ELE539A: Optimization of Communication Systems
Lecture 24: Integer Programming and Applications
Conclusions and Projects

Professor M. Chiang
Electrical Engineering Department, Princeton University

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Lecture Outline

• Integer programming and modelling
• Branch and bound
• Optical network routing and wavelength assignment
• Summary of topics covered in the course
• Final projects
• Conclusions
Integer Programming

Some variables can only assume discrete values:

- Boolean constrained
- Integer constraints

Useful for many applications:

- Some resources cannot be infinitesimally divided
- Only discrete levels of control

In general, integer programming is extremely difficult

- Ad hoc solution: e.g., branch and bound (this lecture)
- Systematic solution: e.g., SOS method
- Understand problem structure (this lecture)
MILP

A special case: mixed integer linear programming

LP with some of the variables constrained to be integers:

\[
\begin{align*}
\text{minimize} & \quad c^T x + d^T y \\
\text{subject to} & \quad Ax + By = b \\
& \quad x, y \geq 0 \\
& \quad x \in \mathbb{Z}^n_+ 
\end{align*}
\]

LP relaxation:

\[
\begin{align*}
\text{minimize} & \quad c^T x + d^T y \\
\text{subject to} & \quad Ax + By = b \\
& \quad x, y \geq 0 
\end{align*}
\]

Optimal value of LP relaxation provides a lower bound (since minimizing over a larger constraint set) that is readily computed
Boolean Constrained LP

LP with some of the variables constrained to be binary:

\[
\begin{align*}
\text{minimize} & \quad c^T x + d^T y \\
\text{subject to} & \quad Ax + By = b \\
& \quad x, y \geq 0 \\
& \quad x_i \in \{0, 1\}
\end{align*}
\]

LP relaxation:

\[
\begin{align*}
\text{minimize} & \quad c^T x + d^T y \\
\text{subject to} & \quad Ax + By = b \\
& \quad x, y \geq 0 \\
& \quad x_i \in [0, 1]
\end{align*}
\]
Modelling Techniques

- **Binary choice:**

  \[
  \text{maximize} \quad c^T x \\
  \text{subject to} \quad w^T x \leq K \\
  x_i \in \{0, 1\}
  \]

  where \( c \) is value vector and \( w \) is weight vector

- **Forcing constraint:**

  If \( x \leq y \) and \( x, y \in \{0, 1\} \), then \( y = 0 \) implies \( x = 0 \)

- **Relationship between variables:**

  \[
  \sum_{i=1}^{n} x_i \leq 1 \quad \text{and} \quad x_i \in \{0, 1\} \quad \text{means at most one of } x_i \text{ can be 1} \\
  y = \sum_{i=1}^{n} a_i x_i, \quad \sum_{i=1}^{n} x_i \leq 1 \quad \text{and} \quad x_i \in \{0, 1\} \quad \text{means that } y \text{ must take a value in } \{a_1, \ldots, a_m\}
  \]
Branch and Bound

Instead of exploring the entire set of feasible integer solutions, which is exponential time, use bounds on optimal cost to avoid exploring certain parts of the set of feasible integer solutions.

Worst case is still exponential time, but sometimes saves searching time.

Split constraint set $F$ into a finite collection of subsets $F_1, \ldots, F_k$ and solve separately each of the problems for $i = 1, \ldots, k$:

$$\text{minimize } c^T x$$

subject to $x \in F_i$
**Branch and Bound**

Assume we can quickly produce a lower bound on a subproblem:

\[ b(F_i) = \min_{x \in F_i} c^T x \]

which is also a lower bound on the original problem.

This lower bound is usually obtained by LP relaxation that takes away integer constraints.

If a subproblem is solved to integral optimality, or its objective function is evaluated using feasible integral solution, an upper bound \( U \) is obtained.

If a subproblem produces a lower bound \( b(F_i) \geq U \), that branch can be ignored.

**Branch and bound**: delete subproblems that are infeasible or produce a lower bound that is larger than an upper bound on the original problem. Subproblems not deleted either can be solved for optimality or be split into more subproblems.
Example

minimize \( x_1 - 2x_2 \)
subject to \( -4x_1 + 6x_2 \leq 9 \)
\( x_1 + x_2 \leq 4 \)
\( x_1, x_2 \geq 0 \)
\( x_1, x_2 \in \mathbb{Z} \)

LP relaxation: \( x = (1.5, 2.5), b(F) = -3.5 \)

Add \( x_2 \geq 3 \) to form \( F_1 \): infeasible, so delete

Add \( x_2 \leq 2 \) to form \( F_2 \): \( x = (3/4, 2), b(F_2) = -3.25 \)

Add \( x_1 \geq 1 \) to form \( F_3 \): \( x = (1, 2), U = -3 \)

Add \( x_1 \leq 0 \) to form \( F_4 \): \( x = (0, 3/2), b(F_4) = -3 \geq U \), so delete

No more subproblem, optimal integer solution \( x^* = (1, 2) \)
Optical Network Routing and Wavelength Assignment

Optical WDM for core backbone networks:

- Establish optical layer connection, lightpath routing (like circuit switching) and wavelength assignment
- **Wavelength continuity** requirement and **Wavelength conversion**: none, partial, full

Traffic arrival models:

- **Static**: minimize link usage cost
- **Dynamic**: minimize blocking probability

Connected undirected graph \( G = (V, E) \). Each edge represents a pair of unidirectional fiber links in opposite directions

Set \( W \) of origin-destination (OD) pairs, where an OD pair is an ordered pair \( w = (i, j) \) of distinct nodes \( i \) and \( j \)
**Notation**

\( r_w \) denotes input traffic of OD pair \( w \): a nonnegative integer representing the given number of lightpath requests of \( w = (i, j) \)

\( P_w \): Set of paths that OD pair \( w \) may use

\( C \): Set of wavelengths/colors available on each link

Link *cost* function (increasing, convex, piecewise linear):

![Link cost function graph](image)

Breakpoint locations have important implications
Formulations

Optimization variables: path flows \( \{x_p \mid w \in W, p \in P_w\} \), where \( x_p \) represents the flow of path \( p \in P_w \) for some \( w \in W \) and takes a nonnegative integer value.

**Integer programming** formulation (NP hard):

minimize \[ \sum_{l \in L} D_l(f_l) \]
subject to
\[ f_l = \sum_{\{p \mid l \in p\}} x_p, \quad \forall l \in L, \]
\[ \sum_{\{p \mid l \in p\}} x_p \leq |C|, \quad \forall l \in L, \]
\[ \sum_{p \in P_w} x_p = r_w, \quad \forall w \in W, \]
\[ x_p \in \mathbb{Z}^+, \quad \forall p \in P_w, w \in W \]

**Piecewise linear programming** relaxation (efficiently solvable):

\( x_p \in \mathbb{R}^+, \quad \forall p \in P_w, w \in W \)
Integral Optimality

Separable ODs in a ring: ring can be separated into two pieces with one piece containing all origins and other other all destinations

For ring networks with separable origins and destinations and feasible solution to relaxed problem, there is an integral optimal solution to relaxed problem that is also optimal to original problem (for both zero and full wavelength conversion cases)
Proof

Suppose unit lightpath requests and no OD pairs share same O or D.

Construct an algorithm that takes an optimal fractional solution of the problem that involves some OD pairs that divide their traffic between alternative paths, and at each iteration produces another optimal solution with fewer OD pairs that divide their traffic.

Suppose in the beginning of an iteration, we have a fractional optimal solution of the relaxed problem that involves \( n \) OD pairs that split their input traffic between two alternative paths, in addition to some other OD pairs that send all their traffic along a single path.

For the \( i^{th} \) OD pair that splits its traffic, denote the flow along the counterclockwise path by \( x_i \), where \( 0 < x_i < 1 \).

\( n \) OD pairs interleave if it is impossible to break a ring into two parts so that there are two OD pairs each entirely contained in one of the two parts. It can be shown that all OD pairs that divide their traffic between alternative paths must interleave.
Suppose origins of the OD pairs arranged consecutively in the clockwise direction. So the corresponding destinations must also be arranged consecutively from 1 to $n$ in the clockwise direction.

Flows of the links that belong to paths used by any of the $n$ OD pairs $1, \ldots, n$ are equal to an integer plus or minus the following $2n$ quantities:

$x_1, (x_1 + x_2), \ldots, (x_1 + x_2 + \cdots + x_{n-1}), (x_1 + x_2 + \cdots + x_{n-1} + x_n), (x_2 + x_3 + \cdots + x_{n-1} + x_n), (x_3 + \cdots + x_{n-1} + x_n), \ldots, x_n, 0$
Another notation: \( a_j^T x, \quad j = 1, \ldots, 2n - 1 \)

Index set \( J(x) = \{ j \in \{1, \ldots, 2n - 1\} | a_j^T x = \text{integer} \} \)

Algorithm iteration \( k \) step 1: Find a nonzero vector \( \Delta x \) satisfying
\[ a_j^T \Delta x = 0, \quad \text{for all } j \in J(x) \]

Change in cost:
\[ \Delta \text{cost} = \alpha \sum_{j \notin J(x)} b_j^k(a_j^T \Delta x) \geq 0 \]
where \( b_j^k \) is slope of linear segment \( k \) in cost function of link \( j \)

\( -\Delta x \) also satisfies \( a_j^T \Delta x = 0 \), so \( \Delta x \) is cost preserving

Algorithm iteration \( k \) step 2: Let \( \alpha \) be the smallest scalar such that, either any one of the components of the vector \( x + \alpha \Delta x \) becomes integer or one of the link flows \( a_j^T (x + \alpha \Delta x) \) for some \( j \notin J(x) \) becomes integer

Replace \( x \) by \( x + \alpha \Delta x \) and \( J(x) \) by \( J(x + \alpha \Delta x) \)

Algorithm iteration \( k \) step 3: If all components of the resulting vector \( x \) are integer, stop
Else, if at least one of the components of the vector $x$ is integer, go to the next iteration $k + 1$. A new optimal solution for the relaxed problem that has fewer fractional components.

Else go back to step 1. A new optimal solution, which yields fewer fractional link flows. This is repeated finite number of times before we get an optimal solution which involves fewer OD pairs that divide their traffic between alternative paths and go to the next iteration.

Since at the end of each iteration, we have an optimal solution with fewer fractional components, this algorithm yields an integer optimal solution for this network in finite number of iterations.
Extensions

We have focused on a special case (separable ring networks with full wavelength conversion). Generalizations:

- General networks: algorithm to round fractional solution (to relaxed problem) to optimal or suboptimal integral solution (to original problem)

  Empirically effective probably because extreme points of relaxed constraint polyhedron are integers in most cases, because of piece-wise linear costs

- Exact penalty approach to partial or no wavelength conversion

- Dynamic programming approach to dynamic arrival traffic models

Open question:

- What kind of empirical traffic statistics lead to integral optimal solution to relaxed formulations?
Lecture Summary

Readings: Chapters 10 and 11 in Bertsimas and Tsitsiklis, *Introduction to Linear Programming*.

End of the Course
Why Take This Course

- Learn the tools and mentality of optimization (surprisingly useful for other study you may engage in later on)
- Learn classic and recent results on optimization of communication systems (over a surprisingly wide range of problems)
- Enhance the ability to do original research in academia or industry
- Have a fun and productive time on the final project
Where We Started

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in C
\end{align*}
\]
**Topics Covered: Methodologies**

- Nonlinear optimization (linear, convex, nonconvex), Pareto optimization, Dynamic programming, Integer programming
- Convex optimization, convex sets and convex functions
- Lagrange duality and KKT optimality condition
- LP, QP, QCQP, SOCP, SDP, GP, SOS
- Gradient method, Newton’s method, interior point method
- Distributed algorithms and decomposition methods
- Nonconvex optimization and relaxations
- Robust LP and QP
- Network flow problems
- NUM: wired, wireless, extensions
Topics Covered: Applications

- **Information theory problems**: channel capacity and rate distortion for discrete memoryless models
- **Detection and estimation problems**: MAP and ML detector, covariance estimation, distribution estimation, multi-user detection, generalized Chebyshev bounds
- **Physical layer signal processing**: DSL spectrum management, wireless MIMO beamforming design, multiple-access linear transceiver design
- **Wireless networks**: resource allocation, power control, joint power and rate allocation
- **Optical networks**: routing and wavelength assignment
- **Network algorithms and protocols**: multi-commodity flow problems, max flow, shortest path routing, network rate allocation, TCP congestion control, IP routing
- **Layering as optimization decomposition**
- **Eigenvalue and norm optimization problems**
Some of the Topics Not Covered

- Circuit switching and optimal routing
- Packet switch architecture and algorithms
- Multi-user information theory problems
- More digital signal processing like blind equalization
- Large deviations theory and queuing theory applications
- System stability and control theoretic improvement of TCP
- Inverse optimization
- Self concordance analysis of interior point method’s complexity
- Ellipsoid method, cutting plane method, simplex method ...
Final Projects: Timeline

April 3 - May 10:

- Talk to me as often as needed
- Start early

April 22: Interim project report due

May 6: Project Presentation Day J323

- 2-2:15pm: Ice cream party
- 2:15-5pm: Student presentations

May 10: Final project report due

Many student continue to extend the project
**Final Projects: Topics**

**Pioter Drubetskoy**: SOS method and signomial programming

**Prashanth Hande**: Wireless NUM for uplink power control

**Jiayue He**: Internet routing optimization

**Shannon Hughes**: SOS method for image processing problems

**Jiaping Liu**: Stochastic NUM or detection optimization

**Hithesh Nama**: Wireless network lifetime maximization

**Chandru Raman**: Contention regulation in wireless MAC

**Chee Wei Tan**: DSL joint scheduling and spectrum management

**Dyana Tanasy**: Convex relaxations for telescope image processing
The Optimization Mentality

- What can be varied and what cannot?
- What’re the objectives?
- What’re the constraints?
- What’s the dual?

Build models, analyze models, prove theorems, compute solutions, design systems, verify with data, refine models ...

Warnings:
- Remember the restrictive assumptions for application needs
- Understand the limitations: discrete, nonconvex, complexity ... (still a lot can be done for these difficult issues)
Optimization of Communication Systems

Three ways to apply optimization theory to communication systems:

- Formulate the problem as an optimization problem
- Interpret a given solution as an optimizer/algorithm for an optimization problem
- Extend the underlying theory by optimization theoretic techniques

A remarkably powerful, versatile, widely applicable and not yet fully recognized framework

Applications in communication systems also stimulate new developments in optimization theory and algorithms
Optimization of Communication Systems
This Is The Beginning

Quote from Thomas Kailath (who may have quoted from others):

A course is not about what you’ve covered, but what you’ve uncovered
To Everyone In ELE539A
You’ve Been Amazing