Estimating Affine Multifactor Term Structure Models Using Closed-Form Likelihood Expansions—Technical Appendix*

Yacine Aït-Sahalia
Department of Economics
Princeton University and NBER†

Robert L. Kimmel
Fisher College of Business
The Ohio State University‡

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Abstract

This technical appendix for *Estimating Affine Multifactor Term Structure Models Using Closed-Form Likelihood Expansions* contains (1) formulae for the log-transition functions for the two one-factor affine models, (2) tables showing Monte Carlo simulations for the one- and two-factor models, and (3) real data estimates for the one- and two-factor models.

A. Appendix: Formulae for the Log-Transition Functions

In this Section, we give the coefficients of the closed-form expansions for the log-transition functions corresponding to the two one-dimensional models. Expansions for all models are available in computer form from the authors upon request.

A.1. The $A_0(1)$ Model

$$C_X^{(-1)}(x|x_0;\theta) = -\frac{(x_1 - x_{10})^2}{2}$$
$$C_X^{(0)}(x|x_0;\theta) = \frac{x_1 - x_{10}}{2} b_{11}(x_1 + x_{10})$$
$$C_X^{(1)}(x|x_0;\theta) = -\frac{b_{11}}{2} - \frac{b_{11}^2}{6}\left(x_1^2 + x_1 x_{10} + x_{10}^2\right)$$
$$C_X^{(2)}(x|x_0;\theta) = -\frac{b_{11}^2}{6}$$
$$C_X^{3}(x|x_0;\theta) = \frac{b_{11}^4}{60}\left(4x_1^2 + 7x_1 x_{10} + 4x_{10}^2\right)$$

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†Princeton, NJ 08544-1021. Phone: (609) 258-4015. E-mail: yacine@princeton.edu.
‡Columbus, OH 43210-1144. Phone: (614) 292-1875. E-mail: kimmel.42@osu.edu.

A.2. The $A_1(1)$ Model

$$
\begin{aligned}
C_X^{(-1)}\left(x|x_0;\theta\right) =& -2\left(\sqrt{x_1}-\sqrt{x_{10}}\right)^2 \\
C_X^{0}\left(x|x_0;\theta\right) =& b_{11}\left(x_1-x_{10}\right)+\left(a_1-\frac{1}{4}\right)\left(\ln x_1-\ln x_{10}\right) \\
C_X^{1}\left(x|x_0;\theta\right) =& -a_1 b_{11}-\frac{\left(a_1-\frac{1}{4}\right)\left(a_1-\frac{3}{4}\right)}{2\sqrt{x_1x_{10}}}-\frac{b_{11}^2}{6}\left(x_1+\sqrt{x_1x_{10}}+x_{10}\right) \\
C_X^{2}\left(x|x_0;\theta\right) =& -\frac{b_{11}^2}{24}-\frac{\left(a_1-\frac{1}{4}\right)\left(a_1-\frac{3}{4}\right)}{8x_1x_{10}} \\
C_X^{3}\left(x|x_0;\theta\right) =& \frac{\left(a_1^2-2a_1+\frac{7}{16}\right)\left(a_1-\frac{9}{16}\right)}{16\left(x_1x_{10}\right)^{\frac{3}{2}}}-\frac{b_{11}^2\left(a_1-\frac{1}{4}\right)\left(a_1-\frac{3}{4}\right)}{8\sqrt{x_1x_{10}}}+\frac{b_{11}^4}{240}\left(4x_1+7\sqrt{x_1x_{10}}+4x_{10}\right)
\end{aligned}
$$

Parameter	$\theta^{(TRUE)}$	$\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$		$\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ (perc. of $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$)		$\hat{\theta}^{(Euler)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)	
		Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
b_{11}	−0.5000	−6.7E − 03	0.041	−0.00%	0.00%	−278.86	1050.38
d_0^M	0.0300	3.1E − 04	0.010	−0.00%	0.00%	23.80	596.96
d_1	0.0300	2.4E − 06	9.4E − 04	0.02%	0.00%	−352814.41	13838.40
λ_1	0.1000	−3.0E − 03	0.32	0.00%	0.00%	−6303.82	1259.66
σ_1	0.0100	−2.0E − 05	3.2E − 04	0.00%	0.00%	−135.98	12.13

Table 1: Monte Carlo Simulations for the $A_0(1)$ Model - Structural Parameters, Weekly Observations

This table reports the results of $1,000$ Monte Carlo simulations for the $A_0(1)$ model, with weekly observations, comparing the distribution of the maximum-likelihood estimator $\hat{\theta}^{(MLE)}$ around the true value of the parameters $\theta^{(TRUE)}$, to the distribution of the differences between the reducible and Euler estimates ($\hat{\theta}^{(2)}$ and $\hat{\theta}^{(Euler)}$, respectively) and the exact MLE $\hat{\theta}^{(MLE)}$. The reducible likelihoods are based on the expansion with $K = 2$ terms. QML estimates are not shown, since for this model, they coincide with the exact MLE. The means and standard deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$ are reported as percentages of the corresponding deviations of $\hat{\theta}^{(MLE)}$ from $\theta^{(TRUE)}$. The means and standard deviations of $\hat{\theta}^{(Euler)}$ from $\hat{\theta}^{(MLE)}$ are reported as multiples of the corresponding deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$. The results show that the difference $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ is always much smaller than the difference $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$ due to the sampling noise. It also shows that the approximate MLE is much closer than the Euler estimate to the exact MLE; the mean and standard deviation of the difference is much smaller for the two-term approximate MLE than for Euler for all parameters.

Parameter	$\theta^{(TRUE)}$	$\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$		$\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ (perc. of $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$)		$\hat{\theta}^{(Euler)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)		$\hat{\theta}^{(QML)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)	
		Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
a_1^M	0.0257	3.5E − 03	3.4E − 03	−0.01%	0.25%	−1202.36	117.11	−900.94	116.17
b_{11}	−0.5000	−0.066	0.24	−0.01%	0.08%	574.32	56.43	362.56	51.38
d_0^M	0.0050	5.5E − 04	3.9E − 03	0.03%	0.05%	−4120.61	610.13	−2959.86	603.99
d_1	0.0225	2.0E − 04	3.0E − 03	0.53%	0.62%	−646.38	42.82	−299.60	42.57
λ_1	−0.1000	−0.063	0.23	−0.00%	0.02%	19617.10	304.38	11804.01	275.47
σ_1	0.0100	−6.5E − 06	3.2E − 04	−0.00%	0.00%	195.32	269.68	181.31	254.29

Table 2: Monte Carlo Simulations for the $A_1(1)$ Model - Structural Parameters, Weekly Observations

This table reports the results of 1,000 Monte Carlo simulations for the $A_1(1)$ model, with weekly observations, comparing the distribution of the maximum-likelihood estimator $\hat{\theta}^{(MLE)}$ around the true value of the parameters $\theta^{(TRUE)}$, to the distribution of the differences between the reducible, Euler, and QML estimates ($\hat{\theta}^{(2)}$, $\hat{\theta}^{(Euler)}$, and $\hat{\theta}^{(QML)}$, respectively) and the exact MLE $\hat{\theta}^{(MLE)}$. The reducible likelihoods are based on the expansion with $K = 2$ terms. The means and standard deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$ are reported as percentages of the corresponding deviations of $\hat{\theta}^{(MLE)}$ from $\theta^{(TRUE)}$. The means and standard deviations of $\hat{\theta}^{(Euler)}$ and $\hat{\theta}^{(QML)}$ from $\hat{\theta}^{(MLE)}$ are reported as multiples of the corresponding deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$. The results show that the difference $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ is always much smaller than the difference $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$ due to the sampling noise. It also shows that the approximate MLE is much closer than the Euler or QML estimate to the exact MLE; the mean and standard deviation of the difference is much smaller for the two-term approximate MLE than for Euler or QML for all parameters. The QML estimates are generally better than the Euler estimates, although not dramatically so.

Parameter	$\theta^{(TRUE)}$	$\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$		$\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ (perc. of $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$)		$\hat{\theta}^{(Euler)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)	
		Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
b_{11}	−0.5000	−0.050	0.14	−0.11%	0.05%	21.81	12.35
b_{21}	1.5000	0.21	0.70	−0.61%	0.18%	38.06	22.42
b_{22}	−2.0000	−0.20	0.42	−0.39%	0.22%	34.86	22.13
d_0^M	0.0200	1.7E − 03	9.6E − 03	−0.19%	0.04%	28.49	20.26
d_1	−0.0500	−2.6E − 04	5.2E − 03	−1.71%	0.02%	196.91	107.14
d_2	0.0300	−1.5E − 03	6.5E − 03	−0.54%	0.05%	−26.94	32.23
λ_1	−0.0500	2.4E − 03	0.35	−0.57%	0.01%	127.86	79.66
λ_2	−0.1000	−0.025	0.43	−0.16%	0.07%	−44.66	14.60
σ_1	0.0100	−1.0E − 06	3.2E − 04	0.25%	0.00%	−9.39	32.84
σ_2	0.0100	8.1E − 07	3.2E − 04	−0.31%	0.00%	−10.82	30.61

Table 3: Monte Carlo Simulations for the $A_0(2)$ Model - Structural Parameters, Weekly Observations

This table reports the results of 1,000 Monte Carlo simulations for the $A_0(2)$ model, with weekly observations, comparing the distribution of the maximum-likelihood estimator $\hat{\theta}^{(MLE)}$ around the true value of the parameters $\theta^{(TRUE)}$, to the distribution of the differences between the reducible and Euler estimates ($\hat{\theta}^{(2)}$ and $\hat{\theta}^{(Euler)}$, respectively) and the exact MLE $\hat{\theta}^{(MLE)}$. The reducible likelihoods are based on the expansion with $K = 2$ terms. QML estimates are not shown, since for this model, they coincide with the exact MLE. The means and standard deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$ are reported as percentages of the corresponding deviations of $\hat{\theta}^{(MLE)}$ from $\theta^{(TRUE)}$. The means and standard deviations of $\hat{\theta}^{(Euler)}$ from $\hat{\theta}^{(MLE)}$ are reported as multiples of the corresponding deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$. The results show that the difference $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ is much smaller, and usually several orders of magnitude smaller, than the difference $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$ due to the sampling noise. It also shows that the approximate MLE is much closer than the Euler estimate to the exact MLE; the mean and standard deviation of the difference is much smaller for the two-term approximate MLE than for Euler for all parameters.

Parameter	$\theta^{(TRUE)}$	$\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$		$\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ (perc. of $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$)		$\hat{\theta}^{(Euler)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)		$\hat{\theta}^{(QML)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)	
		Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
a_1^M	0.0063	2.9E − 03	0.025	−0.01%	0.01%	−419.01	1408.88	−483.59	1334.90
b_{11}	−0.1000	−0.032	0.11	−0.01%	0.01%	614.96	724.23	482.50	702.62
b_{22}	−3.0000	−0.10	0.43	−0.48%	0.26%	34.13	24.56	−0.89	15.47
d_0^M	−0.0010	1.4E − 04	0.025	0.11%	0.01%	−1111.19	1880.07	−1184.04	1784.03
d_1	0.0050	2.7E − 05	1.3E − 03	0.05%	0.01%	−5854.73	1803.23	−4357.55	1734.72
d_2	0.0300	−1.2E − 04	1.2E − 03	−0.61%	0.05%	−1175.56	181.34	−0.56	49.94
λ_1	−0.0500	−0.030	0.11	0.01%	0.00%	−660.72	1284.41	−578.03	1255.09
λ_2	−0.1000	1.53	0.39	−0.01%	0.11%	−179.09	33.60	−0.62	18.11
σ_1	0.0100	−2.0E − 05	3.2E − 04	0.00%	0.00%	−22.85	58.15	−14.71	33.01
σ_2	0.0100	−1.5E − 05	3.1E − 04	0.00%	0.00%	−33.26	62.44	−14.75	39.03

Table 4: Monte Carlo Simulations for the $A_1(2)$ Model - Structural Parameters, Weekly Observations

This table reports the results of 997 Monte Carlo simulations for the $A_1(2)$ model, with weekly observations, comparing the distribution of the maximum-likelihood estimator $\hat{\theta}^{(MLE)}$ around the true value of the parameters $\theta^{(TRUE)}$, to the distribution of the differences between the reducible, Euler, and QML estimates ($\hat{\theta}^{(2)}$, $\hat{\theta}^{(Euler)}$, and $\hat{\theta}^{(QML)}$, respectively) and the exact MLE $\hat{\theta}^{(MLE)}$. 3 of the originally generated 1000 simulations have been deleted from the sample due to numeric stability problems in calculating the exact MLE, $\hat{\theta}^{(MLE)}$. The reducible likelihoods are based on the expansion with $K = 2$ terms. The means and standard deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$ are reported as percentages of the corresponding deviations of $\hat{\theta}^{(MLE)}$ from $\theta^{(TRUE)}$. The means and standard deviations of $\hat{\theta}^{(Euler)}$ and $\hat{\theta}^{(QML)}$ from $\hat{\theta}^{(MLE)}$ are reported as multiples of the corresponding deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$. The results show that the difference $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ is always much smaller than the difference $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$ due to the sampling noise. It also shows that the approximate MLE is much closer than the Euler or QML estimate to the exact MLE; the mean and standard deviation of the difference is much smaller for the two-term approximate MLE than for Euler or QML for all parameters. The QML estimates are generally better than the Euler estimates, although not dramatically so.

Parameter	$\theta^{(TRUE)}$	$\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$		$\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ (perc. of $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$)		$\hat{\theta}^{(Euler)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)		$\hat{\theta}^{(QML)} - \hat{\theta}^{(MLE)}$ (mult. of $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$)	
		Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
a_1^M	0.0063	0.030	0.067	−0.00%	0.00%	−61548.41	59016.00	−109191.92	58132.61
a_2^M	0.1313	0.16	0.33	−0.01%	0.01%	−1481.98	9844.41	−1027.08	8639.94
b_{11}	−0.0300	0.024	0.012	0.00%	0.00%	1977.00	3043.88	766.02	2978.97
b_{22}	−2.5000	0.31	0.31	0.03%	0.05%	−141.14	278.30	−93.94	272.13
d_0^M	−0.0003	−0.015	0.060	−0.00%	0.00%	−588909.39	122653.94	−1030938.04	120722.09
d_1	0.0050	−1.5E − 03	1.6E − 03	−0.00%	0.00%	110.41	3701.16	−688.87	3676.68
d_2	0.0300	−8.7E − 04	8.8E − 03	−0.11%	0.01%	−2443.03	3247.71	−753.68	3307.75
λ_1	−0.0050	0.0283	9.7E − 03	−0.00%	0.01%	−889.32	2054.97	−522.03	2009.45
λ_2	−0.0100	0.40	0.18	0.00%	0.01%	−1240.56	2634.97	−755.71	2647.83
σ_1	0.0100	−1.1E − 05	3.2E − 04	0.00%	0.00%	−24.26	47.48	5.51	41.71
σ_2	0.0100	−2.4E − 05	3.1E − 04	0.00%	0.00%	31.58	40.63	42.08	32.04

Table 5: Monte Carlo Simulations for the $A_2(2)$ Model - Structural Parameters, Weekly Observations

This table reports the results of 1,000 Monte Carlo simulations for the $A_2(2)$ model, with weekly observations, comparing the distribution of the maximum-likelihood estimator $\hat{\theta}^{(MLE)}$ around the true value of the parameters $\theta^{(TRUE)}$, to the distribution of the differences between the reducible, Euler, and QML estimates ($\hat{\theta}^{(2)}$, $\hat{\theta}^{(Euler)}$, and $\hat{\theta}^{(QML)}$, respectively) and the exact MLE $\hat{\theta}^{(MLE)}$. The reducible likelihoods are based on the expansion with $K = 2$ terms. The means and standard deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$ are reported as percentages of the corresponding deviations of $\hat{\theta}^{(MLE)}$ from $\theta^{(TRUE)}$. The means and standard deviations of $\hat{\theta}^{(Euler)}$ and $\hat{\theta}^{(QML)}$ from $\hat{\theta}^{(MLE)}$ are reported as multiples of the corresponding deviations of $\hat{\theta}^{(2)}$ from $\hat{\theta}^{(MLE)}$. The results show that the difference $\hat{\theta}^{(2)} - \hat{\theta}^{(MLE)}$ is always much smaller than the difference $\hat{\theta}^{(MLE)} - \theta^{(TRUE)}$ due to the sampling noise. It also shows that the approximate MLE is much closer than the Euler or QML estimate to the exact MLE; the mean and standard deviation of the difference is much smaller for the two-term approximate MLE than for Euler or QML for all parameters. The QML estimates are generally better than the Euler estimates, although not dramatically so.

Parameter	$A_0\,(1)$		$A_1\,(1)$	
	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.02	(0.22)
b_{11}	-0.107	(0.019)	-0.315	(0.076)
d_0	0.006	(0.044)	0.0030	(0.0012)
d_1	0.02412	(0.00042)	0.00852	(0.00045)
λ_1	-0.52	(0.20)	-0.229	(0.069)
σ_1	0.00890	(0.00026)	0.00891	(0.00026)

Table 6: Parameter Estimates for One Factor Models—Dai-Singleton Market Price of Risk, Reducible Likelihood

This table reports the parameter estimates for the $A_0\,(1)$ and $A_1\,(1)$ models with the reducible likelihood approximation with $K = 2$ terms. Standard errors are reported in parentheses.

Parameter	$A_0\,(1)$		$A_1\,(1)$	
	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.01	(0.63)
b_{11}	-0.36	(0.14)	-0.13	(0.10)
d_0	0.022	(0.019)	0.0014	(0.0012)
d_1	0.02428	(0.00042)	0.00839	(0.00042)
a_1^Q	0.493	(0.087)	2.00	(0.21)
b_{11}^Q	-0.103	(0.019)	-0.098	(0.019)
σ_1	0.00890	(0.00026)	0.00890	(0.00026)

Table 7: Parameter Estimates for One Factor Models—Cheridito-Filipović-Kimmel Market Price of Risk, Reducible Likelihood

This table reports the parameter estimates for the $A_0\,(1)$ and $A_1\,(1)$ models with the reducible likelihood approximation with $K = 2$ terms. Standard errors are reported in parentheses.

Parameter	$A_0\,(1)$		$A_1\,(1)$	
	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.02	(0.24)
b_{11}	-0.107	(0.019)	-0.311	(0.070)
d_0	0.006	(0.044)	0.0031	(0.0014)
d_1	0.02401	(0.00042)	0.00798	(0.00044)
λ_1	-0.52	(0.20)	-0.228	(0.064)
σ_1	0.00890	(0.00026)	0.00891	(0.00026)

Table 8: Parameter Estimates for One Factor Models—Dai-Singleton Market Price of Risk, Euler Likelihood

This table reports the parameter estimates for the $A_0\,(1)$ and $A_1\,(1)$ models with the Euler likelihood approximation. Standard errors are reported in parentheses.

Parameter	$A_0\,(1)$		$A_1\,(1)$	
	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.01	(0.83)
b_{11}	-0.35	(0.14)	-0.13	(0.11)
d_0	0.021	(0.019)	0.0014	(0.0014)
d_1	0.02393	(0.00044)	0.00790	(0.00044)
a_1^Q	0.500	(0.087)	2.12	(0.23)
b_{11}^Q	-0.103	(0.019)	-0.098	(0.019)
σ_1	0.00890	(0.00026)	0.00890	(0.00026)

Table 9: Parameter Estimates for One Factor Models—Cheridito-Filipović-Kimmel Market Price of Risk, Euler Likelihood

This table reports the parameter estimates for the $A_0\,(1)$ and $A_1\,(1)$ models with the Euler likelihood approximation. Standard errors are reported in parentheses.

Parameter	$A_0(1)$		$A_1(1)$	
	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.02	(0.22)
b_{11}	-0.107	(0.019)	-0.315	(0.072)
d_0	0.006	(0.044)	0.0033	(0.0013)
d_1	0.02412	(0.00042)	0.00830	(0.00044)
λ_1	-0.52	(0.20)	-0.231	(0.066)
σ_1	0.00890	(0.00026)	0.00891	(0.00026)

Table 10: Parameter Estimates for One Factor Models—Dai-Singleton Market Price of Risk, QML Likelihood

This table reports the parameter estimates for the $A_0(1)$ and $A_1(1)$ models with the QML likelihood approximation. Standard errors are reported in parentheses.

Parameter	$A_0(1)$		$A_1(1)$	
	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.01	(0.81)
b_{11}	-0.36	(0.14)	-0.13	(0.11)
d_0	0.022	(0.019)	0.0014	(0.0012)
d_1	0.02428	(0.00042)	0.00803	(0.00040)
a_1^Q	0.493	(0.087)	2.08	(0.22)
b_{11}^Q	-0.103	(0.019)	-0.098	(0.019)
σ_1	0.00890	(0.00026)	0.00890	(0.00026)

Table 11: Parameter Estimates for One Factor Models—Cheridito-Filipović-Kimmel Market Price of Risk, QML Likelihood

This table reports the parameter estimates for the $A_0(1)$ and $A_1(1)$ models with the QML likelihood approximation. Standard errors are reported in parentheses.

Parameter	$A_0(2)$		$A_1(2)$—Restricted		$A_1(2)$—Unrestricted		$A_2(2)$—Restricted		$A_2(2)$—Unrestricted	
	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.01	(0.21)	0.00	(0.25)	0.01	(0.21)	0.00	(0.13)
a_2^M	-	-	-	-	-	-	0.12	(0.42)	0.01	(0.58)
b_{11}	-0.1117	(0.0062)	-0.000	(0.055)	-0.000	(0.051)	-0.118	(0.061)	-0.131	(0.082)
b_{12}	-	-	-	-	-	-	-	-	0.000	(0.098)
b_{21}	1.114	(0.056)	-	-	0.295	(0.037)	-	-	0.455	(0.090)
b_{22}	-1.164	(0.041)	-1.847	(0.096)	-1.185	(0.042)	0.00	(0.17)	0.00	(0.22)
d_0	0.005	(0.044)	-0.0000	(0.0073)	-0.0000	(0.0080)	0.0000	(0.0079)	0.0000	(0.0081)
d_1	0.00091	(0.00095)	0.00555	(0.00047)	0.00030	(0.00023)	0.00690	(0.00068)	0.00266	(0.00058)
d_2	0.02581	(0.00051)	0.03093	(0.00096)	0.02581	(0.00051)	0.0357	(0.0038)	0.0171	(0.0026)
λ_1	-0.11	(0.19)	0.070	(0.055)	0.097	(0.050)	-0.044	(0.060)	-0.038	(0.075)
λ_2	-0.80	(0.20)	-0.70	(0.22)	-0.81	(0.20)	1.28	(0.16)	1.02	(0.21)
σ_1	0.00286	(0.00027)	0.00300	(0.00031)	0.00287	(0.00027)	0.00292	(0.00028)	0.00291	(0.00028)
σ_2	0.00433	(0.00045)	0.00453	(0.00051)	0.00433	(0.00045)	0.00432	(0.00044)	0.00431	(0.00043)

Table 12: Parameter Estimates for Two Factor Models—Dai-Singleton Market Price of Risk, Reducible Likelihood

This table reports the parameter estimates for the $A_0(2)$, $A_1(2)$, and $A_2(2)$ models with the reducible likelihood approximation with $K = 2$ terms. For the latter two models, results for both the restricted (i. e., with the parameter restrictions from Table 2 in the paper imposed) and unrestricted versions of the model are reported. Standard errors are reported in parentheses.

Parameter	$A_0(2)$		$A_1(2)$—Restricted		$A_1(2)$—Unrestricted		$A_2(2)$—Restricted		$A_2(2)$—Unrestricted	
	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.0	(1.7)	0.0	(1.7)	-0.00	(0.70)	-0.00	(0.93)
a_2^M	-	-	-	-	-	-	0.0	(1.5)	0.0	(1.8)
b_{11}	-0.17	(0.12)	-0.19	(0.11)	-0.19	(0.11)	-0.15	(0.10)	-1.36	(0.31)
b_{12}	-	-	-	-	-	-	-	-	0.64	(0.31)
b_{21}	-0.32	(0.35)	-	-	0.224	(0.054)	-	-	0.00	(0.42)
b_{22}	-2.29	(0.30)	-0.44	(0.16)	-1.30	(0.25)	0.00	(0.14)	0.00	(0.22)
d_0	0.009	(0.033)	0.008	(0.016)	-0.001	(0.017)	-0.0000	(0.0073)	-0.0000	(0.0078)
d_1	0.0196	(0.0026)	0.00038	(0.00021)	0.00033	(0.00022)	-0.00041	(0.00087)	-0.0020	(0.0014)
d_2	0.0168	(0.0031)	0.02533	(0.00051)	0.02582	(0.00054)	0.00723	(0.00091)	0.00629	(0.00078)
a_1^Q	0.61	(0.15)	1.68	(0.34)	1.68	(0.34)	0.97	(0.17)	0.77	(0.20)
a_2^Q	0.40	(0.26)	-3.17	(0.89)	-0.21	(0.71)	0.5	(1.1)	0.5	(1.1)
b_{11}^Q	-0.14	(0.19)	-0.1002	(0.0066)	-0.0995	(0.0065)	-0.15	(0.15)	-0.35	(0.18)
b_{12}^Q	-1.20	(0.16)	-	-	-	-	0.036	(0.093)	0.094	(0.028)
b_{21}^Q	-0.02	(0.15)	0.276	(0.037)	0.273	(0.037)	1.42	(0.39)	2.11	(0.59)
b_{22}^Q	-1.12	(0.20)	-1.172	(0.040)	-1.186	(0.043)	-1.09	(0.16)	-0.88	(0.18)
σ_1	0.00286	(0.00028)	0.00286	(0.00027)	0.00286	(0.00027)	0.00287	(0.00027)	0.00287	(0.00027)
σ_2	0.00433	(0.00045)	0.00433	(0.00045)	0.00434	(0.00045)	0.00433	(0.00044)	0.00433	(0.00045)

Table 13: Parameter Estimates for Two Factor Models—Cheridito-Filipović-Kimmel Market Price of Risk, Reducible Likelihood

This table reports the parameter estimates for the $A_0(2)$, $A_1(2)$, and $A_2(2)$ models with the reducible likelihood approximation with $K = 2$ terms. For the latter two models, results for both the restricted (i. e., with the parameter restrictions from Table 2 in the paper imposed) and unrestricted versions of the model are reported. Standard errors are reported in parentheses.

Parameter	$A_0(2)$		$A_1(2)$—Restricted		$A_1(2)$—Unrestricted		$A_2(2)$—Restricted		$A_2(2)$—Unrestricted	
	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.01	(0.21)	0.00	(0.26)	0.01	(0.22)	0.00	(0.11)
a_2^M	-	-	-	-	-	-	0.12	(0.53)	0.01	(0.60)
b_{11}	-0.1117	(0.0062)	-0.000	(0.055)	-0.000	(0.050)	-0.118	(0.061)	-0.131	(0.077)
b_{12}	-	-	-	-	-	-	-	-	0.000	(0.090)
b_{21}	1.110	(0.056)	-	-	0.291	(0.036)	-	-	0.464	(0.076)
b_{22}	-1.163	(0.041)	-1.849	(0.096)	-1.189	(0.042)	0.00	(0.15)	0.00	(0.21)
d_0	0.005	(0.044)	-0.0000	(0.0074)	-0.0000	(0.0082)	0.0000	(0.0085)	0.0000	(0.0078)
d_1	0.00211	(0.00092)	0.00551	(0.00047)	0.00063	(0.00021)	0.00692	(0.00069)	0.00284	(0.00047)
d_2	0.02463	(0.00050)	0.02867	(0.00096)	0.02461	(0.00050)	0.0332	(0.0041)	0.0160	(0.0024)
λ_1	-0.12	(0.19)	0.070	(0.054)	0.096	(0.050)	-0.043	(0.061)	-0.037	(0.074)
λ_2	-0.83	(0.20)	-0.76	(0.22)	-0.84	(0.20)	1.25	(0.15)	1.01	(0.19)
σ_1	0.00286	(0.00027)	0.00300	(0.00031)	0.00287	(0.00027)	0.00292	(0.00027)	0.00291	(0.00027)
σ_2	0.00433	(0.00045)	0.00454	(0.00051)	0.00433	(0.00045)	0.00431	(0.00043)	0.00431	(0.00043)

Table 14: Parameter Estimates for Two Factor Models—Dai-Singleton Market Price of Risk, Euler Likelihood

This table reports the parameter estimates for the $A_0(2)$, $A_1(2)$, and $A_2(2)$ models with the Euler likelihood approximation. For the latter two models, results for both the restricted (i. e., with the parameter restrictions from Table 2 in the paper imposed) and unrestricted versions of the model are reported. Standard errors are reported in parentheses.

Parameter	$A_0\,(2)$		$A_1\,(2)$—Restricted		$A_1\,(2)$—Unrestricted		$A_2\,(2)$—Restricted		$A_2\,(2)$—Unrestricted	
	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.0	(1.7)	0.0	(1.7)	-0.00	(0.78)	-0.0	(1.2)
a_2^M	-	-	-	-	-	-	0.0	(1.4)	0.0	(1.7)
b_{11}	-0.17	(0.12)	-0.20	(0.11)	-0.19	(0.11)	-0.23	(0.11)	-1.23	(0.28)
b_{12}	-	-	-	-	-	-	-	-	0.73	(0.27)
b_{21}	-0.30	(0.31)	-	-	0.207	(0.049)	-	-	0.00	(0.32)
b_{22}	-2.12	(0.25)	-0.44	(0.15)	-1.23	(0.22)	0.00	(0.14)	-0.00	(0.21)
d_0	0.009	(0.032)	0.008	(0.016)	-0.002	(0.017)	-0.0000	(0.0074)	-0.0000	(0.0083)
d_1	0.0193	(0.0024)	0.00038	(0.00020)	0.00056	(0.00022)	-0.00016	(0.00058)	-0.00094	(0.00081)
d_2	0.0151	(0.0028)	0.02487	(0.00050)	0.02449	(0.00050)	0.00713	(0.00079)	0.00642	(0.00074)
a_1^Q	0.62	(0.15)	1.74	(0.37)	1.74	(0.37)	1.04	(0.13)	0.94	(0.18)
a_2^Q	0.45	(0.28)	-3.29	(0.94)	-0.25	(0.76)	0.5	(1.2)	0.5	(1.3)
b_{11}^Q	-0.13	(0.17)	-0.1002	(0.0065)	-0.0993	(0.0065)	-0.10	(0.11)	-0.26	(0.13)
b_{12}^Q	-1.11	(0.14)	-	-	-	-	0.000	(0.086)	0.083	(0.044)
b_{21}^Q	-0.02	(0.15)	0.275	(0.038)	0.272	(0.038)	1.34	(0.28)	1.68	(0.39)
b_{22}^Q	-1.12	(0.18)	-1.176	(0.040)	-1.191	(0.044)	-1.15	(0.11)	-0.98	(0.13)
σ_1	0.00286	(0.00028)	0.00286	(0.00027)	0.00286	(0.00027)	0.00287	(0.00027)	0.00287	(0.00027)
σ_2	0.00432	(0.00045)	0.00433	(0.00045)	0.00434	(0.00045)	0.00433	(0.00044)	0.00432	(0.00045)

Table 15: Parameter Estimates for Two Factor Models—Cheridito-Filipović-Kimmel Market Price of Risk, Euler Likelihood

This table reports the parameter estimates for the $A_0\,(2)$, $A_1\,(2)$, and $A_2\,(2)$ models with the Euler likelihood approximation. For the latter two models, results for both the restricted (i. e., with the parameter restrictions from Table 2 in the paper imposed) and unrestricted versions of the model are reported. Standard errors are reported in parentheses.

Parameter	$A_0\,(2)$		$A_1\,(2)$—Restricted		$A_1\,(2)$—Unrestricted		$A_2\,(2)$—Restricted		$A_2\,(2)$—Unrestricted	
	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.01	(0.21)	0.00	(0.26)	0.01	(0.22)	0.00	(0.11)
a_2^M	-	-	-	-	-	-	0.12	(0.51)	0.01	(0.62)
b_{11}	-0.1117	(0.0062)	-0.000	(0.055)	-0.000	(0.050)	-0.120	(0.061)	-0.130	(0.078)
b_{12}	-	-	-	-	-	-	-	-	0.000	(0.087)
b_{21}	1.114	(0.056)	-	-	0.291	(0.036)	-	-	0.473	(0.072)
b_{22}	-1.164	(0.041)	-1.859	(0.099)	-1.189	(0.042)	0.00	(0.15)	0.00	(0.20)
d_0	0.005	(0.044)	-0.0000	(0.0075)	-0.0000	(0.0082)	0.0000	(0.0084)	0.0000	(0.0079)
d_1	0.00091	(0.00095)	0.00550	(0.00047)	0.00031	(0.00022)	0.00689	(0.00069)	0.00294	(0.00049)
d_2	0.02581	(0.00051)	0.03089	(0.00096)	0.02581	(0.00051)	0.0318	(0.0037)	0.0157	(0.0021)
λ_1	-0.11	(0.19)	0.070	(0.054)	0.097	(0.050)	-0.045	(0.061)	-0.037	(0.075)
λ_2	-0.80	(0.20)	-0.70	(0.22)	-0.81	(0.20)	1.28	(0.15)	1.03	(0.19)
σ_1	0.00286	(0.00027)	0.00300	(0.00031)	0.00287	(0.00027)	0.00292	(0.00028)	0.00291	(0.00027)
σ_2	0.00433	(0.00045)	0.00454	(0.00051)	0.00433	(0.00045)	0.00432	(0.00043)	0.00430	(0.00043)

Table 16: Parameter Estimates for Two Factor Models—Dai-Singleton Market Price of Risk, QML Likelihood

This table reports the parameter estimates for the $A_0\,(2)$, $A_1\,(2)$, and $A_2\,(2)$ models with the QML likelihood approximation. For the latter two models, results for both the restricted (i. e., with the parameter restrictions from Table 2 in the paper imposed) and unrestricted versions of the model are reported. Standard errors are reported in parentheses.

Parameter	$A_0(2)$		$A_1(2)$—Restricted		$A_1(2)$—Unrestricted		$A_2(2)$—Restricted		$A_2(2)$—Unrestricted	
	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.	Estimate	Std. Dev.
a_1^M	-	-	0.0	(1.6)	0.0	(1.6)	−0.00	(0.71)	−0.0	(1.0)
a_2^M	-	-	-	-	-	-	0.0	(1.4)	0.0	(1.7)
b_{11}	−0.17	(0.12)	−0.19	(0.12)	−0.19	(0.11)	−0.28	(0.11)	−1.51	(0.32)
b_{12}	-	-	-	-	-	-	-	-	0.70	(0.27)
b_{21}	−0.32	(0.35)	-	-	0.225	(0.055)	-	-	0.00	(0.42)
b_{22}	−2.31	(0.30)	−0.45	(0.16)	−1.31	(0.25)	0.00	(0.14)	0.00	(0.22)
d_0	0.009	(0.033)	0.007	(0.016)	−0.001	(0.017)	−0.0000	(0.0071)	−0.0000	(0.0081)
d_1	0.0196	(0.0026)	0.00039	(0.00021)	0.00034	(0.00022)	−0.00018	(0.00060)	−0.0015	(0.0010)
d_2	0.0168	(0.0031)	0.02534	(0.00051)	0.02582	(0.00054)	0.00717	(0.00074)	0.00629	(0.00069)
a_1^Q	0.61	(0.15)	1.69	(0.36)	1.69	(0.37)	0.99	(0.13)	0.80	(0.18)
a_2^Q	0.40	(0.26)	−3.20	(0.90)	−0.21	(0.71)	0.5	(1.2)	0.5	(1.3)
b_{11}^Q	−0.14	(0.19)	−0.0999	(0.0065)	−0.0992	(0.0065)	−0.10	(0.11)	−0.28	(0.14)
b_{12}^Q	−1.21	(0.16)	-	-	-	-	0.000	(0.083)	0.078	(0.035)
b_{21}^Q	−0.02	(0.15)	0.276	(0.039)	0.274	(0.039)	1.39	(0.29)	2.04	(0.49)
b_{22}^Q	−1.12	(0.20)	−1.177	(0.040)	−1.191	(0.044)	−1.14	(0.11)	−0.95	(0.15)
σ_1	0.00286	(0.00028)	0.00286	(0.00027)	0.00286	(0.00027)	0.00287	(0.00027)	0.00287	(0.00027)
σ_2	0.00433	(0.00045)	0.00433	(0.00045)	0.00434	(0.00045)	0.00433	(0.00044)	0.00433	(0.00045)

Table 17: Parameter Estimates for Two Factor Models—Cheridito-Filipović-Kimmel Market Price of Risk, QML Likelihood

This table reports the parameter estimates for the $A_0(2)$, $A_1(2)$, and $A_2(2)$ models with the QML likelihood approximation. For the latter two models, results for both the restricted (i. e., with the parameter restrictions from Table 2 in the paper imposed) and unrestricted versions of the model are reported. Standard errors are reported in parentheses.