_{21}x_1 - \kappa_{22}x_2 \\ -\kappa_{31}x_1 - \kappa_{32}x_2 - \kappa_{33}x_3 \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Parameters: $\kappa_{11}, \kappa_{21}, \kappa_{22}, \kappa_{31}, \kappa_{32}, \kappa_{33}$.

3.4.7 Model AY13

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} \kappa_{11}(\theta_1 - x_1) \\ \kappa_{21}(\theta_1 - x_1) - \kappa_{22}x_2 - \kappa_{23}x_3 \\ \kappa_{31}(\theta_1 - x_1) - \kappa_{32}x_2 - \kappa_{33}x_3 \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} x_1^{1/2} & 0 & 0 \\ 0 & (1 + \beta_{21}x_1)^{1/2} & 0 \\ 0 & 0 & (1 + \beta_{31}x_1)^{1/2} \end{pmatrix}$$

Parameters: $\theta_1, \kappa_{11}, \kappa_{21}, \kappa_{22}, \kappa_{23}, \kappa_{31}, \kappa_{32}, \kappa_{33}, \beta_{21}, \beta_{31}$.

3.4.8 Model AY23

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} \kappa_{11}(\theta_1 - x_1) + \kappa_{12}(\theta_2 - x_2) \\ \kappa_{21}(\theta_1 - x_1) + \kappa_{22}(\theta_2 - x_2) \\ \kappa_{31}(\theta_1 - x_1) + \kappa_{32}(\theta_2 - x_2) - \kappa_{33}x_3 \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} x_1^{1/2} & 0 & 0 \\ 0 & x_2^{1/2} & 0 \\ 0 & 0 & (1 + \beta_{31}x_1 + \beta_{32}x_2)^{1/2} \end{pmatrix}$$

Parameters: $\theta_1, \theta_2, \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \kappa_{31}, \kappa_{32}, \kappa_{33}, \beta_{31}, \beta_{32}$.

3.4.9 Model AY33

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} \kappa_{11}(\theta_1 - x_1) + \kappa_{12}(\theta_2 - x_2) + \kappa_{13}(\theta_3 - x_3) \\ \kappa_{21}(\theta_1 - x_1) + \kappa_{22}(\theta_2 - x_2) + \kappa_{23}(\theta_3 - x_3) \\ \kappa_{31}(\theta_1 - x_1) + \kappa_{32}(\theta_2 - x_2) + \kappa_{33}(\theta_3 - x_3) \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} x_1^{1/2} & 0 & 0 \\ 0 & x_2^{1/2} & 0 \\ 0 & 0 & x_3^{1/2} \end{pmatrix}$$

Parameters: $\theta_1, \theta_2, \theta_3, \kappa_{11}, \kappa_{12}, \kappa_{13}, \kappa_{21}, \kappa_{22}, \kappa_{23}, \kappa_{31}, \kappa_{32}, \kappa_{33}$.

3.5 Semi-Affine Models (Dimensions 1 to 3)

3.5.1 Model SA11

$$\mu(x) = a + bx + d\sqrt{x}$$

$$\sigma(x) = \sqrt{x}$$

Parameters: a, b, d

3.5.2 Model SA12

$$\mu(x_1, x_2) = \begin{pmatrix} a + b_{11}x_1 + d_1\sqrt{x_1} \\ b_{21}x_1 + b_{22}x_2 + d_2\sqrt{1 + fx_1} \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_1} & 0 \\ 0 & \sqrt{1 + fx_1} \end{pmatrix}$$

Parameters: $a, b_{11}, d_1, b_{21}, b_{22}, d_2, f$

3.5.3 Model SA22

$$\mu(x_1, x_2) = \begin{pmatrix} a_1 + b_{11}x_1 + b_{12}x_2 + d_1\sqrt{x_1} \\ a_2 + b_{21}x_1 + b_{22}x_2 + d_2\sqrt{x_2} \end{pmatrix}$$

$$\Sigma(x_1, x_2) = \begin{pmatrix} \sqrt{x_1} & 0 \\ 0 & \sqrt{x_2} \end{pmatrix}$$

Parameters: $a_1, b_{11}, b_{12}, d_1, a_2, b_{21}, b_{22}, d_2$

3.5.4 Model SA13

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} a + b_{11}x_1 + d_1\sqrt{x_1} \\ b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + d_2\sqrt{1 + e_1x_1} \\ b_{31}x_1 + b_{32}x_2 + b_{33}x_3 + d_3\sqrt{1 + e_2x_1} \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} \sqrt{x_1} & 0 & 0 \\ 0 & \sqrt{1 + e_1x_1} & 0 \\ 0 & 0 & \sqrt{1 + e_2x_1} \end{pmatrix}$$

Parameters: $a, b_{11}, d_1, b_{21}, b_{22}, b_{23}, d_2, e_1, b_{31}, b_{32}, b_{33}, d_3, e_2$.

3.5.5 Model SA23

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} a_1 + b_{11}x_1 + b_{12}x_2 + d_1\sqrt{x_1} \\ a_2 + b_{21}x_1 + b_{22}x_2 + d_2\sqrt{x_2} \\ b_{31}x_1 + b_{32}x_2 + b_{33}x_3 + d_3\sqrt{1 + e_1x_1 + e_2x_2} \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} \sqrt{x_1} & 0 & 0 \\ 0 & \sqrt{x_2} & 0 \\ 0 & 0 & \sqrt{1 + e_1x_1 + e_2x_2} \end{pmatrix}$$

Parameters: $a_1, b_{11}, b_{12}, d_1, a_2, b_{21}, b_{22}, d_2, b_{31}, b_{32}, b_{33}, d_3, e_1, e_2$.

3.5.6 Model SA33

$$\mu(x_1, x_2, x_3) = \begin{pmatrix} a_1 + b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + d_1\sqrt{x_1} \\ a_2 + b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + d_2\sqrt{x_2} \\ a_3 + b_{31}x_1 + b_{32}x_2 + b_{33}x_3 + d_3\sqrt{x_3} \end{pmatrix}$$

$$\Sigma(x_1, x_2, x_3) = \begin{pmatrix} \sqrt{x_1} & 0 & 0 \\ 0 & \sqrt{x_2} & 0 \\ 0 & 0 & \sqrt{x_3} \end{pmatrix}$$

Parameters: $a_1, b_{11}, b_{12}, b_{13}, d_1, a_2, b_{21}, b_{22}, b_{23}, d_2, a_3, b_{31}, b_{32}, b_{33}, d_3$.