ELE539A: Optimization of Communication Systems
Lecture 16: NUM Extension and Layering as Optimization Decomposition

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Lecture Outline

- Heterogeneous congestion control protocol
- TCP/IP optimization
- Two samples in active current research. Many open problems in this area

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NUM Extensions

Utility function:
- Nonconcave (Lecture 22)
- Coupled
- Not a function of rate
- Nonsmooth (Lecture 18)

Constraint:
- Nonlinear constraints (Lecture 19)
- Integer constraints

Pricing:
- Per-user differential pricing (This lecture)
Heterogeneous Congestion Control Protocol

$j$: index users. $l$: index links

$p$: link price. $x$: source rate

$g$: link algorithm. $f$: source algorithm

$m^j_l$: price mapping function

\[
\dot{p}_l = g_l \left( \sum_{j: l \in L(j)} x_j(t), p_l(t) \right)
\]

\[
\dot{x}_j = f_j \left( x_j(t), \sum_{l \in L(j)} m^j_l(p_l(t)) \right)
\]

Properties much more complicated. Focus on equilibrium
Notation

\(N^j\) sources using protocol \(j\), indexed by \((j, i)\). Total number of sources is \(N := \sum_j N^j\)

\(L \times N^j\) 0–1 routing matrix \(R^j\) for type \(j\) sources: \(R^j_{li} = 1\) iff source \((j, i)\) uses link \(l\)

Overall routing matrix: \(R = \begin{bmatrix} R^1 & R^2 & \cdots & R^J \end{bmatrix}\)

Effective price and intrinsic price: \(p^j_l = m^j_l(p_l)\)

Path effective price: \(q^j_i = \sum_l R^j_{li} p^j_l = \sum_l R^j_{li} m^j_l(p_l)\)

Source rate: \(x^j_i(q^j_i) = \left[\left(U^j_i\right)^{-1} (q^j_i)\right]^+\)

Link load: \(y^j(p) = R^j x^j(p), y(p) = Rx(p)\)
Network Equilibrium

Given a network \((c, m, R, U)\), an equilibrium \(p\) is such that

\[
P(y(p) - c) = 0, \quad y(p) \leq c, \quad p \geq 0
\]

where \(P\) is \(\text{diag}(p)\)

Consider a reduced system consisting of only active constraint links

\[
\hat{J}(\hat{p}) = \frac{\partial \hat{y}(\hat{p})}{\partial \hat{p}}
\]

\[
= \sum_j \hat{R}^j \frac{\partial x^j(\hat{p})}{\partial \hat{q}^j} \left( \hat{R}^j \right)^T \frac{\partial \hat{m}^j(\hat{p})}{\partial \hat{p}}
\]

\[
\frac{\partial x^j}{\partial \hat{q}^j} = \text{diag} \left( \left( \frac{\partial^2 U_i^j}{\partial (x_i^j)^2} \right)^{-1} \right)
\]

\[
\frac{\partial \hat{m}^j}{\partial \hat{p}} = \text{diag} \left( \frac{\partial \hat{m}_l^j}{\partial \hat{p}_l} \right)
\]
Existence of Equilibrium

Assumption 1. Utility functions $U^j_i$ are increasing, strictly concave, and twice continuously differentiable. Price mapping functions $m^j_l$ are continuously differentiable and strictly increasing with $m^j_l(0) = 0$

Assumption 2. For any $\epsilon > 0$, there exists a number $p_{\text{max}}$ such that if $p_l > p_{\text{max}}$ for link $l$, then $x^j_i(p) < \epsilon$ for all $(j, i)$ with $R^j_{li} = 1$

Theorem: There exists an equilibrium $p^*$ for any network $(c, m, R, U)$

But there can be many equilibria
Example With Infinite Number of Equilibria

Concave quadratic utilities

All of the following are equilibrium prices

\[ p_1 = p_3 = 1/8 + \epsilon \]
\[ p_2 = 1/4 - 2\epsilon \quad \text{where} \quad \epsilon \in [0, 1/24] \]
Regular Networks

An equilibrium $p^*$ induces active constraint set $\hat{L}$, i.e., $p^*$ is solution of $\hat{y}(\hat{p}) = \hat{c}$

$p^*$ is locally unique if $J(p)$ is nonsingular at $p^*$

A network $(c, m, R, U)$ is regular if all its equilibrium prices are locally unique

**Theorem**: For any routing $R$, utilities $U$, and price mapping functions $m$, the set of link capacities $c$ for which the network is not regular has measure zero in $\mathbb{R}^L$

Proof: use Sard’s Theorem

**Theorem**: Number of equilibria for a regular network is finite

**Implications**: almost all networks have finite number of equilibria for heterogeneous congestion control protocols
Index Theorem

Assumption 3: When link price is small enough, the total flow rate on the link exceeds link capacity

Now focus on regular networks satisfying Assumptions 1, 2, and 3

Define index $I(p)$ of an equilibrium price as 1 if $\det(J(p)) > 0$, and $-1$ if $\det(J(p)) < 0$

Theorem:

$$\sum_{p \in E} I(p) = (-1)^L$$

Proof: use Poincare-Hopf Index Theorem from differential topology
**Global Uniqueness**

**Theorem:** Number of equilibria is odd

Proof: both $I(p)$ and $(-1)^L$ are odd

An equilibrium is **locally stable** if $J(p)$ is stable, *i.e.*, all eigenvalues have negative real part

**Theorem:** If all equilibria are locally stable, then there is a **globally unique** equilibrium

Proof: locally stable equilibrium has index $(-1)^L$

**Implications:** a local property of algorithms (local stability) implies a global property of a network (global uniqueness)
Example Revisited

Same topology and routing as in last Example, but using $\alpha$ utilities:

$$\frac{x^{1-\alpha}}{1-\alpha}$$

<table>
<thead>
<tr>
<th>Equilibria($p_1, p_2, p_3$)</th>
<th>Eigenvalues</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.135, 0.23, 0.135)</td>
<td>$-0.21, -17.43, -26.73$</td>
<td>-1</td>
</tr>
<tr>
<td>(0.1419, 0.2060, 0.1419)</td>
<td>0.21, $-12.32, -22.40$</td>
<td>1</td>
</tr>
<tr>
<td>(0.165, 0.17, 0.165)</td>
<td>$-12.41, -1.67, -0.67$</td>
<td>-1</td>
</tr>
</tbody>
</table>
Special Cases for Global Uniqueness

**Corollary**: For linear and link-independent price mapping functions, there is a unique equilibrium

**Corollary**: For a network with at most two congested links, there is a unique equilibrium

**Corollary**: For a line network, there is a unique equilibrium

\[
\begin{align*}
&x^1 \\
&x^2 \\
&\cdots \\
&x^{L+1}
\end{align*}
\]
Layering as Optimization Decomposition

- Network
- Generalized NUM
- Layers
- Decomposed subproblems
- Interfaces
- Functions of primal or dual variables
- Layering
- Decompositions
Layering as Optimization Decomposition

- How to layer? How not to layer?
- Separation theorem?

- Both reverse engineering and forward engineering
- Systematic study of architectural principles and tradeoffs of layering
- Vertical and horizontal decomposition

How many different ways to decompose?
- Infinite!
- How you write down the problem constraints decomposition possibilities!
**Layering as Optimization Decomposition**

**Generalized NUM.** An example formulation

\[
\text{maximize } \sum_s U_s(x_s, P_e, s) + \sum_j V_j(w_j)
\]

subject to \( Rx \preceq c(w, P_e) \),
\( x \in C_1(P_e) \cap C_2(F) \),
\( R \in \mathcal{R}, \ F \in \mathcal{F}, \ w \in \mathcal{W} \)

Optimization variables: \( x, w, P_e, R, F \)

- **Application layer.** Utility functions \( U_i \) and \( V_j \)
- **Transport layer.** End-to-end throughput \( x_s \)
- **Network layer.** Routing matrix can be designed by varying \( R \)
- **Link layer.** Contention matrix \( F \) can be designed
- **Physical layer.** Logical link capacities \( c \) as functions of decoding error probabilities \( P_e \)
Samples

- HTTP/TCP (Ongoing work)
- **TCP/IP shortest path routing** (This lecture)
- TCP/IP other routing models (Ongoing work)
- TCP/MAC (Ongoing work)
- **TCP/PHY** (Lecture 19)
- Multicommodity flow routing/PHY (Lecture 19)
- MAC/PHY (Ongoing work)

- HTTP/TCP/IP/MAC/PHY (Ongoing work)
**TCP/IP Interaction**

**Assumptions:**
- TCP: dual-based congestion control
- IP: single, dynamic, shortest path routing
- TCP timescale much shorter than IP timescale

**Goals:**
- Utility attained at equilibrium
- Stability of interactions
- Implications to routing and link capacity provisioning
Notation

Path topology constant: $K^i$ acyclic paths for source $i$ represented by a $L \times K^i$ 0-1 matrix $H^i$:

$$H^i_{lj} = \begin{cases} 1, & \text{if path } j \text{ of source } i \text{ uses link } l \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{H}^i$: set of all columns of $H^i$ (all the available paths to $i$ under single-path routing)

$L \times K = \sum_i K^i$ overall topology matrix: $H = [H^1 \ldots H^N]$

Path selection variable: $w^i$: $K^i \times 1$ vector where the $j$th entry represents the fraction of $i$'s flow on its $j$th path. $w^i_j \geq 0 \ \forall j$, and $1^T w^i = 1$

Collect vectors $w^i$, $i = 1, \ldots, N$, into $K \times N$ block-diagonal matrix $W$

Set of single path routing:

$\mathcal{W}_s = \{W | W = \text{diag}(w^1, \ldots, w^N) \in \{0, 1\}^{K \times N}, 1^T w^i = 1 \}$

Set of multipath routing:
\( \mathcal{W}_m = \{ W | W = \text{diag}(w^1, \ldots, w^N) \in [0, 1]^{K \times N}, 1^T w^i = 1 \} \)

**Routing:** \( L \times N \) routing matrix \( R = HW \)

Set of all single-path routing matrices: \( \mathcal{R}_s = \{ R | R = HW, W \in \mathcal{W}_s \} \)

Set of all multipath routing matrices: \( \mathcal{R}_m = \{ R | R = HW, W \in \mathcal{W}_m \} \)

Single-path routing matrix in \( \mathcal{R}_s \) is an 0-1 matrix:

\[
R_{li} = \begin{cases} 
1, & \text{if link } l \text{ is in a path of source } i \\
0, & \text{otherwise.}
\end{cases}
\]

Multipath routing matrix in \( \mathcal{R}_m \):

\[
R_{li} \begin{cases} 
> 0, & \text{if link } l \text{ is in a path of source } i \\
= 0, & \text{otherwise.}
\end{cases}
\]

Path of source \( i \): \( r^i = [R_{1i} \ldots R_{Li}]^T \), the \( i \)th column of \( R \)
**TCP/IP Equilibrium**

Link cost at time $t$: $d_l(t) = a p_l(t) + b \tau_l$

Static component: $\tau_l$. Dynamic component: $p_l(t)$

TCP/IP equilibrium $R^*, x^*, p^*$ satisfies

$$r^i(t + 1) = \arg \min_{r^i \in \mathcal{H}_i} \sum_l (a p_l(t) + b \tau_l) r^i_l, \quad \text{for all } i$$

$$\sum_l R_{li}(t) p_l(t) = U'_i(x_i(t)) \quad \text{for all } i$$

$$\sum_i R_{li}(t) x_i(t) \begin{cases} \leq c_l & \text{if } p_l(t) \geq 0 \\ = c_l & \text{if } p_l(t) > 0 \end{cases} \quad \text{for all } l$$

$$x(t), p(t) \geq 0$$
Properties of Equilibrium

Case 1: $a = 0$ and $b > 0$ Purely static routing
- Trivial: Equilibrium always exist
- Open: May not solve any joint optimization problem

Case 2: $a > 0$ and $b = 0$ Purely dynamic routing
- Joint NUM solved by TCP/IP equilibrium iff no duality gap
- No ‘cost of not splitting’ iff no duality gap
- Joint NUM is NP-hard

Case 3: $a > 0$ and $b > 0$ General case
- Open: Not even sure about existence of equilibrium
Joint NUM Problem

Primal problem of NUM over $R$ and $x$:

$$\max_{R \in \mathcal{R}_s} \max_{x \geq 0} \sum_i U_i(x_i) \quad \text{s. t.} \quad Rx \leq c$$

Dual problem

$$\min_{p \geq 0} \sum_i \max_{x_i \geq 0} \left( U_i(x_i) - x_i \min_{r^i \in \mathcal{H}^i} \sum_l R_{li} p_l \right) + \sum_l c_l p_l$$

$r^i$ is the $i$th column of $R$ with $r^i_l = R_{li}$

Is TCP/IP (with purely dynamic routing) solving the above problems?

Yes, if TCP/IP equilibrium exists
Characterization

Theorem: TCP/IP equilibrium exists iff no duality gap, in which case it solves joint NUM

\[
V_{sp} = \max_{R \in \mathcal{R}_s, x \geq 0} \min_{p \geq 0} L(R, x, p)
\]

\[
V_{sd} = \min_{p \geq 0} \max_{R \in \mathcal{R}_s, x \geq 0} L(R, x, p)
\]

\[
V_{mp} = \max_{R \in \mathcal{R}_m, x \geq 0} \min_{p \geq 0} L(R, x, p)
\]

\[
V_{md} = \min_{p \geq 0} \max_{R \in \mathcal{R}_m, x \geq 0} L(R, x, p)
\]

Theorem: \( V_{sp} \leq V_{sd} = V_{md} = V_{mp} \)

Duality gap comes from single-path integer constraint
Special Case: Ring Network

Equilibrium always exists. But there is a tradeoff between
- Network utility achieved
- Stability of routing
Utility Stability Tradeoff

Continuous ring network model: routing is represented by a scalar

**Theorem:** Purely dynamic routing maximizes utility but routing oscillates between 0 and \( N \) (except when starting from equilibrium)

Needs to set \( b = 1 \) and investigate effect of \( a \) on stability

**Theorem:** For sufficiently small \( a \), routing is stable. For sufficiently large \( a \), routing oscillates. Utility attained converges to optimal utility as \( a \to \infty \)
Lecture Summary

• Equilibrium of heterogeneous congestion control protocol
• TCP/IP cross layer interaction
