1 Representation of states

Pure states of systems in QM are represented by vectors. (We’ll only be concerned with pure states for now.) A vector space is a set of elements equipped with certain mathematical structure. The structure defines operations of adding vectors to other vectors, multiplying vectors by scalars, and so on. (For a detailed treatment of vector spaces, see chapter 1 of Hughes. For a less technical treatment, see chapter 2 of Albert.)

Example: the pure spin states of an electron are represented by vectors belonging to a two-dimensional vector space. Elements of this space include $z^+$ (pronounced “spin-up in the z direction”), $z^-$ (pronounced “spin-down in the z direction”), $1/\sqrt{2} (z^+ + z^-)$.

The vector spaces we’ll be dealing with are equipped with an inner product, which is an operation that takes two vectors $v, v'$ and
returns a scalar, denoted $\langle v, v' \rangle$. When the inner product between two vectors equals zero, we say that the vectors are orthogonal. The norm of a vector $v$ (think: length), is denote by $|v|$ and equals $\sqrt{\langle v, v \rangle}$.

An orthonormal basis for a vector space is a set of vectors $\{b_1, b_2, \cdots \}$ such that:

- Each basis element has unit length
- Distinct basis elements are orthogonal
- Any vector in the space can be written as a weighted sum of the basis vectors

Heuristic: think of an orthonormal basis as determining a set of scaled axes for the space. Why does it make sense that the dimensionality of a vector space is defined to be the number of elements in a basis for it?

The set $\{z^+, z^-\}$ is a basis for the space that represents spin states. That means any vector in the space can be written in the form $az^+ + bz^-$. Once we fix a basis, we can write vectors as columns of numbers. For example, fixing the basis above, we can write $az^+ + bz^-$ as $\begin{pmatrix} a \\ b \end{pmatrix}$. Therefore, we write $z^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
\[ z^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ 1/\sqrt{2} \left( z^+ + z^- \right) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \]

Define \( x^+ = 1/\sqrt{2} (z^+ + z^-) \), \( x^- = 1/\sqrt{2} (z^+ - z^-) \). Notice that we can think of \( x^+ \) as a weighted sum ("superposition") of \( z^+ \) and \( z^- \). (Which sum?) But equally well we can think of \( z^+ \) as a superposition of \( x^+ \) and \( x^- \). (How?)

Notice that \( \{x^+, x^-\} \) is also a basis for the spin space (proof?).

2 Experimental outcomes

None of this so far connects up to experiment, since we haven’t said what experimental results are predicted given a quantum system whose state is represented by a particular vector.

In the simplest case, each experimental setup is associated with a basis (i.e., choice of scaled axes) in the appropriate state space. For example, a z-spin measurement setup (e.g., a Stern-Gerlach magnet and associated target screen) is associated with this basis \( \{z^+, z^-\} \). And each experimental outcome is associated with a member of that basis. For example, in the z-spin case, the “up” outcome is associated with \( z^+ \) and the “down” outcome is associated with \( z^- \).
**Born’s rule** connects up system states with measurement outcomes, as follows. We always represent the (pure) state of a system with a vector of unit length. Suppose that we are given such a vector $v$, and a measurement with associated basis $\{z^+, z^-\}$. To get the probability that the outcome of the measurement will be “up”, project $v$ onto the axis associated with the “up” outcome. Then square the length of the resulting vector. (For now, think of projection as the familiar geometrical operation that maps a vector to the “shadow” it casts on an axis, assuming that light shines down perpendicular to the axis.)

Given a vector $az^+ + bz^-$, what is a condition on $a, b$ that guarantees that the vector has unit length?

Given a vector $az^+ + bz^-$, what is the probability that an electron in a state represented by that vector will produce an outcome “up” on a z-spin measurer?

Now we have enough theory on the table to see why an electron with spin state $z^+$ is sure to go up through a z-measurer (why?).

And we can see why an electron with spin state $z^+$ is 50% likely to go up through an $x^+$ measurer (why?).

Is there any spin state for an electron to be in that would dispose it to go up for certain through an $x$-measurer (if measured in that way), and also up for certain through a z-measurer (if measured in
that way)?

3 “Interpretations”

As Frank Arntzenius has pointed out, the things that are often called “competing interpretations” of quantum mechanics are better understood as competing physical theories, that happen to have nearly the same empirical predictions.

4 Naive collapse interpretation

Here is one such “interpretation”, or theory. According to the Naive collapse theory, there are two ways for the state of the world to change over time. The state changes in the first way when no measurements are being done. This way involves continuous changes in the vector that represents the world’s state. We will have more to say about this way later.

The state changes in the second way when a measurement is being done. This change is called a “collapse”, and involves a discontinuous change in the vector that represents the state of a system. Here is an example. Suppose that an electron in state $z^+$ is subjected to an $x$-measurement. Then the measurement has a 50% chance of resulting in an “up” outcome in which case the state of the electron instanta-
neously becomes $x^+$, and a 50% chance of resulting in a “down” outcome in which case the state of the electron instantaneously becomes $x^-$. More generally, when a measurement represented by a particular basis is done, then the state vector of the system jumps to one or another of the basis elements. The probability that the system in state $v$ jumps to basis element $b_i$ equals the squared length of the result of projecting $v$ onto $b_i$.

5 Problems with naive collapse interpretation


6 Multi-part systems

Multiparticle classical systems: to get two-particle state space, use *cartesian product* of individual state spaces. Example: two point-particles in 3-space. State space of each particle has 6 dimensions: 3 position and 3 momentum. State space of composite system has $6 + 6 = 12$ dimensions. Each element can be thought of as a pair $(s_1, s_2)$, where $s_1$ represents the state of the first particle and $s_2$ represents the
state of the second.

In quantum case, we do not do the same thing. Instead, the state space for the combined system is the tensor product of the state-spaces for the individual systems. Taking the tensor product of two 6-dimensional spaces produces a space with $6 \cdot 6 = 36$ dimensions. More generally: the dimensionality of the tensor product space is the product of the dimensionalities of the component spaces.

Basic idea of the tensor product construction is best given by an example: composite system consisting of spin states of two electrons. (Here I follow the Rieffel article.) Each electron has a spin space associated with it, with basis $\{z^+, z^-\}$. The tensor product space has basis $\{z^+ \otimes z^+, z^+ \otimes z^-, z^- \otimes z^+, z^- \otimes z^-\}$. In other words, every vector in that space can be written in the form:

$$az^+ \otimes z^+ + bz^+ \otimes z^- + cz^- \otimes z^+ + dz^- \otimes z^-$$

Example: the state $z^+ \otimes z^-$ might be pronounced “the first electron has $z$-spin up and the second electron has $z$-spin down. We would write this state down with respect to the above basis as:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The tensor product operation distributes over linear combinations:
\[ v \otimes (av_0 + bv_1) = av \otimes v_0 + bv \otimes v_1 \]

\[ (av_0 + bv_1) \otimes v = av_0 \otimes v + bv_1 \otimes v \]

Another example: the state \( z^+ \otimes x^+ \) equals \( z^+ \otimes 1/\sqrt{2} (z^+ + z^-) \).

(How would you write this down in the above basis?)

The tensor product space has an inner product that satisfies this condition:

\[ \langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \langle w, w' \rangle. \]

Note that not every element of the product space can be written in the form \( v \otimes w \). (Example of such an element: \( 1/\sqrt{2} (z^+ \otimes z^+ + z^- \otimes z^-) \). Why can’t this state be written in the above form?)

### 7 Linear operators and dynamics

A linear operator maps vectors to vectors in a way that distributes over linear combinations, as follows. For any vectors \( v, v' \):

\[ L(av + bv') = aL(v) + bL(v'). \]
One special sort of linear operator is a **unitary operators**. A unitary operator is a linear operators that *preserve inner products*. In other words, for any \( v, v' \):

\[
\langle v, v' \rangle = \langle U(v), U(v') \rangle
\]

It follows that unitary operators preserve the lengths of vectors: for any vector \( v \), the length of \( v \) equals the length of \( U(v) \) (why?). It also follows that every unitary operator \( U \) has the following property: if \( \{b_1, b_2, \cdots\} \) is an orthonormal basis for the space in question, then so is \( \{U(b_1), U(b_2), \cdots\} \). (Why?)

Heuristic: think of unitary operators as *rotations*. (This heuristic reflects a simplification that we will go into later.)

Unitary operators are important because they represent how quantum systems change state over time (except during collapses, if there are such). It is an important fact that this sort of evolution of quantum systems is linear:

\[
U(av + bv') = aU(v) + bU(v').
\]