Minimax Universal Sampling for Compound Multiband Channels

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Capacity of Undersampled Channels

- Point-to-point channels

**Issue:** wideband systems preclude Nyquist-rate sampling!
Capacity of Undersampled Channels

- **Point-to-point channels**

  ![Diagram](image)

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- **Sub-Nyquist sampling well explored in Signal Processing**
  - Landau-rate sampling, compressed sensing, etc.
  - **Objective metric:** MSE
Capacity of Undersampled Channels

- **Point-to-point channels**

  ![Diagram of a communication channel](image)

  **Issue:** wideband systems preclude Nyquist-rate sampling!

- **Sub-Nyquist sampling** well explored in Signal Processing
  - Landau-rate sampling, compressed sensing, etc.
  - **Objective metric:** MSE

- **Question:** which sub-Nyquist samplers are optimal in terms of **CAPACITY**?
Prior work: Channel-specific Samplers

- Consider linear time-invariant sub-sampled channels

\[ x(t) \xrightarrow{H(f)} \eta(t) \xrightarrow{\mathcal{P}(\cdot)} y(t) \xrightarrow{} y[n] = y(t_n) \]
Prior work: Channel-specific Samplers

- Consider linear time-invariant sub-sampled channels

- The channel-optimized sampler (optimized for a single channel)
  - (1) a filter bank followed by uniform sampling
  - (2) a single branch of and modulation and filtering with uniform sampling
Prior work: Channel-specific Samplers

- Consider linear time-invariant sub-sampled channels

- The channel-optimized sampler *(optimized for a single channel)*
  - (1) a filter bank followed by uniform sampling
  - (2) a single branch of and modulation and filtering with uniform sampling

- Suppresses Aliasing

- *No need to use non-uniform sampling grid!*

![Diagram of channel-optimized sampler](image)
Universal Sampling for Compound Channels

The channel-optimized sampler suppresses aliasing

- What if there are a collection of channel realizations?
Universal Sampling for Compound Channels

The channel-optimized sampler suppresses aliasing

- What if there are a collection of channel realizations?

- **Universal (channel-blind) Sampling**
  ---- A sampler is typically integrated into the hardware
  ---- Need to operate *independently* of instantaneous realization
Sub-optimality of Channel-optimized Samplers

Consider 2 possible channel realizations ........

(a) Effective channel gain

(b) Effective channel gain
Sub-optimality of Channel-optimized Samplers

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optimal sampler for (a) — Far from optimal!
Sub-optimality of Channel-optimized Samplers

Consider 2 possible channel realizations ..........

(a) Effective channel gain

(b) Effective channel gain

optimal sampler for (a) ➔ Far from optimal!

- No single linear sampler can maximize capacity for all realizations!
- **Question:** how to design a universal sampler robust to different channel realizations
Robustness Measure: Minimax Capacity Loss

- Consider a channel state $s$ and a sampler $Q$:

  - maximum capacity $C_s$
  - achievable rate under $Q$: $C_s^{Q}$

**Capacity Loss:**

$$L_s^Q := C_s - C_s^{Q}$$
Robustness Measure: Minimax Capacity Loss

• Consider a channel state $s$ and a sampler $Q$:

- Maximum capacity $C_s$
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Capacity Loss: $L_s^Q := C_s - C_s^Q$

Minimax Capacity Loss:

$$\min_Q \max_s L_s^Q$$

accounting for all channel states $s$
Robustness Measure: Minimax Capacity Loss

- Consider a channel state $s$ and a sampler $Q$:

  - Maximum capacity $C_s$
  - Achievable rate under $Q$: $C_s^Q$

  **Capacity Loss:**
  \[ L_s^Q := C_s - C_s^Q \]

  **Minimax Capacity Loss:**
  \[ \min_Q \max_s L_s^Q \]

  Optimize over a large class of samplers accounting for all channel states $s$

  \[ Q^* = \arg \min_Q \max_s L_s^Q \quad -- \quad \text{Minimax Sampler} \]
Minimax Universal Sampling

Capacity

Nyquist-rate Capacity

Capacity under Minimax Sampler

State: s
Minimax Universal Sampling

- A sampler that minimizes the worse-case capacity loss due to universal sampling

$$Q^* = \arg \min_Q \max_s C_s - C_s^Q$$
Minimax Universal Sampling

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A sampler that maximizes compound channel capacity

\[ \hat{Q} = \arg \max_Q \min_s C_s^Q \]

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A sampler that minimizes the worse-case capacity loss due to universal sampling

\[ Q^* = \arg \min_Q \max_s C_s - C_s^Q \]
Focus on Multiband Channel Model

A class of channels where at each time only a fraction of bandwidths are active.

$k$ out of $n$ subbands are active.
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Sparsity ratio: $\beta := k/n$

Undersampling ratio: $\alpha := f_s/W$
Focus on Multiband Channel Model

A class of channels where at each time only a fraction of bandwidths are active.

$k$ out of $n$ subbands are active.

$m$-branch sampling with modulation and filtering:

\[
F_i(f) \times S_i(f) \rightarrow y_i(t) \rightarrow y_i[n]
\]

Sparsity ratio: $\beta := \frac{k}{n}$

Undersampling ratio: $\alpha := \frac{f_s}{W}$
**Converse: Landau-rate Sampling** \((\alpha=\beta)\)

Sparsity ratio: \(\beta := k/n\)

Undersampling ratio: \(\alpha := m/n = f_s/W\)

**Theorem (Converse):** The minimax capacity loss *per Hertz* obeys:

\[
\inf_{Q} \max_{s \in \binom{[n]}{k}} L^Q_s \geq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \frac{2}{\sqrt{\text{SNR}_{\text{min}}} } - \frac{\log n}{n} \right\}
\]

\(\mathcal{H}(\beta) := -\beta \log \beta - (1 - \beta) \log(1 - \beta)\)
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At high SNR and large \(n\),

\(\minimax\ capacity\ loss\ determined\ by\ subband\ uncertainty\)
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\]

**Key observation for the proof:**

\[
\sum_{s} \exp(L_s^Q) \approx \text{constant}
\]
Converse: Landau-rate Sampling \((\alpha=\beta)\)

Sparsity ratio: \(\beta := k/n\)
Undersampling ratio: \(\alpha := m/n = f_s/W\)

**Theorem (Converse):** The minimax capacity loss per Hertz obeys:

\[
\inf_Q \max_{s \in [n]^k} L_s^Q \geq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \frac{2}{\sqrt{\text{SNR}_{\text{min}}} } - \frac{\log n}{n} \right\}
\]

**Key observation for the proof:**

\[
\sum_s \exp(L_s^Q) \approx \text{constant}
\]

The minimax sampler achieves *equivalent loss* across all channel states.
Achievability: Landau-rate Sampling \((\alpha=\beta)\)

-\(\text{Sparsity ratio: } \beta := k/n\)
-\(\text{Undersampling ratio: } \alpha := m/n = f_s/W\)

-\textbf{Deterministic optimization is NP-hard (non-convex).}
Achievability: Landau-rate Sampling \((\alpha=\beta)\)

Sparsity ratio: \(\beta := k/n\)

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- **Deterministic** optimization is NP-hard (non-convex).

- **Hope:** random sampling

Fourier transform of periodic sequence is a spike-train
Achievability: Landau-rate Sampling ($\alpha=\beta$)

Sparsity ratio: $\beta := k/n$

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- **Deterministic** optimization is NP-hard (non-convex).
- **Hope:** random sampling

Fourier transform of periodic sequence is a spike-train

A sampling system is called **independent random sampling** if the coefficients of the spike-train are independently and randomly generated.
Achievability: Landau-rate Sampling \((\alpha=\beta)\)

Sparsity ratio: \(\beta := k/n\)
Undersampling ratio: \(\alpha := m/n = f_s/W\)

**Theorem (Achievability):** The capacity loss per Hertz under independent random sampling is

\[
\forall s \in \binom{[n]}{k} : \quad L_s^Q \leq \frac{1}{2} \left\{ \mathcal{H}(\beta) + \frac{5 \log k}{n} + \frac{\beta}{\text{SNR}_{\text{min}}} \right\}
\]

with probability exceeding \(1 - e^{-\Omega(n)}\).
Implications: Landau-rate Sampling \( (\alpha=\beta) \)

**Theorem (Converse):**
\[
\inf_Q \max_{s \in \binom{[n]}{k}} L^Q_s \geq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \frac{2}{\sqrt{\text{SNR}_{\min}}} - \frac{\log n}{n} \right\}
\]

**Theorem (Achievability):** Under independent random sampling (with zero mean and unit variance), with exponentially high probability,
\[
\forall s \in \binom{[n]}{k} : \quad L^Q_s \leq \frac{1}{2} \left\{ \mathcal{H}(\beta) + \frac{5 \log k}{n} + \frac{\beta}{\text{SNR}_{\min}} \right\}
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Implications: Landau-rate Sampling \((\alpha=\beta)\)

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\inf_{Q} \max_{s \in \binom{[n]}{k}} L_{s}^{Q} \geq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \frac{2}{\sqrt{\text{SNR}_{\text{min}}} \log n} \right\}
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\]

- Random sampling is **Minimax**
- Sharp concentration – exponentially high probability
Implications: Landau-rate Sampling (α=β)

Theorem (Converse):
\[ \inf_Q \max_{s \in \{[n]\}} L^Q_s \geq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \frac{2}{\sqrt{\text{SNR}_{\min}}} \log n \right\} \]

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- Random sampling is \textbf{Minimax}
- Sharp concentration – exponentially high probability
- \textbf{Universality phenomena:}
  - A large class of distributions can work!
    - Gaussian, Bernoulli, uniform…
  - No need for i.i.d. randomness
    - can be a mixture of Gaussian, Bernoulli, uniform…
Capacity Loss for Multiband Channels

Capacity

Nyquist-rate Capacity

minimax capacity loss

Capacity under Minimax Sampler

State: $s$
Minimax sampling yields *equivalent capacity loss* over all possible channel realizations when SNR and $n$ are large!
Converse: Super-Landau Sampling \((\alpha > \beta)\)

Sparsity ratio: \(\beta := k/n\)

Undersampling ratio: \(\alpha := m/n = f_s/W\)

Theorem (Converse): The minimax capacity loss per Hertz obeys:

\[
\inf_Q \max_{s \in \binom{[n]}{k}} L^Q_s \geq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \alpha \mathcal{H}\left(\frac{\beta}{\alpha}\right) - \frac{2}{\sqrt{\text{SNR}_{\text{min}}}} - \frac{\log n}{n} \right\}
\]

- Capacity gain due to oversampling is

\[
\frac{1}{2} \alpha \mathcal{H}\left(\frac{\beta}{\alpha}\right)
\]
Achievability: Super-Landau Sampling \( (\alpha > \beta) \)

- Sparsity ratio: \( \beta := k/n \)
- Undersampling ratio: \( \alpha := m/n = f_s/W \)

Gaussian sampling \( \rightarrow \) Gaussian modulation coefficients.
Achievability: Super-Landau Sampling \((\alpha > \beta)\)

Sparsity ratio: \(\beta := k/n\)

Undersampling ratio: \(\alpha := m/n = f_s/W\)

Gaussian sampling \(\Rightarrow\) Gaussian modulation coefficients

Theorem (Achievability): If \(\alpha + \beta < 1\), then the capacity loss per Hertz under i.i.d. Gaussian random sampling is

\[
\forall s \in \binom{[n]}{k}: \quad L_s^Q \leq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \alpha \mathcal{H} \left( \frac{\beta}{\alpha} \right) + O \left( \frac{\log^2 n}{\sqrt{n}} \right) + \frac{\beta}{\text{SNR}_{\text{min}}} \right\}
\]

with probability exceeding \(1 - e^{-\Omega(n)}\).
Implications: super-Landau sampling ($\alpha=\beta, \alpha+\beta<1$)

Theorem (Converse): The minimax capacity loss per Hertz obeys:

$$\inf_Q \max_{s \in \binom{[n]}{k}} L_s^Q \geq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \alpha \mathcal{H} \left( \frac{\beta}{\alpha} \right) - \frac{2}{\sqrt{\text{SNR}_{\text{min}}}} \frac{\log n}{n} \right\}$$

Theorem (Achievability): Under i.i.d. Gaussian random sampling, with exponentially high probability

$$\forall s \in \binom{[n]}{k}: \quad L_s^Q \leq \frac{1}{2} \left\{ \mathcal{H}(\beta) - \alpha \mathcal{H} \left( \frac{\beta}{\alpha} \right) + O \left( \frac{\log^2 n}{\sqrt{n}} \right) + \frac{\beta}{\text{SNR}_{\text{min}}} \right\}$$

- Gaussian sampling is **Minimax**!
- Sharp concentration: exponentially high probability
- **Universality phenomena not shown**...
  - We have only shown the results for i.i.d. Gaussian sampling
Concluding Remarks

- **Minimax Capacity Loss**
  - A new *metric* to characterize *robustness against different channel realizations*
  - For multiband channels, it depends only on undersampling factor and sparsity ratio

- **The power of random sampling**
  - Near-optimal in an overall sense (*minimax*)
  - Large random samplers behave *in deterministic ways* (sharp concentration + universality)

- **A Non-Asymptotic analysis of random channels**