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## Dynamic equilibrium and volatility in financial asset markets

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### Abstract

This paper develops and estimates a continuous-time model of a financial market where investors' trading strategies and the specialist's rule of price adjustments are best response to each other. We examine how far modeling market microstructure in a purely rational framework can go in explaining alleged asset pricing anomalies. The model produces some major findings of the empirical literature: excess volatility of the market price compared to the asset's fundamental value, serially correlated volatility, contemporaneous volume–volatility correlation, and excess kurtosis of price changes. We implement a nonlinear filter to estimate the unobservable fundamental value, and avoid the discretization bias by computing the exact conditional moments of the price and volume processes over time intervals of any length. © 1998 Elsevier Science S.A.

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## 1. Introduction

Microeconomic theory is essentially concerned with the study of market equilibrium. Agents make plans, in general as a result of solving individual optimization problems. Then certain variables, typically prices, are assumed to take the values required for those plans to be mutually consistent. The resulting prices are called equilibrium prices. All in all, very little is said about the mechanism(s) that make possible, compute, and finally implement such an equilibrium in the first place. Even less is said about how the economy would return to equilibrium if the equilibrium were shifted for any reason. Equilibrium changes are treated in a comparative statics framework, where the values of some parameters and consequently the equilibrium are shifted. The price moves to the new equilibrium, and trade can only occur in equilibrium.

Casual observation of financial markets, however, reveals that this paradigm may not translate very well in markets where (i) prices move constantly, (ii) trading occurs at very high frequency and at every price and (iii) the volatility of the prices is a crucial element (see Black, 1986). We attempt to model the price formation process to reflect these empirical features. Investors trade for portfolio reasons a speculative asset. The specialist posts a price at which he is required to take the opposite side of the order submitted. After executing the order, the specialist can adjust the price. Observing the price being posted, the investor then acts to rebalance optimally his portfolio, trading occurs, the specialist revises optimally the market price to maximize his expected trading profits and so on. Instead of prices, strategies constitute an equilibrium. An equilibrium is a specialist's pricing rule and a sequence of investors' trading strategies that are mutually best response to each other. The specialist acts as a Stackelberg leader: he receives the investors' demand function before setting his price.

We show that in equilibrium, the specialist (who controls the price adjustment process) finds it optimal to add volatility to the price he posts compared to the exogenous 'fundamental value (or price)' of the asset, which is defined as the expected value of the sum of discounted future dividends. The spread between the market and fundamental value of the asset drives the expected return of the asset. By adding volatility, the specialist reduces the holdings of risk-averse investors when high returns are expected, which benefits him since he is taking the opposite side of the investors' trades. The model involves nonlinear strategies but is nevertheless solved in closed form. The price and volume processes determined in equilibrium provide some microeconomic foundations for the statistical specifications of price and volume processes in the literature (see Epps and Epps, 1976; Tauchen and Pitts, 1983; Karpoff, 1987).

We then revisit some of the classical empirical asset pricing 'anomalies'. The model can generate in a fully optimizing world many effects, such as excess volatility or mean reversion in asset prices, which were sometimes interpreted as

sure signs of market irrationality (e.g. Shiller, 1981; Summers, 1986). There is no exogenous source of noise such as noise traders in Kyle (1985), or pure exogenous noise as in Campbell and Kyle (1993). A nice feature of this model is that these relevant issues in asset pricing can be interpreted as simple hypothesis tests on one or more of the three parameters of the model:  $\alpha$  which measures how market prices mean-revert to the stochastic fundamental price,  $\beta$  which measures how the volatility of the market price relates to the spread between market and fundamental prices and finally  $\sigma$  which measures the volatility of the fundamental price itself.

The estimation of the model presents several challenges. A state variable, the fundamental price, is unobservable to the econometrician, and must therefore be estimated along with the parameters. The dynamics of the state variables produced by the model are nonlinear. We are interested in testing hypotheses which involve time-series features of the fundamental price: for example, is the fundamental price less volatile than the observed market price? We propose to use the tools of conditionally optimal filtering to achieve this task. Next, we derive closed-form expressions for the conditional moments of the joint processes determining the market and fundamental prices, and the trading volume, conditioned on their history. These conditional moments however cannot be used to form moment conditions in a GMM framework, because the conditioning set contains variables unobservable to the econometrician (the history of the fundamental price process). We therefore compute, again in closed form, the moments of the joint processes conditioned only on the observable market price and volume processes. These moments are the basis used to construct a GMM estimator of the parameters of the model. Despite the fact that the model is written in continuous time and the data are sampled at discrete time intervals, the estimator is free of discretization bias.

The paper is organized as follows. Section 2 presents the model and solves for the equilibrium price and volume processes. Section 3 focuses on the estimation strategy. Section 4 examines the empirical implications of the model. Section 5 concludes.

## 2. Equilibrium dynamics

### 2.1. Price formation

Consider a financial exchange where a stock is traded by a specialist and a price-taker risk-averse investor. In this model, the single specialist enjoys monopoly power on a given stock, a situation typical of organized exchanges. By contrast, in dealer markets, multiple markets makers are supposed to compete on a given stock. However there exists some empirical evidence to justify modeling the market power of market makers even in a dealer market (for

example, collusion and other non-competitive behavior among market makers on NASDAQ is suggested by Christie and Schultz (1994), and Christie et al., (1994). The single investor assumption is a proxy for a continuum of small identical investors, each one of them price-taker. The investor can buy and sell the stock as well as lend or borrow at the constant risk-free rate  $r$ . There are no constraints on borrowing or short sales, and the stock carries unlimited liability. Throughout the paper  $Z_i$   $i = 1, 2$ , denote standard Brownian Motions.

Between  $t$  and  $t + dt$  each share of the stock pays an exogenously-determined stochastic dividend  $D_t dt$ . We assume that  $D_t$  is a martingale following  $dD_t = r\sigma dZ_{2t}$  with  $\sigma$  constant. We define the stock's fundamental value or price  $\bar{p}_t$  to be the expected value of the sum of discounted future dividends.<sup>1</sup> We have that

$$\bar{p}_t = E_t \left[ \int_t^{+\infty} e^{-r(\tau-t)} D_\tau d\tau \right] = \int_t^{+\infty} e^{-r(\tau-t)} E_t[D_\tau] d\tau = (1/r) D_t, \quad (1)$$

since  $E_t[D_\tau] = D_t$ . The fundamental price therefore follows the dynamics:

$$d\bar{p}_t = \sigma dZ_{2t}. \quad (2)$$

A change in  $\bar{p}_t$  reflects the arrival of new information regarding the future cash flows generated by the stock. We attempt in this paper to model microstructure effects and do not assume that the stock necessarily trades for  $\bar{p}_t$ . Instead let  $p_t$  be the market price of the stock at time  $t$ . Define the price spread:

$$s_t \equiv \bar{p}_t - p_t. \quad (3)$$

For simplicity, take the initial value as  $s_0 = 0$ . The form of the dynamics (2) is common knowledge, and  $\bar{p}_t$  is revealed at every instant to all market participants. At instant  $t$ , the investor desires to hold  $q_t$  shares, and having observed  $\bar{p}_t$ , submits his demand function to the specialist. The investor is allowed to condition his trades on price. The specialist then executes the buy or sell order received and takes the opposite side of the trade. NYSE Rule 104 specifies that: 'The specialist must take or supply stock as necessary (...)' (NYSE, 1995). In practice, many trades get executed with no formal intervention by the specialist, a fact which is ignored here. In the model, the specialist must hold  $-q_t$  shares (the stock is assumed to be in zero net supply). He may then revise the price by an amount  $dp_t$ . The stock then pays its instantaneous rate of return. We denote by

<sup>1</sup> We may interpret  $\bar{p}_t$  as the price that would prevail for the asset in a pure competitive economy with risk-neutral agents (Lucas, 1978). We are not assuming this setup here.

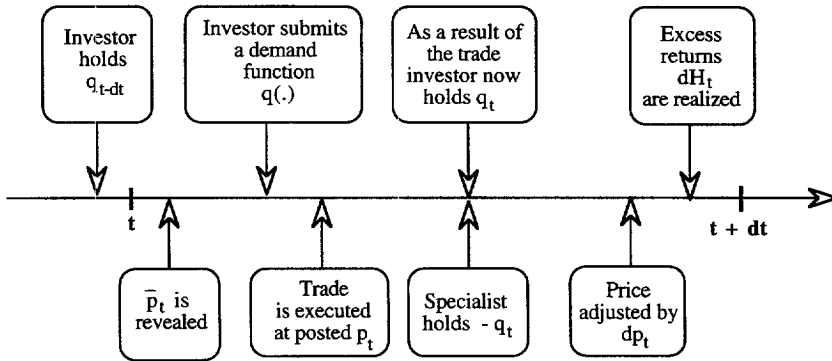


Fig. 1. Chronology of events.

$F_t$  the information set consisting of the sequence of past and present fundamental and market prices.<sup>2</sup> Fig. 1 summarizes the price formation in this market.

### 2.2. Equilibrium concept

We model this market’s microstructure as a stochastic differential game, with the specialist acting as a Stackelberg leader. The investor takes the price adjustment rule as given and determines his optimal holding of the stock, i.e., his best response, by maximizing his expected utility. Knowing the demand function of the investor, i.e., the investor’s best response to his choice of price adjustment, the specialist determines the price adjustment by maximizing his expected profit. This is an equilibrium in strategies (not prices or quantities): each player’s choice of an optimal strategy is a control problem in which he takes into account the effect of his actions on the state, both directly and indirectly through the influence of the state on the strategies of his opponent. We make all of this more specific below.

### 2.3. Optimization by the specialist

In deciding how to revise the price, the specialist faces two constraints – like most monopolies, he is regulated. First, he must provide price continuity. The price continuity rate measures the percentage of all trades occurring with no

<sup>2</sup> Formally,  $F_t$  is the augmentation of the increasing family of  $\sigma$ -fields generated by the stochastic processes  $\{p_s, \bar{p}_s | 0 \leq s \leq t\}$ .

change in price or a one-tick change (1/8 on the NYSE). Second, the exchange expects the specialist to stabilize price movements. The specialist performance is partly assessed through the stabilization rate, the percentage of shares purchased by the specialist at prices below or sold at prices above the last different price. The NYSE fixes a minimal monthly stabilization rate for its specialists. Table 1 reports these two rates for the NYSE.

We incorporate these two requirements in the model by constraining the possible price revision  $dp_t$  that the specialist can choose. If the stock's fundamental value were constant, we could possibly model price continuity as a constraint that sets the drift of  $dp_t$  to zero. However because  $\bar{p}_t$  is stochastic in the model, we interpret the price continuity rule as a requirement that  $p_t$  be held as constant as possible – but only after adjusting to the new level of  $\bar{p}_t$ . Suppose that news affecting the stock are released, that is a large realization of  $dZ_{2t}$  occurs, so that the fundamental value changes substantially between  $t$  and  $t + dt$ . The specialist is still expected to provide price continuity, but it seems natural for him to be allowed (and even, encouraged) to respond to changes in  $\bar{p}_t$ . The NYSE Specialist's Job Description stipulates that the specialist should 'initiate trading in each security as soon as market conditions allow, at a price that reflects a thorough, professional assessment of market conditions at the time'. This could be viewed as saying that the specialist, faced with a shock to  $\bar{p}_t$ , is expected to do whatever possible to move in an orderly fashion the market price in relation to the new value of  $\bar{p}_t$ . The job description does encourage specialists to learn about the companies (i.e., their  $\bar{p}_t$ ): 'in order to establish a positive professional relationship with Exchange-listed companies, [the specialist should] contact during each quarter one or more senior officials [of the company] of the rank of Corporate Secretary or above'.

To capture this effect, a tractable assumption is to set the drift of  $dp_t$  to have the simple linear form:  $\alpha(\bar{p}_t - p_t)dt = \alpha s_t dt$ . In order to satisfy price continuity we constrain the specialist to not further adjust the price deterministically. The specialist however controls the price adjustment through its volatility  $v_i$ :

$$dp_t = \alpha(\bar{p}_t - p_t)dt + v_t dZ_{1t}. \quad (4)$$

Table 1  
Price continuity and stabilization (Source: NYSE 1994 Fact Book)

Year	Price continuity rate (%)	Price stabilization rate (%)
1994	97.4	76.3
1993	97.1	77.6
1992	96.4	78.3
1991	95.9	80.9
1990	95.8	83.1

Controlling  $v_t$  is how the specialist exploits his monopoly power. The specialist cannot predict the dividend shock between  $t$  and  $t + dt$ , and hence we assume that  $E[dZ_{1t}dZ_{2t}] = 0$ .

Given that price adjustment, the instantaneous excess return provided by the stock is given by

$$dH_t = D_t dt + dp_t - rp_t dt = \{r(\bar{p}_t - p_t)\} dt + dp_t = (\alpha + r)s_t dt + v_t dZ_{1t}. \quad (5)$$

The drift of the excess return depends on the price spread  $s_t$ , which from (4) has the dynamics:

$$ds_t = -\alpha s_t dt + \sigma dZ_{2t} - v_t dZ_{1t}. \quad (6)$$

Future excess returns are therefore (partly) predictable, and investors will choose their trading strategies by taking into account (5) and (6).

We further interpret price continuity and stabilization as limiting the specialist's ability to set an arbitrarily large volatility. For example, according to the NYSE Rule 104, 'the maintenance of a fair and orderly market implies the maintenance of price continuity with reasonable depth, and the minimization of the effects of temporary disparity between supply and demand' (NYSE (1995), 2104.10). Instead of setting an artificial upper limit to the admissible volatility choice, we assume that this objective is achieved by implementing an incentive scheme. The specialist receives a net monetary transfer  $T$  (positive or negative) from the exchange depending upon the level of volatility that he decides to set. Large volatility choices get penalized ( $T < 0$ ), low volatility ones rewarded ( $T > 0$ ). Other things equal, for small volatility values, the transfer is higher when the specialist must react to a larger change in  $\bar{p}_t$ , i.e., when  $s_t$  is large in absolute value. This reflects the added difficulty for the specialist of doing business following a large change in what the NYSE Rules call 'market conditions'. Let  $T(v_t, s_t) dt$  be the amount of the transfer from the exchange to the specialist between  $t$  and  $t + dt$ . We therefore want the transfer function  $T$  to satisfy:

$$\partial T / \partial v_t < 0 \quad (7)$$

and

$$\lim_{v_t \rightarrow +\infty} T < 0 \quad \lim_{v_t \rightarrow 0} T > 0 \quad (8)$$

and be such that in equilibrium the specialist earns zero expected total profit. This arrangement can be seen as a simple proxy to represent more complex institutional schemes.

We now propose a particular case of a transfer function which has the advantage of allowing for a closed-form solution to the equilibrium and satisfies

(7), (8) and the zero profit condition:

$$T(v_t, s_t) = \frac{(\alpha + r)^2 s_t^2}{ar} \left[ \frac{\sigma^2 + v_t^2}{(3\sigma^2 + \beta s_t^2 + v_t^2)^2} + \frac{1}{v_t^2} - \frac{1}{4(2\sigma^2 + \beta s_t^2)} \right]. \tag{9}$$

Note that if the investor is infinitely risk-averse ( $a = \infty$ ) he never holds the stock, no trade ever takes place, and the transfer function is identically zero. A plot of the transfer function is provided in Fig. 2.

The specialist chooses his optimal volatility level, hence his price adjustment rule (3), in order to maximize his total expected profit. Total profit consists of both the transfer from the exchange and the specialist’s trading revenue:

$$\max_{\{v_t, \tau \geq t\}} E_t \left[ \int_t^{+\infty} e^{-r\tau} \{T(v_\tau, s_\tau) d\tau + d\Pi_\tau\} \right], \tag{10}$$

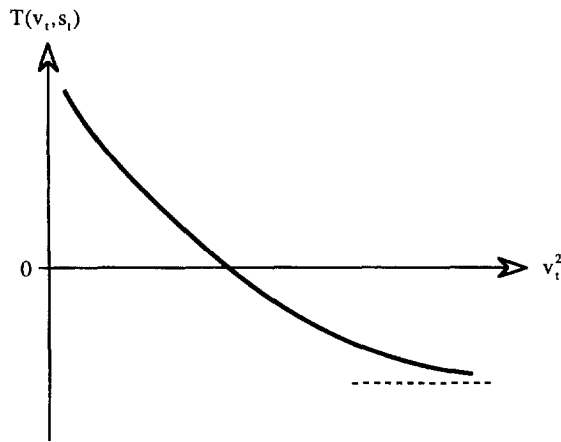


Fig. 2. (a) Effect of a change in volatility on the transfer function.

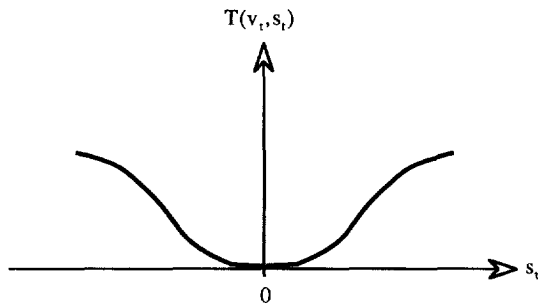


Fig. 2. (b) Effect of a change in price spread on the transfer function for low values of volatility.



where  $d\Pi_t$  is the trading revenue derived from his stock holdings between  $t$  and  $t + dt$ . As indicated, the specialist recognizes that the number of shares  $q_t$  demanded by the investor will depend on his choice of volatility. The specialist is not allowed to trade independently for his own account, but instead must clear the market according to the investor's desires and therefore holds  $-q_t$  shares of the stock. Given the investor's demand function  $q_t = q(v_t, s_t)$ , the specialist's trading revenue is

$$d\Pi_t = -q(v_t, s_t) dH_t. \tag{11}$$

From (5) and  $E_t[dZ_{it}] = 0$ , it follows that

$$E_t \left[ \int_t^{+\infty} e^{-r\tau} d\Pi_\tau \right] = E_t \left[ \int_t^{+\infty} e^{-r\tau} \{ -q(v_\tau, s_\tau) \beta s_\tau \} d\tau \right] \tag{12}$$

To summarize: the specialist chooses  $\{v_t\}$  to maximize (10), subject to the dynamics of the state variable  $s_t$  given by (6). We now describe how the investor determines his demand function  $q_t = q(v_t, s_t)$ .

#### 2.4. Optimization by investors

The investor's portfolio holdings  $q_t$  and consumption  $c_t$  at date  $t$  are determined by maximizing his expected utility. In doing so, the investor takes as given the rule of price adjustment (4) determined by the specialist. To avoid wealth effects, we assume that the investor derives exponential utility  $u(c_t) \equiv -\exp(-ac_t)$  from his consumption stream, with Arrow–Pratt coefficient of absolute risk aversion  $a$  and discount factor  $\rho$ . We further assume that the investor's aversion for volatility goes beyond the effect captured by the concavity of  $u(c_t)$ . He dislikes volatility so much that, at each date, his utility derived from consumption is reduced by a factor  $\exp(-\gamma/v_t^2)$ , making his overall utility function state-dependent. Since his utility is negative, states of nature where this factor takes large (resp. small) values, i.e., state where volatility  $v_t$  is high (resp. low), correspond to a large (resp. small) reduction in his utility level. Here  $\gamma$  is a constant parameter.

Thus the investor's objective is

$$\max_{\{c_t, q_t, \tau \geq t\}} E_t \left[ \int_t^{+\infty} -\exp(-\rho\tau - ac_\tau + \gamma/v_\tau^2) d\tau \right] \tag{13}$$

subject to (6) and the dynamics of his wealth:

$$\begin{aligned} dW_t &= rW_t dt - c_t dt + q_t dH_t \\ &= \{rW_t + (\alpha + r)s_t q_t - c_t\} dt + q_t v_t dZ_{1t}. \end{aligned} \tag{14}$$

When formulating his demand for the stock, the investor will exploit the predictability of the (risky) excess returns (5) to the maximum extent permitted by his own risk aversion. At time  $t + dt$  each investor will trade the quantity  $dq_t$  required to insure that his investment  $q_{t+dt}$  in the stock is optimal at that point in time, given his holdings  $q_t$  an instant earlier, that is:  $dq_t = q_{t+dt} - q_t$ .

### 2.5. Computation of the equilibrium

We first solve the investor's optimization problem, obtaining his best response function to every possible choice of volatility by the specialist. Then we find the optimal volatility choice by the specialist (acting as a Stackelberg leader) given the investor's best response function. The result is summarized as:

*Proposition 1. In equilibrium, the specialist, who as a Stackelberg leader takes into account the investor's best response, chooses volatility optimally as*

$$v_t = v(s_t) = \sqrt{\sigma^2 + \beta s_t^2}. \quad (15)$$

The investor's equilibrium holdings of the risky asset are given by

$$q_t = q(s_t, v(s_t)) = \frac{(\alpha + r)s_t}{ar(\sigma^2 + \beta s_t^2)}, \quad (16)$$

where  $\beta \equiv \sigma^2(\alpha + r)^2 / (2ar\gamma)$ .

*Proof. See the appendix.*

The basic intuition for the result is the following: when  $s_t > 0$  (resp.  $s_t < 0$ ), positive (resp. negative) excess returns are expected, and the investor desires to hold (resp. short) more shares of the stock: see (16). This is detrimental to the specialist, who must take the opposite side of the trade: see (10). By inputting volatility into the price process when  $s_t \neq 0$  (see Fig. 3), the specialist manages to reduce the holdings of investors when excess returns are expected. Since investors are risk averse, they indeed respond to the extra volatility by demanding fewer shares of the stock than they would under constant volatility (see Fig. 4). The specialist exploits optimally the rent derived from his monopoly power. In equilibrium, he makes zero expected profit. The Exchange exactly compensates him for having to face the trades of an investor who can optimally exploit the predictability of future returns.

### 2.6. Empirical implications

We now show that the equilibrium produces the stylized facts cited in the literature as the main characteristics of the joint distribution of price and volume

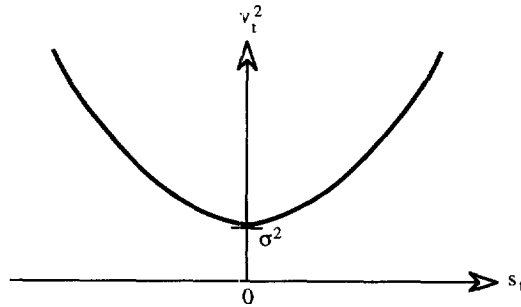


Fig. 3. Variance of the price adjustment in equilibrium.

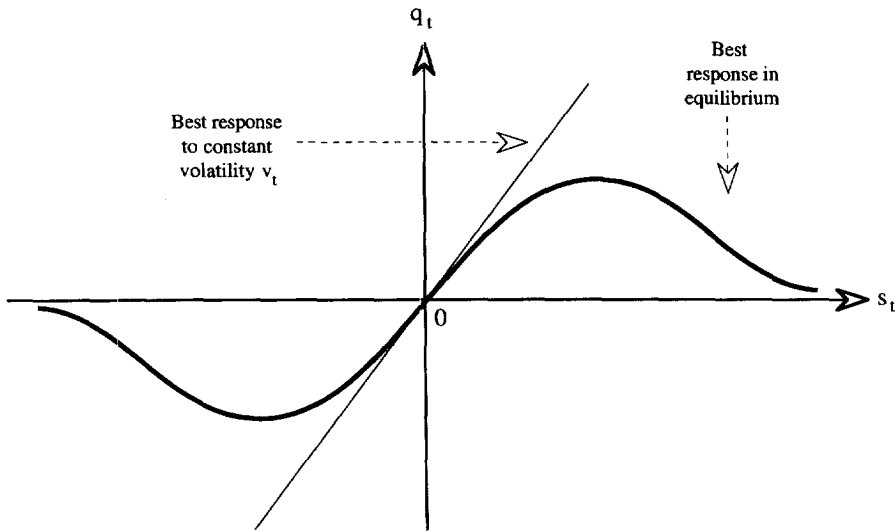


Fig. 4. Investor's asset holdings in equilibrium.

data (see Karpoff, 1987; Gallant et al., 1992): serial correlation in the conditional volatility of price changes (the ARCH effect), contemporaneous correlation between trading volume and absolute changes in prices, and excess kurtosis of price changes.<sup>3</sup>

<sup>3</sup> Skewness of price changes, while less apparent in the distribution estimated from the time series of stock prices, is however typically present in the distribution of stock price changes implicit in the prices of traded stock options (the 'smile' effect).

In order to examine these effects, we need to derive some properties of the moments of  $s_t$ :

*Proposition 2.* (i) *The price spread  $s_t$  is strictly stationary and the first four conditional moments of  $s_{t+\Delta}$  given  $s_t$  are given in closed-form for any time interval  $\Delta > 0$  by*

$$\begin{aligned}
 E[s_{t+\Delta} | s_t] &= s_t e^{-\alpha\Delta}, \\
 E[s_{t+\Delta}^2 | s_t] &= \frac{2\sigma^2}{2\alpha - \beta} + \left( s_t^2 - \frac{2\sigma^2}{2\alpha - \beta} \right) e^{-(2\alpha - \beta)\Delta}, \\
 E[s_{t+\Delta}^3 | s_t] &= \left( \frac{6\sigma^2 s_t}{2\alpha - 3\beta} \right) e^{-\alpha\Delta} + \left( s_t^3 - \frac{6\sigma^2 s_t}{2\alpha - 3\beta} \right) e^{-(3\alpha - 3\beta)\Delta}, \\
 E[s_{t+\Delta}^4 | s_t] &= \left( \frac{12\sigma^4}{(2\alpha - 3\beta)(2\alpha - \beta)} \right) + 12\sigma^2 \left( \frac{s_t^2}{(2\alpha - 5\beta)} - \frac{2\sigma^2}{(2\alpha - \beta)} \right) e^{-(2\alpha - \beta)\Delta} \\
 &\quad + \left( s_t^4 - \frac{12\sigma^2(\sigma^2 - (2\alpha - 3\beta)s_t^2)}{(2\alpha - 5\beta)(2\alpha - 3\beta)} \right) e^{-2(2\alpha - 3\beta)\Delta}.
 \end{aligned}
 \tag{17}$$

(ii) *The stationary unconditional distribution  $\pi(s)$  of  $s_t$  is given by*

$$\pi(s) = 1 / \left[ (2\sigma^2 + \beta s^2)^{1 + \alpha/\beta} \int_{-\infty}^{+\infty} \{1/(2\sigma^2 + \beta u^2)^{1 + \alpha/\beta}\} du \right].
 \tag{18}$$

*It admits finite moments up to order  $n$  if and only if  $2\alpha > (n - 1)\beta$ . Assuming  $2\alpha > 3\beta$ , the first four unconditional moments of  $s_t$  are*

$$\begin{aligned}
 E[s_t] &= 0, & E[s_t^2] &= 2\sigma^2/(2\alpha - \beta), & E[s_t^3] &= 0, \\
 E[s_t^4] &= 12\sigma^4/((2\alpha - \beta)(2\alpha - 3\beta))
 \end{aligned}
 \tag{19}$$

*Proof.* See Appendix.

Serial correlation in price volatility follows from the fact that  $s_t$  is serially correlated (from (6)), and thus so is  $v_t$  (from (15)). Specifically, by Itô’s Lemma, the conditional variance of price changes follows:

$$dv_t^2 = ((2\alpha + \beta)\sigma^2 - (2\alpha - \beta)v_t^2) dt + 2\sqrt{\beta}\sqrt{v_t^2 - \sigma^2}(\sigma dZ_{2t} - v_t dZ_{1t})
 \tag{20}$$

and is therefore mean-reverting around its mean value  $E[v_t^2] = \sigma^2(2\alpha + \beta)/(2\alpha - \beta)$  at speed  $(2\alpha - \beta)$ .

As for the second effect, define the trading volume as the change  $|dq_t|$  in the investor's holdings. Because the Brownian increments  $dZ_i$  are Gaussian random variables it follows that<sup>4</sup>

$$\text{corr}(|dp_t|, |dq_t|) = \left(1 - \frac{2}{\pi}\right) \left(1 - \sqrt{1 - \text{corr}(dp_t, dq_t)^2}\right) \geq 0. \tag{21}$$

Therefore, the magnitude of the price change is positively correlated with the trading volume. According to the Wall Street adage, it 'takes volume to move prices' which is what (21) demonstrates for this model.

Finally, to show that the model generates a leptokurtic distribution for price changes, we compute the conditional moments of price changes (see Fig. 5). We derive below as part of the estimation procedure the exact expressions of the conditional moments of  $(p_{t+\Delta}, \bar{p}_{t+\Delta})$  given  $(p_t, \bar{p}_t)$ . While the moments are available in closed-form, the excess kurtosis can most easily be seen from the first terms in the Taylor series expansions:

$$\begin{aligned} E[(p_{t+\Delta} - p_t) | \bar{p}_t, p_t] &= \alpha s_t \Delta + o(\Delta), \\ E[(p_{t+\Delta} - p_t)^2 | \bar{p}_t, p_t] &= (\sigma^2 + \beta s_t^2) \Delta + o(\Delta), \\ E[(p_{t+\Delta} - p_t)^3 | \bar{p}_t, p_t] &= 3\alpha s_t (\sigma^2 + \beta s_t^2) \Delta^2 + o(\Delta^2), \\ E[(p_{t+\Delta} - p_t)^4 | \bar{p}_t, p_t] &= 3(\sigma^2 + \beta s_t^2)^2 \Delta^2 + o(\Delta^2). \end{aligned} \tag{22}$$

We then derive the unconditional moments of price changes by applying the law of iterated expectations and using (19):

$$\begin{aligned} E[(p_{t+\Delta} - p_t)] &= o(\Delta), \\ E[(p_{t+\Delta} - p_t)^2] &= \sigma^2 \left(\frac{2\alpha + \beta}{2\alpha - \beta}\right) \Delta + o(\Delta), \\ E[(p_{t+\Delta} - p_t)^3] &= o(\Delta^2), \\ E[(p_{t+\Delta} - p_t)^4] &= 3\sigma^4 \left(\frac{4\alpha^2 + 3\beta^2}{(2\alpha - 3\beta)(2\alpha - \beta)}\right) \Delta^2 + o(\Delta^2). \end{aligned} \tag{23}$$

The price changes exhibit excess kurtosis. Indeed, we can see from (23) that  $E[(p_{t+\Delta} - p_t)^4] / (E[(p_{t+\Delta} - p_t)^2])^2 > 3$  if and only if  $\beta > 0$  (i.e., if the specialist optimally adjusts volatility in response to investors' trades). The model therefore reproduces these three major empirical facts: serially correlated price volatility,

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<sup>4</sup>This is a standard result for the correlation of the absolute value of two jointly Gaussian random variables; see e.g., the appendix in Wang (1994) for a derivation.

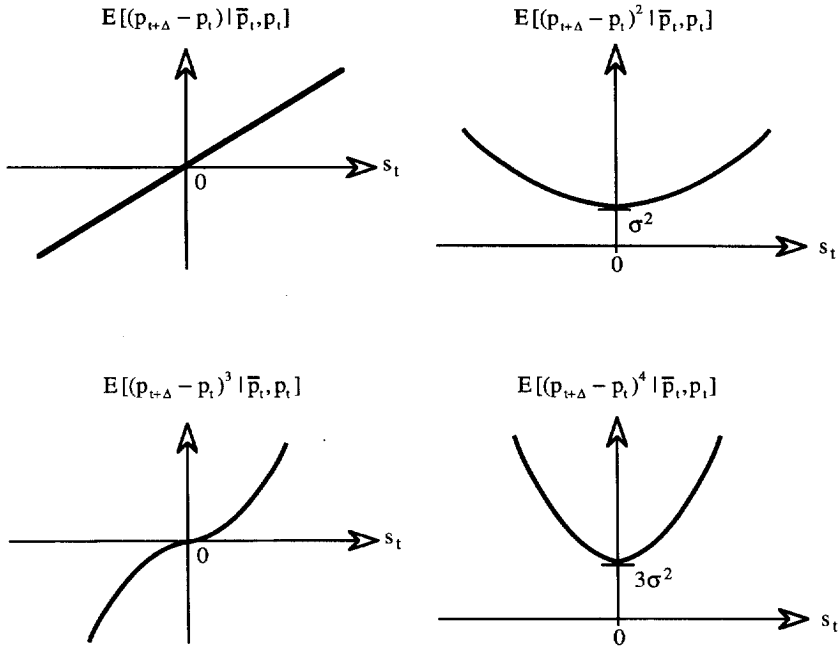


Fig. 5. Conditional moments of price changes.

correlation between trading volume and absolute price changes, and excess kurtosis of price changes.

### 3. Estimation of the equilibrium price and volume processes

The objective of this section is to estimate jointly the system of stochastic differential equations specifying the evolution of the price and volume variables. We first concentrate on the price dynamics alone. Recall that  $s_t \equiv \bar{p}_t - p_t$  and that Proposition 1 produced the following stochastic dynamics:

$$\begin{aligned}
 dp_t &= \alpha(\bar{p}_t - p_t) dt + \sqrt{\sigma^2 + \beta(\bar{p}_t - p_t)^2} dZ_{1t}, \\
 d\bar{p}_t &= \sigma dZ_{2t}.
 \end{aligned}
 \tag{24}$$

The interest rate level  $r$  is assumed to be observable. Direct estimation of the parameter vector  $\theta \equiv (\alpha, \sigma, \beta)$  in the system (24) is not feasible because  $\bar{p}_t$  is unobservable to the econometrician. Only the market price  $p_t$  is observed.

We propose the following approach to estimate the system. We start by determining the dynamics of the first two conditional moments of  $\bar{p}_t$  given  $p_t$  (Proposition 3). Second, we derive expressions of the conditional moments of  $p_{t+A}$  given  $p_t$  and  $\bar{p}_t$  (Proposition 4). We then use the law of iterated expectations to obtain the conditional moments of  $p_{t+A}$  given  $p_t$ , using our estimates of the moments  $\bar{p}_t$  given  $p_t$ . Finally we estimate the parameters by the generalized method of moments using the conditional moments of  $p_{t+A}$  given  $p_t$  that were just derived, and the unconditional moments of  $p_t$ . The same procedure is repeated once transaction volume data are introduced – instead of conditioning on the market price alone, we then condition on the full set of observables: market price and volume (Proposition 5).

### 3.1. Estimation of the fundamental value from data on market price only

The first step is to filter the fundamental price at time  $t$  from observations of the market price up to that time, assuming full knowledge of the joint dynamics (24), i.e., for a given set of parameter values  $\theta$ . Intuitively, knowledge of the joint dynamics of the prices, associated with observations of the market price up to  $t$  collected in  $\mathcal{G}_t \equiv \{p_s/0 \leq s \leq t\}$ , should reveal information on the unobservable underlying fundamental price. If we minimize the conditional expected estimation error  $R_t \equiv E[(\hat{p}_t - \bar{p}_t)^2 | \mathcal{G}_t]$ , the resulting estimate  $\hat{p}_t$  of  $\bar{p}_t$  is clearly given by the conditional expectation of the fundamental price given the market price observations:  $\hat{p}_t = E[\bar{p}_t | \mathcal{G}_t]$ .

The theory of nonlinear optimal filtering for diffusion processes is described in detail in Lipster and Shiryaev (1977, Chapter 8). However the equations of optimal filtering can seldom be solved. Other than some specific cases, they can only be solved for linear dynamics, yielding the Kalman–Bucy (1960) filter. Furthermore, the equations can be formulated only when the volatility of the observable process dynamics do not depend on the unobservable process.<sup>5</sup> This is a major hindrance in our problem since the instantaneous volatility  $v_t = \sqrt{\sigma^2 + \beta(\bar{p}_t - p_t)^2}$  depends upon both the observable process  $p_t$  on the unobservable process  $\bar{p}_t$ . To address these limitations, multiple suboptimal filtering methods have subsequently been developed. Pugachev and Sinitsyn (1987) present an account of these developments. We use an extended Kalman–Bucy filter.<sup>6</sup> This filter is obtained by expanding the dynamics (24) through a Taylor series expansion of their drift and diffusion for  $\bar{p}_t$  in the vicinity of its filtered value  $\hat{p}_t$ . In the extended Kalman–Bucy filter, the expansions of the diffusion terms are limited to the first order term. We solve for the filter and obtain:

<sup>5</sup> See Eq. (8.44) page 307 in Lipster and Shiryaev (1997).

<sup>6</sup> See Pugachev and Sinitsyn (1987), Section 8.3.2.

*Proposition 3.* (i) *The extended nonlinear Kalman–Bucy filter follows the stochastic differential equation:*

$$d\hat{p}_t = \frac{\alpha R_t}{\sigma^2 + \beta(\hat{p}_t - p_t)^2} \{dp_t - \alpha(\hat{p}_t - p_t) dt\}, \quad (25)$$

where the conditional estimation error  $R_t$  follows the Riccati equation:

$$dR_t = \left\{ \sigma^2 - \frac{\alpha^2 R_t^2}{\sigma^2 + \beta(\hat{p}_t - p_t)^2} \right\} dt. \quad (26)$$

(ii) *In the special case where  $\beta = 0$ , i.e., when the specialist does not input additional volatility to the market price relative to the fundamental price, the filter (25) reduces to the linear Kalman–Bucy filter with dynamics:  $d\hat{p}_t = (\alpha R_t / \sigma^2) \{dp_t - \alpha(\hat{p}_t - p_t) dt\}$  where  $R_t = (\sigma^2 / \alpha)(e^{2\alpha t} - 1) / (e^{2\alpha t} + 1)$ . The steady state limit is:  $R = \sigma^2 / \alpha$ .*

*Proof.* See appendix.

Note that, as required, (25) and (26) make it possible to compute  $\hat{p}_t$  recursively at any  $t$  from a record of observations on the past market price changes. It is interesting to explore how the optimal estimator of the fundamental price is updated as new information becomes available. Eq. (25) shows that the estimator  $\hat{p}_t$  is updated in response to the market price change  $dp_t$  that was just recorded. But it is only revised in response to the increment in the market price that is uncorrelated with its past values, that is  $dp_t - \alpha(\hat{p}_t - p_t) dt$ . Indeed  $n_t \equiv p_t - \int_0^t \alpha(\hat{p}_s - p_s) ds$  is an innovation process<sup>7</sup> for the market price process  $p_t$ . In other words, the estimator of the fundamental price changes only when new unpredictable information arrives. When a change in the market price  $dp_t$  is observed, part of it is attributed to a change in the fundamental price (and consequently used to update the fundamental price estimate), and part of it to noise.

Finally, an appealing feature of the estimator is that the estimation error made when replacing the unknown fundamental price by its filtered estimate is bounded above by a finite constant, and therefore does not grow as a power of  $t$  when  $t$  increases. As could be expected, less noise in the processes ( $\sigma$  smaller) and/or more reversion of  $p$  to  $\bar{p}$  ( $\alpha$  larger) make the error smaller. Note also that if  $dZ_1$  and  $dZ_2$  were correlated with correlation coefficient  $\rho_{12}$  then in the case

<sup>7</sup> A random process  $n_t$  is an innovation process for a process  $p_t$  iff: (i) at any  $t \geq 0$ ,  $n_t$  is a functional of  $\mathcal{G}_t \equiv \{p_s | 0 \leq s \leq t\}$  and (ii) at any  $0 \leq \tau \leq t \leq s$ , the increment  $n_s - n_\tau$  is uncorrelated with  $p_\tau$ .



$\beta = 0$  the conditional estimation error would become  $R_t = (\sigma^2/\alpha)((1 + \rho_{12})e^{2\alpha t} - 1)/((1 + \rho_{12})/(1 - \rho_{12})e^{2\alpha t} - 1)$ , with steady state limit  $R = (1 - \rho_{12})(\sigma^2/\alpha)$ . Therefore,  $R$  is smaller when the correlation coefficient  $\rho_{12}$  between the stochastic components of the respective fundamental and market price increments is closer to one. Intuitively, as  $\rho_{12}$  gets closer to one, movements in the market price tend to reflect more closely movements in the underlying fundamental price process, so it becomes easier to estimate the value of the fundamental price by observing only the market price. In the limit of perfect correlation, observing the market price fully reveals the fundamental price at every instant.

### 3.2. Parameter estimation

In this section, we construct a consistent and asymptotically normal estimator of the unknown parameter values in (24).

*Proposition 4. The conditional means and (co)variances of the pair  $(p_{t+\Delta}, \bar{p}_{t+\Delta})$  given  $(p_t, \bar{p}_t)$  are given in closed-form for any time interval  $\Delta > 0$  by*

$$\begin{aligned}
 E[\bar{p}_{t+\Delta} | \bar{p}_t, p_t] &= \bar{p}_t, \\
 E[p_{t+\Delta} | \bar{p}_t, p_t] &= e^{-\alpha\Delta}(p_t - \bar{p}_t) + \bar{p}_t, \\
 E[\bar{p}_{t+\Delta}^2 | \bar{p}_t, p_t] &= \bar{p}_t^2 + \sigma^2\Delta \\
 E[\bar{p}_{t+\Delta} p_{t+\Delta} | \bar{p}_t, p_t] &= e^{-\alpha\Delta}(\sigma^2/\alpha + \bar{p}_t(p_t - \bar{p}_t)) + (\bar{p}_t^2 - \sigma^2/\alpha) + \sigma^2\Delta \quad (27) \\
 E[p_{t+\Delta}^2 | \bar{p}_t, p_t] &= e^{-(2\alpha - \beta)\Delta}((p_t - \bar{p}_t)^2 - 2\sigma^2/(2\alpha - \beta)) + 2e^{-\alpha\Delta}(\sigma^2/\alpha + \bar{p}_t(p_t - \bar{p}_t)) \\
 &\quad + (\bar{p}_t^2 - 2(\sigma^2/\alpha)(\alpha - \beta)/(2\alpha - \beta)) + \sigma^2\Delta.
 \end{aligned}$$

*Proof. See appendix.*

We have obtained in Proposition 3 both  $E[\bar{p}_t | p_t]$  and  $E[\bar{p}_t^2 | p_t]$ . We now apply the law of iterated expectations to derive the conditional mean and variance of the market price at time  $t + \Delta$ , conditioned only on the observed variable at  $t$ :

$$E[p_{t+\Delta} | p_t] = e^{-\alpha\Delta}(p_t - E[\bar{p}_t | p_t]) + E[\bar{p}_t | p_t], \quad (28)$$

$$\begin{aligned}
 E[p_{t+\Delta}^2 | p_t] &= e^{-(2\alpha - \beta)\Delta}(E[(p_t - \bar{p}_t)^2 | p_t] - 2\sigma^2/(2\alpha - \beta)) \\
 &\quad + 2e^{-\alpha\Delta}(\sigma^2/\alpha + E[\bar{p}_t(p_t - \bar{p}_t) | p_t]) \quad (29) \\
 &\quad + (E[\bar{p}_t^2 | p_t] - 2(\sigma^2/\alpha)(\alpha - \beta)/(2\alpha - \beta)) + \sigma^2\Delta.
 \end{aligned}$$

The unconditional variance of the price process is given by

$$E[p_t^2] - E[p_t]^2 = \sigma^2 \{ 2e^{-\alpha/\alpha} - 2e^{-(2\alpha - \beta)t} / (2\alpha - \beta) - (1/\alpha)(\alpha - \beta) / (2\alpha - \beta) + t \}. \tag{30}$$

The parameter vector  $\theta$  can now be estimated. In (28) and (29), the estimates for  $E[\bar{p}_t | p_t]$  and  $E[\bar{p}_t^2 | p_t]$  are given by  $\hat{p}_t$  and  $E[\bar{p}_t^2 | p_t] = R_t + \hat{p}_t^2$ , respectively, which are both functions of  $p_t$  and  $(\alpha, \beta, \sigma)$  as determined in Proposition 3 by (25) and (26). We then use GMM to estimate  $(\alpha, \beta, \sigma)$  using the moment conditions for the stationary price changes:  $E[(p_{t+A} - p_t) | p_t]$  and  $E[(p_{t+A} - p_t)^2 | p_t]$ , computed from (28) and (29). Each conditional moment contributes two orthogonality conditions: itself, and its product with  $p_t$ . The estimator is consistent and asymptotically normal from the classical results of Hansen (1982), since price changes are stationary and ergodic; the parameters belong to  $R$ ; expectations exist and are finite for all values of the parameters in a compact subset; the weighting matrix sequence converges almost surely. In the empirical work that follows, the GMM weighting matrix is chosen optimally.

3.3. Estimating the equilibrium price from market price and transaction volume information

When volume data are available, the fundamental price estimator can be improved by filtering out  $\bar{p}_t$  from both  $p_t$  and  $q_t$ , instead of  $p_t$  only. By Itô's Lemma:

$$dq_t = \frac{\partial q_t}{\partial s_t} ds_t + \frac{1}{2} \frac{\partial^2 q_t}{\partial s_t^2} (\sigma^2 + v_t^2) dt, \tag{31}$$

where  $q_t$  is given in equilibrium by (16), and the price spread follows (6). Note that this expression yields the market depth defined as the order flow required to change the market price by one dollar, i.e.,  $1/\delta_t$  in the equation  $E[dp_t] = \delta_t E[dq_t]$ .

While the model determines exactly the transaction volume equation (31), we estimate the following simplified expression, derived by retaining in (16) only the first order term for small price spreads  $s_t$ :

$$dq_t = -\lambda x s_t dt + \lambda(\sigma dZ_{2t} - v_t dZ_{1t}). \tag{32}$$

*Proposition 5. The conditional mean and variance of the observed transaction volume  $q_{t+A} - q_t$  are*

$$\begin{aligned} E[(q_{t+A} - q_t) | p_t, \bar{p}_t] &= -\lambda(\bar{p}_t - p_t)(e^{-\alpha A} - 1), \\ E[(q_{t+A} - q_t)^2 | p_t, \bar{p}_t] &= \lambda^2(e^{-(2\alpha - \beta)A} - 2e^{-\alpha A} + 1)(\bar{p}_t - p_t)^2 \\ &\quad + 2\sigma^2 \lambda^2(1 - e^{-(2\alpha - \beta)A}) / (2\alpha - \beta). \end{aligned} \tag{33}$$

*Proof. See appendix.*

As for the market price process, apply the law of iterated expectations to obtain  $E[(q_{t+\Delta} - q_t) | p_t]$  and  $E[(q_{t+\Delta} - q_t)^2 | p_t]$ , as functions of  $p_t$  and the filtered fundamental price and its conditional estimation error. Then use the resulting moments to form orthogonality conditions, and add them to those derived from (27).

In the extended Kalman–Bucy filter, the fundamental price estimate is now revised by a linear combination of the innovations in both the market price change  $dp_t$  and the transaction volume  $dq_t$ . It can be shown that the steady state limit of the conditional estimation error is  $\tilde{R} < R$ . In other words, the estimator of the fundamental price obtained from data on both the market price and transaction volume is always more precise than the estimator obtained from observations on the market price only.

#### 4. Applications and empirical implementation

##### 4.1. The data

The implications of the model are examined on transaction data for AMR, the parent company of American Airlines, during the month of January 1993 (Monday 1/4 to Friday 1/29: 20 trading days). The source for the data is the NYSE TAQ Database, which lists all trades and quotes for the NYSE, Amex and NASDAQ and the regional exchanges. These transactions are submitted by participants on exchanges. We sign trades here as a buy or sell trade based on their relative proximity to the bid and ask quotes prevailing 5 seconds before (see Lee and Ready (1991) for various approaches to signing transaction volume.) We also aggregate successive trades of the same sign into a single trade. This trade is time-stamped at the average of the individual trade times, weighted by trade size. The aggregation results in a final sample of 2229 trades. Table 2 reports the descriptive statistics for this sample. Figs. 6 and 7 plot AMR's signed transaction volume and the time between trades.

During the sample period, the main events affecting the fundamental value of AMR stock (to be viewed as large realizations of the Brownian Motion  $Z_2$  determining the change in  $\bar{p}$ ) have been gathered from the *Wall Street Journal* and are given in Table 3.

##### 4.2. Estimation of the equilibrium dynamics

The estimation procedure has two distinct steps. All the parameters of the structural system (24) are estimated and the best estimate of the fundamental price is computed at every date. The parameter estimates are reported in Table 4. Fig. 8 reports the observed changes in the market price ( $dp_t$ ) used to construct the fundamental price filter in (25). Fig. 9 gives the market price of

Table 2  
Descriptive statistics

	Market price (Dollars)	Market price change (Dollars)	Trading volume (Number of shares)	Time between trades (min)
Mean	66.970	-0.001	-86	3.579
Standard deviation	2.189	0.165	15,350	5.889
Minimum	61.750	-0.875	-162,000	0.017
25% percentile	65.000	-0.125	-2,100	0.617
50% percentile	67.375	0.125	100	1.667
75% percentile	68.750	0.125	2,000	4.200
Maximum	70.250	1.000	173,000	108.517
Lag 1 autocorrelation	0.996	-0.499	-0.129	0.085

Note: Same-side trades (buy or sell) are aggregated for the purpose of computing trading volume and time between trades

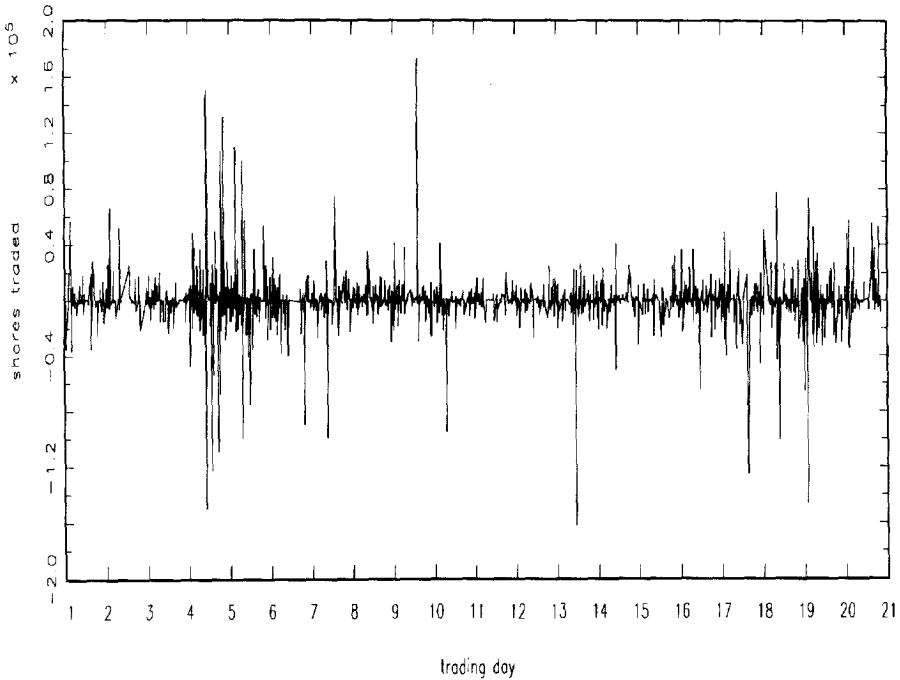


Fig. 6. AMR signed transaction volume.

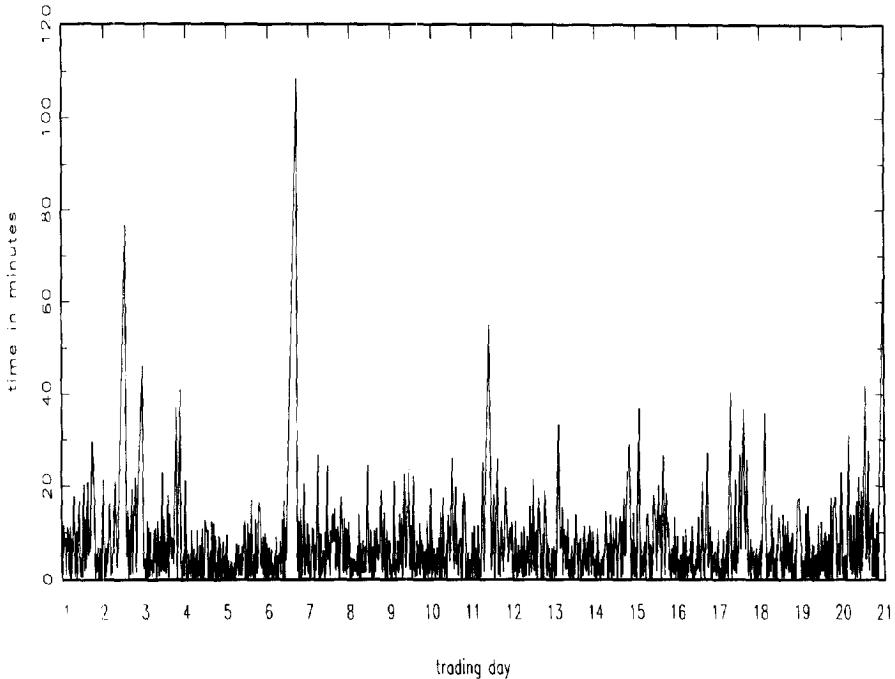


Fig. 7. Time between trades.

Table 3  
Potential shocks affecting the fundamental value of AMR during January 1993

Date	Trading day	Event
Jan.7	4	UAL announces drastic cost-cutting measures. Along with other airline stocks, AMR jumps 7/8 to close at 68 3/4.
Jan.12	7	Northwest Airlines launched a new fare war. AMR off 1 5/8 at 67 3/4.
Jan.14	9	Threat of wholesale fare war recedes. AMR leaps 2 to 69.
Jan.18	11	County NatWest increases its 1992 estimated loss for AMR from continuing operations to \$4.50 from \$2.80 a share.
Jan.20	13	AMR reports its worst loss ever (\$935m) for 1992 and announces spending cuts of \$300m for 1993. AMR stock down 3/4 to 67 1/8.
Jan.25	16	Possible OPEC production cuts announced. Fuel prices up. AMR loses 1 7/8 to 64 1/8.
Jan.26	17	AMR announces delayed delivery of eight Boeing jets. AMR down 3/8 to 64.
Jan.28	19	Delta and UAL announce large 1992 losses. AMR down 1 1/4 to 62 1/4 along with the other airlines' stocks.

Table 4  
Parameter estimates

Parameter		Coefficient	95% confidence interval
Price processes	$\alpha$	$1.36 \times 10^{-1}$	[0.027, 0.25]
	$\sigma^2$	$6.28 \times 10^{-1}$	[0.47, 0.79]
	$\beta$	$8.41 \times 10^{-2}$	[0.008, 0.16]
Volume process	$\lambda$	$4.50 \times 10^{+5}$	[17,351, 72,649]

Note: The price processes are given by (24). The trading volume process is given by (32). In these estimates the unit of time is one trading day. Prices are measured in dollars, and volume in number of shares traded.

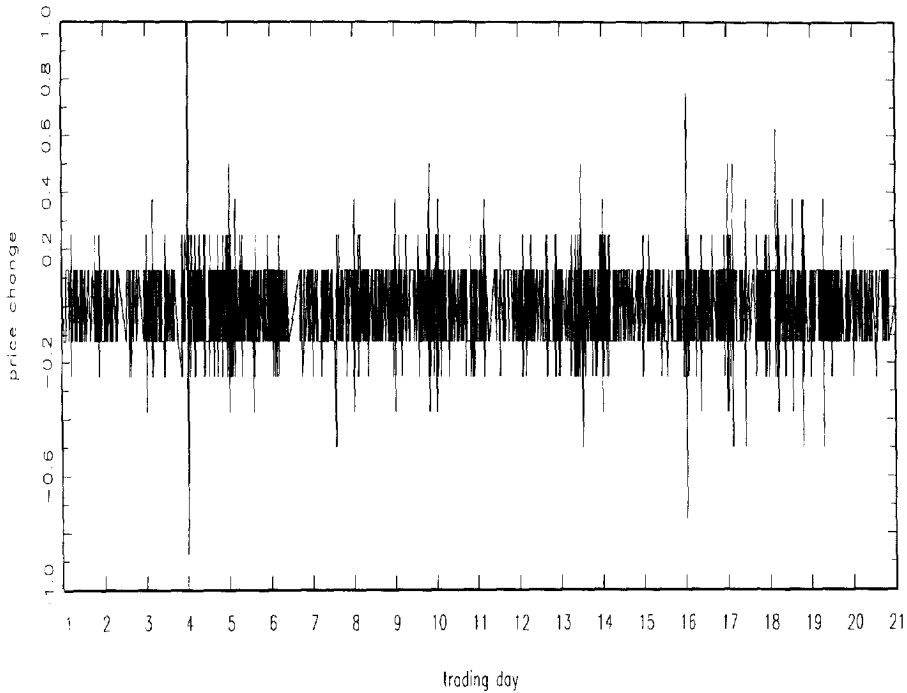


Fig. 8. AMR market price change.

AMR and the estimated fundamental price. It can be seen on the graph that large differences between the two prices tend to precede an adjustment in the market price that is consistent with the model. When the market price is significantly below the estimated fundamental price at  $t$ , the market price tends

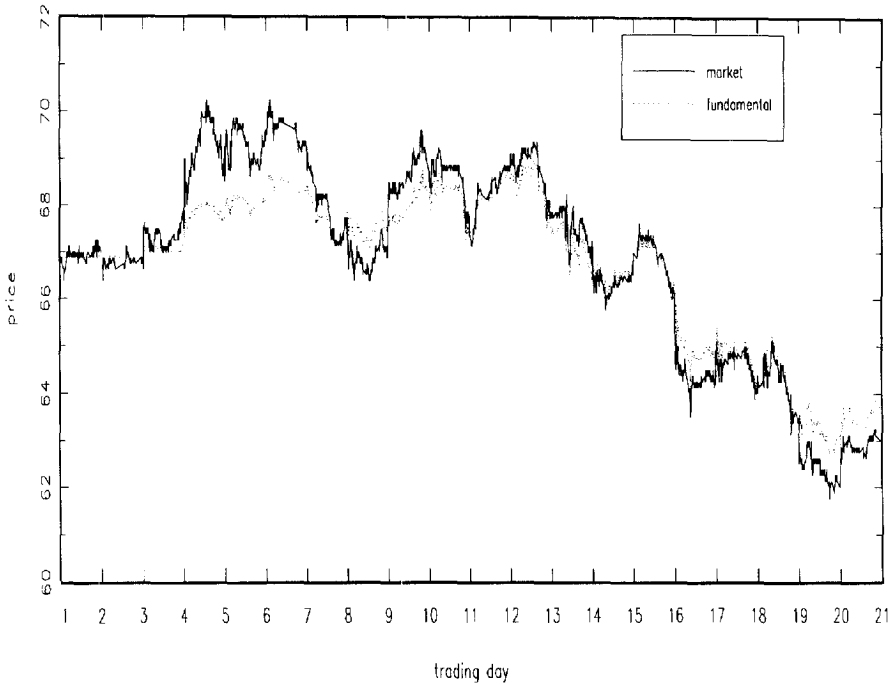


Fig. 9. Market and estimated fundamental prices.

to go up subsequently, and vice versa. This suggests graphically some predictive power, a question which is examined more carefully below.

The accuracy of the filter is determined by the conditional estimation error  $R_t$  in (26), which we plot in Fig. 10. As discussed in Section 3.1, the conditional estimation error is bounded above, a fact which becomes apparent on the graph. We finally report in Fig. 11 the stationary distribution of the price spread  $s_t$ , given by (18). This is the unconditional distribution of the difference between the market and fundamental prices. It measures the likelihood, in steady state, of observing deviations of any given magnitude between the market and fundamental prices. To illustrate the kurtosis effect that was demonstrated in Proposition 2, we also graph in Fig. 11 a Normal density with mean and variance set to match those of the actual density of the price spread. This provides a graphical illustration of the leptokurtic behavior of the price spread implied by the model.

A nice feature of this model is that relevant issues or ‘anomalies’ in asset pricing can be interpreted as simple hypothesis tests on one or more of the three parameters of the model,  $\alpha$ ,  $\beta$  and  $\sigma^2$ , and we now focus on each of them.

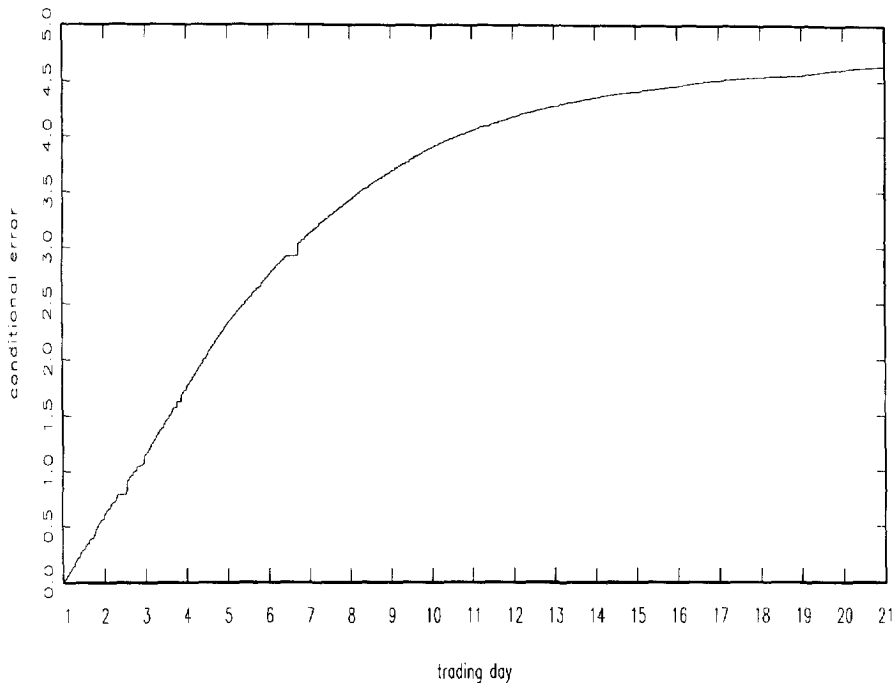


Fig. 10. Fundamental price conditional estimation error.

#### 4.3. Mean reversion in stock prices

The extent to which stock prices tend to revert to their mean over long horizons has been the subject of long-standing attention in the finance literature, as part of a broader study of departures from the random walk hypothesis. On the empirical side, the investigation of mean-reversion in stock prices has generally focused on the autocorrelation at various frequencies of security returns. The idea that market prices would fluctuate around fundamental values, defined as the discounted cash flows that the stock gives title to, dates back at least to the classical books by Graham and Dodd (1934) and Williams (1938). This line of research includes Cowles (1933), Kendall (1953), Summers (1976), Fama and French (1988), Lo and MacKinlay (1988) and Poterba and Summers (1988). The empirical findings concentrate on long horizon returns and generally find significant mean reversion – the setting here is naturally different, since we focus on transaction data and short horizons.

In this model, the market price can be interpreted as mean-reverting at every instant to the stochastic target level  $\bar{p}_t$  (recall (24)). Because the fundamental



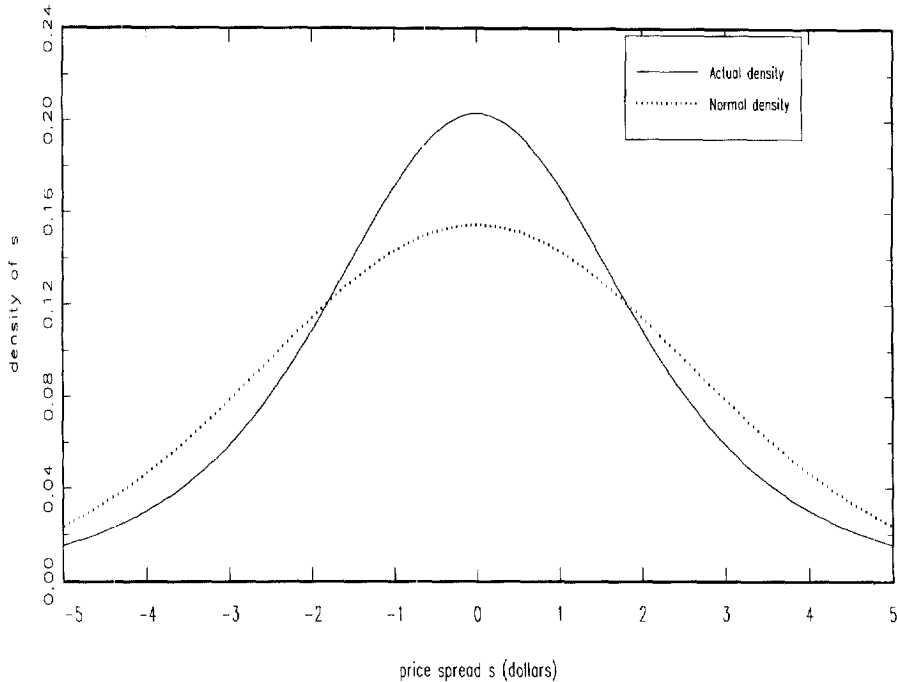


Fig. 11. Stationary distribution of the price spread.

price is itself a stochastic process, we propose two nested definitions of mean-reversion: (i) no mean-reversion in the strong sense, corresponding to the joint hypothesis  $H_s$ :  $\alpha = 0$  and  $\sigma^2 = 0$ , and (ii) no mean-reversion in the weak sense, corresponding to the hypothesis  $H_w$ :  $\alpha = 0$  only. In this framework, define a market price series to be strongly mean-reverting if the joint hypothesis  $H_s$ , as well as the two individual hypotheses, are rejected, and weakly mean-reverting if only  $H_w$  is rejected. A strongly mean-reverting market price reverts to a non-stochastic value – which is the usual definition of mean-reversion. A weakly mean-reverting series reverts to a randomly changing level – which is the prediction made by the model. A market price series that is not weakly mean-reverting is essentially a random walk.

The two null hypotheses can be tested in the GMM framework spelled out in Proposition 4. We use the Wald statistics. The test results are in Table 5. Both  $H_s$  and the two single hypotheses (in particular  $H_w$ ) are rejected at the 95% level. In other words, the market price mean-reverts at a speed  $\alpha \neq 0$  to the fundamental price, which evolves stochastically ( $\sigma^2 \neq 0$ ).

Table 5  
Test statistics

Economic issue	Null hypothesis	Wald statistics
Mean reversion of prices?	$\alpha = 0$ and $\sigma^2 = 0$	64.21
	$\sigma^2 = 0$	59.14
Predictability of price changes?	$\alpha = 0$	5.95
Do prices move too much?	$\beta = 0$	4.62
Mean reversion of variance?	$2\alpha - \beta = 0$	4.17

Note: The Wald Statistics are distributed as  $\chi^2_2$  for the joint hypothesis on  $(\alpha, \sigma^2)$ , and  $\chi^2_1$  for the others. The 95% critical values are respectively 5.99 and 3.84

#### 4.4. The predictability of stock price changes

A closely related question is whether stock price changes are predictable. In this model, the fundamental price estimate makes it possible to predict market price changes at every instant, by replacing the unobservable drift of the market price process in (24) with the estimated drift, yielding an estimate at  $t$  of the change between  $t$  and  $t + dt$ :

$$\hat{E}[dp_t | \mathcal{G}_t] = \hat{\alpha}(\hat{p}_t - p_t) dt \quad (34)$$

with the same notation as before:  $\hat{\alpha}$  is the unconstrained estimate and  $\hat{p}_t = E[\bar{p}_t | \mathcal{G}_t]$ . At every instant  $t$  (sufficiently far from the beginning of the sample), observations on  $\{p_s | 0 \leq s \leq t\}$  and the corresponding volume are used to form  $\hat{\alpha}$  and  $\hat{p}_t$ . Then the expected market price change is computed according to (34). The forecasts can be compared to the random walk forecasts:

$$\hat{\hat{E}}[dp_t | \mathcal{G}_t] = 0. \quad (35)$$

The null hypothesis  $H_p: \alpha = 0$  is rejected at the 95% level, and therefore (34) should be used instead of (35) in order to predict market price changes. An alternative easily interpretable test of forecasting power could be conducted by counting the proportion  $\hat{\pi}$  of instants in the sample for which (34) is closer to the actual market price change recorded between  $t$  and  $t + dt$  than the random walk (35). Under the null that market changes are unpredictable given the past and present market prices in  $\mathcal{G}_t$ , the probability that (34) be closer to the true market price change than (35) is  $\pi = \frac{1}{2}$ , as the predicted departure from (35) given by (34) is uncorrelated to the actual departure under the null. We would test  $H_\pi: \pi = 1/2$  vs.  $\bar{H}_\pi: \pi > \frac{1}{2}$ . The sample probability  $\hat{\pi}$  would have the binomial distribution  $\sqrt{T}(\hat{\pi} - \frac{1}{2}) \rightarrow N(0, \frac{1}{4})$  under the null.

#### 4.5. Do market prices move too much?

To examine whether market prices move too much to be justified by changes in the fundamental price, it suffices to test the null hypothesis  $H_v: \beta = 0$  since the market price volatility is given by  $v_t = \sqrt{\sigma^2 + \beta s_t^2}$ . Under  $H_v$ , the market price has the same volatility  $\sigma$  as the fundamental price. Note that more complex hypotheses regarding the joint distribution of  $(\bar{p}_t, p_t)$  can be tested, since the model specifies this entire distribution. The null  $H_v$  is rejected at the 95% level, suggesting that market prices are indeed more volatile than fundamental prices. Note that this is not a statement that the filtered fundamental price is smoother than the market prices. At the filtering stage (Proposition 3), the value of  $\beta$  is not yet determined and could well have been zero. The parameters are only estimated at the GMM stage (Proposition 4), and as the parameter values are updated so is the filter. In addition,  $\beta \neq 0$  implies that market price changes are leptokurtic (see (23)). We can also see from (20) that the conditional variance of price changes is serially correlated and mean-reverts at speed  $(2\alpha - \beta)$ , which is significantly different from zero as can be seen in Table 5.

The evidence that  $\beta \neq 0$  in this model is compatible with the classical result of Shiller (1981) and LeRoy and Porter (1981), who found that the volatility of stock prices was too high to be justified by changes in future dividends (see Cochrane, 1991 for a critique). Shiller showed that under his assumptions the market price must have a lower volatility than the perfect foresight price, so finding the opposite result yielded the conclusion that markets are inefficient or irrational. An important difference is that there is no constraint here on the relative size of the market and fundamental price volatilities. Rejecting  $H_v$  has no implications for market efficiency in this model. Every market participant is rational, yet market prices could be more volatile than fundamental values.

### 5. Conclusions and extensions

The results suggest that it is possible to replicate, in a fully rational world, some of the empirical findings that have been labeled as market anomalies. Its simple empirical implications are all validated by the data: excess volatility of the market price ( $\beta \neq 0$ ), and mean reversion ( $\alpha \neq 0$ ) to a stochastic target ( $\sigma^2 \neq 0$ ). In addition, these non-zero parameter values imply that price changes are leptokurtic, their conditional volatility exhibits serial correlation, and trading volume and absolute changes in prices are contemporaneously correlated.

There is no question however that this model is simplistic, and that many other puzzling empirical regularities are beyond its scope. In that respect, three

extensions can be considered. First, the theoretical model could be extended to incorporate asymmetric information (Detemple, 1986; Gennotte, 1986; Wang 1993). For example, the fundamental price could be revealed only to the specialist. The investor would receive a signal on the fundamental price and then trade based on the signal, in addition to the price set by the specialist. In equilibrium, the market price would not be a sufficient statistics for all relevant market information, and the resulting model would have a noisy rational equilibrium flavor (Grossman and Stiglitz, 1980; Hellwig, 1980; Diamond and Verrecchia, 1981).

Secondly, the model could incorporate a bid-ask spread, as a source of profit to the specialist (see Glosten and Milgrom, 1985). This could serve as an additional motivation for our assumption that the specialist is constrained in setting the drift of the price adjustment, but controls its volatility. With a bid-ask spread, the specialist would have an incentive to ‘move prices around’, i.e., control the price volatility, as liquidity investors would then readjust their portfolios more often – thereby generating bid-ask profits for the specialist. Thirdly, the model predicts that market price changes exhibit additional volatility compared to those of the fundamental value of the stock. Instead of relying on a particular parametrization, this excess volatility function could be estimated nonparametrically as in Ait-Sahalia (1996).

## Appendix

*Proof of Proposition 1.* The investor takes as given the volatility function  $v_t \equiv v(s_t)$  and computes his optimal asset holdings. His optimization problem has the two control variables  $c_t$  and  $q_t$ , and the two state variables are  $W_t$  and  $s_t$ . Let  $V(t, W_t, s_t)$  be its value function. The Bellman Equation is

$$0 = \sup_{\{c_t, q_t\}} \left\{ -e^{-\rho t - \alpha c_t - \gamma/v_t^2} + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial W_t} \{rW_t + (\alpha + r)s_t q_t - c_t\} + \frac{\partial V}{\partial s_t} \{-\alpha s_t\} \right. \\ \left. + \frac{1}{2} \frac{\partial^2 V}{\partial W_t^2} \{q_t^2 v_t^2\} + \frac{\partial^2 V}{\partial W_t \partial s_t} \{-q_t v_t^2\} + \frac{1}{2} \frac{\partial^2 V}{\partial s_t^2} \{\sigma^2 + v_t^2\} \right\} \quad (\text{A.1})$$

since  $\text{var}(dW_t) = \text{var}(q_t v_t dZ_{1t}) = q_t^2 v_t^2 dt$ ,  $\text{var}(ds_t) = \text{var}(\sigma dZ_{2t} - v_t dZ_{1t}) = \{\sigma^2 + v_t^2\} dt$  and  $\text{cov}(dW_t, ds_t) = \text{cov}(q_t v_t dZ_{1t}, \sigma dZ_{2t} - v_t dZ_{1t}) = -q_t v_t^2 dt$ .

The first-order conditions give the investor's optimal investment strategy  $q_t$ :

$$\frac{\partial V}{\partial W_t} (\alpha + r)s_t + \frac{\partial^2 V}{\partial W_t^2} q_t v_t^2 - \frac{\partial^2 V}{\partial W_t \partial s_t} v_t^2 = 0 \quad (\text{A.2})$$

and consumption policy  $c_t$ :

$$ae^{-\rho t - ac_t - \gamma/v_t^2} - \frac{\partial W}{\partial W_t} = 0. \tag{A.3}$$

Replacing the control variables by their optimal values in the Bellman Equation leads to a partial differential equation to be satisfied by the value function. The solution can be found in the form:  $V(t, W_t, s_t) = -(1/r)\exp(-\rho t - arW_t - ah(s_t, v_t))$ , where  $h$  is to be determined as part of the solution. Since  $v_t \equiv v(s_t)$ , define  $h'$  (and similarly  $h''$ ) as the total derivatives of  $h$  with respect to  $s_t$ . We obtain that

$$q(s_t, v_t) = \frac{(\alpha + r)s_t}{arv_t^2} + \frac{h'(s_t, v_t)}{r} \quad \text{and} \quad c_t = rW_t + h(s_t, v_t) - \gamma/v_t^2. \tag{A.4}$$

The Bellman equation evaluated at the optimal policies becomes

$$0 = r - \rho - ar\{(\alpha + r)s_t q_t - h(s_t, v_t) + \gamma/v_t^2\} + \alpha x h'(s_t, v_t) s_t + \frac{1}{2} a^2 r^2 q_t^2 v_t^2 - a^2 r h'(s_t, v_t) q_t v_t^2 + \frac{1}{2} \{a^2 h''(s_t, v_t) - ah''(s_t, v_t)\} \{\sigma^2 + v_t^2\}. \tag{A.5}$$

The solution for  $h$  in the value function is chosen to satisfy the transversality condition  $\lim_{t \rightarrow \infty} E\{V(t, W_t, s_t)\} = 0$ . The optimal investment policy follows from replacing  $h'(s_t)$  by its value.

Consider now the specialist. He acts as a Stackelberg leader. His objective is to maximize over  $\{v_\tau, \tau \geq t\}$ :

$$E_t \left[ \int_t^{+\infty} e^{-r\tau} \{T(v_\tau, s_\tau) - q(v_\tau, s_\tau)(\alpha + r)s_\tau\} d\tau \right] \tag{A.6}$$

taking into account that the investor will react optimally to his choice of volatility. Let  $V(t, s_t) = e^{-\rho t} k(s_t)$  be the specialist's value function. His Bellman Equation is

$$\begin{aligned} 0 = & \sup_{\{v_t\}} \left\{ \{T(v_t, s_t) - q(v_t, s_t)(\alpha + r)s_t\} - \rho k(s_t) + k'(s_t) \right. \\ & \left. \{-\alpha s_t\} + \frac{1}{2} k''(s_t) \{\sigma^2 + v_t^2\} \right\} \\ = & \sup_{\{v_t\}} \left\{ \frac{\beta^2 s_t^2}{ar} \left[ \frac{\sigma^2 + v_t^2}{(3\sigma^2 + \beta s_t^2 + v_t^2)^2} - \frac{1}{4(2\sigma^2 + \beta s_t^2)} \right] - \frac{(\alpha + r)s_t h'(s_t, v_t)}{r} \right. \\ & \left. - \rho k(s_t) + k'(s_t) \{-\alpha s_t\} + \frac{1}{2} k''(s_t) \{\sigma^2 + v_t^2\} \right\}, \tag{A.7} \end{aligned}$$

where  $h$  is determined as part of (i) by (A.5). The optimal control  $v_t$  to be set by the specialist solves the first-order condition:

$$\frac{(\alpha + r)^2 s_t^2}{ar} \frac{2v_t(\sigma^2 + \beta s_t^2 - v_t^2)}{(3\sigma^2 + \beta s_t^2 + v_t^2)^3} - \frac{(\alpha + r)s_t \partial h'(s_t, v_t)/\partial v}{r} + k''(s_t)v_t = 0. \tag{A.8}$$

We now show that the game admits  $v_t^2 = \sigma^2 + \beta s_t^2$  as its solution. We need to show that  $v_t^2 = \sigma^2 + \beta s_t^2$  solves (A.8). If  $v_t^2 = \sigma^2 + \beta s_t^2$ , then the solution  $h$  of the investor's Bellman Equation (A.5) is a constant determined by solving:

$$0 = -\frac{(\alpha + r)^2 s_t^2}{2} + (\sigma^2 + \beta s_t^2)(r - \rho + ar\{h - \gamma/(\sigma^2 + \beta s_t^2)\}), \tag{A.9}$$

i.e.,

$$\begin{cases} -(\alpha + r)^2/2 + \beta(r - \rho + arh) = 0 \\ \sigma^2(r - \rho + arh) - ar\gamma = 0 \end{cases} \Leftrightarrow h = \frac{1}{ar} \left\{ \frac{(\alpha + r)^2}{2\beta} + \rho - r \right\} \tag{A.10}$$

since we recall that  $\gamma = \sigma^2(\alpha + r)^2/(2ar\beta)$ . With  $h$  determined, the first-order condition of the specialist's problem becomes

$$\frac{(\alpha + r)^2 s_t^2}{ar} \frac{2v_t(\sigma^2 + \beta s_t^2 - v_t^2)}{(3\sigma^2 + \beta s_t^2 + v_t^2)^3} + k''(s_t)v_t = 0, \tag{A.11}$$

which has the solution  $v_t^2 = \sigma^2 + \beta s_t^2$ , with  $k(s_t) = 0$ . This function  $k$  satisfies the transversality condition  $\lim_{t \rightarrow \infty} E\{e^{-rt}k(s_t)\} = 0$ . Note also that in equilibrium the specialist makes zero profit. Finally, since  $h'(s_t, \sqrt{\sigma^2 + \beta s_t^2}) = 0$ , the investor's optimal holdings of the risky asset are given in equilibrium by  $q(s_t, v_t) = (\alpha + r)s_t/(arv_t^2)$ , i.e.,

$$q(s_t) = (\alpha + r)s_t/(ar\{\sigma^2 + \beta s_t^2\}). \tag{A.12}$$

*Proof of Proposition 2.* (i) Define  $\tau(s) \equiv \exp\{\int^s 2\alpha u/(\sigma^2 + \beta u^2) du\}$ ,  $m(s) \equiv 1/((\sigma^2 + \beta s^2)\tau(s))$ , the scale measure  $T(s) \equiv \int^s \tau(u) du$  and the speed measure  $M(s) \equiv \int^s m(u) du$ . The process  $s_t$  is strictly stationary on  $D = (-\infty, +\infty)$  and both  $+\infty$  and  $-\infty$  are entrance boundaries if and only if at both boundaries the scale measure  $T(s)$  diverges, the speed measure  $M(s)$  and the cross-integral  $N \equiv \int^s T(v) dM(v)$  both converge (see Karlin and Taylor, 1981, 15.6). These properties are satisfied here since near infinity  $\tau(s) \propto s^{2\alpha/\beta}$ .

Let  $p(t, s_t, |s_0)$  be the conditional density of  $s_t$  given  $s_0$ . To compute the conditional moments, we define the moment generating function  $\phi(t, \theta) \equiv E[e^{-\theta s_t} | s_0]$ . For notational simplicity the dependence of  $\phi$  on  $s_0$  is omitted since

$s_0$  is held fixed in what follows. We have that

$$\begin{aligned} \frac{\partial \phi(t, \theta)}{\partial t} &= \int_{-\infty}^{+\infty} e^{-\theta s_t} \frac{\partial p(t, s_t | s_0)}{\partial t} ds_t \\ &= \int_{-\infty}^{+\infty} e^{-\theta s_t} \left\{ \frac{\partial}{\partial s_t} [\alpha s_t p(t, s_t | s_0)] + \frac{1}{2} \frac{\partial^2}{\partial s_t^2} [(2\sigma^2 + \beta s_t^2) p(t, s_t | s_0)] \right\} ds_t \\ &= \int_{-\infty}^{+\infty} e^{-\theta s_t} \{ \theta \alpha s_t + \frac{1}{2} \theta^2 (2\sigma^2 + \beta s_t^2) \} p(t, s_t | s_0) ds_t, \end{aligned} \tag{A.13}$$

where we have successively applied the Kolmogorov forward equation (see, for example, Karlin and Taylor, 1981, 15.5) and integrated by parts. Therefore, we have obtained that

$$\frac{\partial \phi(t, \theta)}{\partial t} = -\theta \alpha \frac{\partial \phi(t, \theta)}{\partial \theta} + \frac{1}{2} \theta^2 \left( 2\sigma^2 \phi(t, \theta) + \beta \frac{\partial^2 \phi(t, \theta)}{\partial \theta^2} \right), \tag{A.14}$$

where

$$(-1)^n \left[ \frac{\partial^n \phi(t, \theta)}{\partial \theta^n} \right]_{\theta=0} = E[s_t^n | s_0] \equiv C_n(t) \text{ and } \phi(t, 0) = 1. \tag{A.15}$$

Differentiate both sides of this equality with respect to  $\theta$ , and evaluate the result at  $\theta = 0$ , to obtain

$$\begin{aligned} C'_1(t) &= -\alpha C_1(t), \\ C'_2(t) &= -(2\alpha - \beta) C_2(t) + 2\sigma^2, \\ C'_3(t) &= -3(\alpha - \beta) C_3(t) + 6\sigma^2 C_1(t), \\ C'_4(t) &= -2(2\alpha - 3\beta) C_4(t) + 12\sigma^2 C_2(t). \end{aligned} \tag{A.16}$$

Solving these first-order ordinary differential equations with the initial conditions  $C_n(0) = s_0^n$  yields the conditional moments  $E[s_t^n | s_0]$  for  $n = 1, 2, 3$  and 4.

(ii) In equilibrium, the price spread dynamics are given by (6) with the volatility (15):  $ds_t = -\alpha s_t dt + \sigma dZ_{2t} - \sqrt{\sigma^2 + \beta s_t^2} dZ_{1t}$ . The stationary distribution of  $s_t$  is determined from the drift  $-\alpha s_t$  and the diffusion  $(2\sigma^2 + \beta s_t^2)$ , with the normalization constant  $\xi$  determined to insure that the density integrates to one:

$$\begin{aligned} \pi(s) &= \frac{\xi}{(2\sigma^2 + \beta s^2)} \exp \left\{ \int_{-\infty}^s \frac{-2xu}{(2\sigma^2 + \beta u^2)} du \right\} \\ &= \frac{\xi}{(2\sigma^2 + \beta s^2)^{1+\alpha/\beta}} \\ &= \frac{1}{(2\sigma^2 + \beta s^2)^{1+\alpha/\beta} \int_{-\infty}^{+\infty} \{1/(2\sigma^2 + \beta u^2)^{1+\alpha/\beta}\} du}. \end{aligned} \tag{A.17}$$

Therefore near infinity  $\pi(s) \propto (\beta s^2)^{-1-\alpha/\beta}$  and  $\int^s u^n \pi(u) du \propto \beta^{-1-\alpha/\beta} s^{n-1-2\alpha/\beta}$  converges if and only if  $2\alpha > (n - 1)\beta$ . The unconditional moments  $U_n \equiv E[s_t^n]$  (independent of  $t$  by stationarity) can be computed either directly from the expression of the unconditional density  $\pi(s)$ , or more easily by appealing to the ergodicity of the process. That is:  $\pi(s) = \lim_{t \rightarrow +\infty} p(s, t | s_0)$  and hence  $U_n = \lim_{t \rightarrow +\infty} C_n(t)$ . The result for  $U_n$  is immediate given the expression of the conditional moments  $C_n(t)$ .

*Proof of Proposition 3.* (i) The extended Kalman–Bucy filter is derived from the following joint dynamics for the (unobservable) equilibrium and (observable) market prices:

$$d\bar{p}_t = \sigma dZ_{2t} \text{ and } dp_t = \alpha(\bar{p}_t - p_t)dt + \sqrt{\sigma^2 + \beta(\hat{\bar{p}}_t - p_t)^2} dZ_{1t} \tag{A.18}$$

(see Pugachev and Sinitsyn, 1987, p. 448).

Our objective is to find the stochastic differential equation followed by:  $\hat{f} \equiv E[f(\bar{p}_t) | \mathcal{G}_t]$  with  $\bar{p}_t$  and  $p_t$  given above. Using results from the theory of optimal filtering (e.g., Pugachev and Sinitsyn, 1987, (15) p. 388), it can be shown that  $\hat{f}_t \equiv E[f(\bar{p}_t) | \mathcal{G}_t]$  follows the following stochastic differential equation:

$$\begin{aligned} d\hat{f}_t = & E \left[ \frac{1}{2} \frac{d^2 f(\bar{p}_t)}{d\bar{p}_t^2} \sigma^2 \middle| \mathcal{G}_t \right] dt + E \left[ f(\bar{p}_t) \{ \alpha(\bar{p}_t - \hat{\bar{p}}_t) \right. \\ & \left. + \frac{df(\bar{p}_t)}{d\bar{p}_t} \rho_{12} \sigma \sqrt{\sigma^2 + \beta(\hat{\bar{p}}_t - p_t)^2} \middle| \mathcal{G}_t \right] \\ & \times (\sigma^2 + \beta(\hat{\bar{p}}_t - p_t)^2)^{-1} \{ dp_t - \alpha(\hat{\bar{p}}_t - p_t) dt \}, \end{aligned}$$

where at this stage we have allowed for the sake of generality the two Brownian motions to be correlated:  $E[dZ_{1t} dZ_{2t}] = \rho_{12} dt$ .

Apply this to  $f(z) \equiv z$ , hence  $\hat{f}_t \equiv E[f(\bar{p}_t) | \mathcal{G}_t] = \hat{\bar{p}}_t$ , to obtain the stochastic differential equations (25) for  $\hat{\bar{p}}_t$ . Then apply it to  $f(z) \equiv z^2$ , and subtract  $\hat{\bar{p}}_t^2$ , to obtain the equation (26) for the conditional estimation error  $R_t$ .

(ii) In the special case where  $\beta = 0$ , the extended Kalman–Bucy filters reduces to the optimal Kalman–Bucy linear filter. The conditional estimation error then follows a deterministic Riccati equation:

$$\frac{dR_t}{dt} = \sigma^2 - \left( \frac{\alpha R_t + \rho_{12} \sigma^2}{\sigma} \right)^2 \tag{A.19}$$

with initial condition  $R_0 = 0$ . It is immediate to verify that the solution of the Riccati equation is

$$R_t = (1 + \rho_{12}) \frac{\sigma^2}{\alpha} \left\{ \exp(2\alpha t) - 1 \right\} \bigg/ \left\{ \left( \frac{1 + \rho_{12}}{1 - \rho_{12}} \right) \exp(2\alpha t) + 1 \right\} \tag{A.20}$$



conditioning on the initial fundamental value  $\bar{p}_0$  assumed known, that is  $R_0 = 0$ . Its steady-state solution is

$$R \equiv \lim_{t \rightarrow +\infty} R_t = (1 - \rho_{12}) \frac{\sigma^2}{\alpha}. \tag{A.21}$$

*Proof of Proposition 4.* Let  $p(t, \bar{p}_t, p_t | \bar{p}_0, p_0)$  be the conditional density of the pair  $(\bar{p}_t, p_t)$  given  $(\bar{p}_0, p_0)$  and define the moment generating function  $\phi(t, \bar{\theta}, \theta) \equiv E[e^{-\bar{\theta}\bar{p}_t - \theta p_t} | \bar{p}_0, p_0]$ . Applying the multidimensional Kolmogorov forward equation, we have that

$$\begin{aligned} \frac{\partial \phi}{\partial t} = & -\theta\alpha \left( \frac{\partial \phi}{\partial \theta} - \frac{\partial \phi}{\partial \bar{\theta}} \right) + \frac{1}{2} \theta^2 \left( \sigma^2 \phi + \beta \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial \bar{\theta}^2} - 2 \frac{\partial^2 \phi}{\partial \theta \partial \bar{\theta}} \right) \right) \\ & + \frac{1}{2} \bar{\theta}^2 \sigma^2 \phi, \end{aligned} \tag{A.22}$$

where

$$(-1)^{n+m} \left[ \frac{\partial^n \phi(t, \bar{\theta}, \theta)}{\partial \bar{\theta}^n \partial \theta^m} \right]_{\bar{\theta}=\theta=0} = E[\bar{p}_t^n p_t^m | \bar{p}_0, p_0] \equiv C_{nm}(t) \text{ and } \phi(t, 0, 0) = 1. \tag{A.23}$$

Differentiate both sides of this equality with respect to  $\theta$ , and evaluate the result at  $\bar{\theta} = \theta = 0$ , to obtain

$$\begin{aligned} C'_{10}(t) &= 0, \\ C'_{01}(t) &= -\alpha C_{01}(t) + \alpha C_{10}(t), \\ C'_{20}(t) &= \sigma^2, \\ C'_{11}(t) &= -\alpha C_{11}(t) + \alpha C_{20}(t), \\ C'_{02}(t) &= -(2\alpha - \beta) C_{02}(t) + 2(\alpha - \beta) C_{11}(t) + \beta C_{20}(t) + \sigma^2. \end{aligned} \tag{A.24}$$

Solving this system of first-order ordinary differential equations with the initial conditions  $C_{nm}(0) = \bar{p}_0^n p_0^m$  yields the described conditional moments given by (28) and (29). The solution is obtained sequentially: first solve for  $C_{10}$ , then use  $C_{10}$  to find  $C_{01}$  from the second equation in the system, etc.

*Proof of Proposition 5.* Using the same method as in the proof of Proposition 4, for the two state variables  $s_t$  and  $q_t$  with respective dynamics (6) and (32), we obtain

$$\begin{aligned} E[q_{t+\Delta} | s_t, q_t] &= q_t - \lambda s_t (e^{-\alpha\Delta} - 1) \\ E[q_{t+\Delta}^2 | s_t, q_t] &= q_t^2 - 2\lambda (e^{-\alpha\Delta} - 1) q_t s_t + \lambda^2 (e^{-(2\alpha - \beta)\Delta} - 2e^{-\alpha\Delta} + 1) s_t^2 \\ &\quad + 2\sigma^2 \lambda^2 / (2\alpha - \beta). \end{aligned} \tag{A.25}$$

The expressions for the conditional moments of the observed volume, i.e., the change in the investor's holding between  $t$  and  $t + \Delta$ ,  $q_{t+\Delta} - q_t$ , follow.

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