ELE539A: Optimization of Communication Systems
Lecture 17: Optimal Transceiver Design for Multi-Access Communications

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Lecture Outline

- Channel model
- MMSE transceiver design
- SDP formulation
- SOCP formulation

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Two-User Multi-Access Communication Channel

Mathematical model: \( x = H_1 F_1 s_1 + H_2 F_2 s_2 + \rho n, \quad \rho > 0. \)
Linear detection: \( s_i = \text{sign} (G_i x), \quad i = 1, 2. \)

Given the channel matrices, \( H_1, H_2, \) design transceivers \( F_1, F_2, G_1, G_2. \)
Mean Square Error

- Let $e_i$ denote the error vector (before making the hard decision) for user $i$, $i = 1, 2$. Then

$$e_1 = G_1 x - s_1 = G_1 (H_1 F_1 s_1 + H_2 F_2 s_2 + \rho n) - s_1$$

$$= (G_1 H_1 F_1 - I) s_1 + G_1 H_2 F_2 s_2 + \rho G_1 n.$$  

- This further implies

$$E(e_1 e_1^\dagger) = (G_1 H_1 F_1 - I) (G_1 H_1 F_1 - I)^\dagger + (G_1 H_2 F_2) (G_1 H_2 F_2)^\dagger + \rho^2 G_1 G_1^\dagger \quad (1)$$

- Similarly, we have

$$E(e_2 e_2^\dagger) = (G_2 H_2 F_2 - I) (G_2 H_2 F_2 - I)^\dagger + (G_2 H_1 F_1) (G_2 H_1 F_1)^\dagger + \rho^2 G_2 G_2^\dagger.$$
Formulation: MMSE Equalizer Case

- Goal: design a set of transmitting matrix filters $\mathbf{F}_i$ and a set of matrix equalizers $\mathbf{G}_i$ to minimize total mean squared error:

$$\text{MSE} = \text{tr}(E(e_1 e_1^\dagger)) + \text{tr}(E(e_2 e_2^\dagger))$$

is minimized

- Power constraints on the transmitting matrix filters:

$$\text{tr}(\mathbf{F}_1 \mathbf{F}_1^\dagger) \leq p_1, \quad \text{tr}(\mathbf{F}_2 \mathbf{F}_2^\dagger) \leq p_2$$

- We first eliminate the variables $\mathbf{G}_1, \mathbf{G}_2$: the MMSE equalizers
Formulation: MMSE Equalizer Case

By minimizing $E(e_1e_1^\dagger)$ with respect to $G_1$, we obtain the following MMSE equalizer for user 1: $G_1 = F_1^\dagger H_1^\dagger W$, where

$$W = \left( H_1 F_1 F_1^\dagger H_1^\dagger + H_2 F_2 F_2^\dagger H_2^\dagger + \rho^2 I \right)^{-1}.$$ 

Substituting this into $E(e_1e_1^\dagger)$ gives:

$$E(e_1e_1^\dagger) = -F_1^\dagger H_1^\dagger W H_1 F_1 + I.$$ 

Similarly, the MMSE equalizer $G_2$ for user 2 is given by $G_2 = F_2^\dagger H_2^\dagger W$ and resulting minimized (with respect to $G_2$) mean square error for user 2 is given by:

$$E(e_2e_2^\dagger) = -F_2^\dagger H_2^\dagger W H_2 F_2 + I.$$
**Total MSE**

Substituting into the above expression gives rise to

\[
\text{MSE} = \text{tr}(E(e_1 e_1^\dagger)) + \text{tr}(E(e_2 e_2^\dagger))
\]

\[
= -\text{tr} \left( F_1^\dagger H_1^\dagger WH_1 F_1 \right) - \text{tr} \left( F_2^\dagger H_2^\dagger WH_2 F_2 \right) + 2n
\]

\[
= -\text{tr} \left( WH_1 F_1 F_1^\dagger H_1^\dagger \right) - \text{tr} \left( WH_2 F_2 F_2^\dagger H_2^\dagger \right) + 2n
\]

\[
= -\text{tr} \left( W(H_1 F_1 F_1^\dagger H_1^\dagger + H_2 F_2 F_2^\dagger H_2^\dagger) \right) + 2n
\]

\[
= \rho^2 \text{tr}(W) + n,
\]

where the last step follows from the definition of $W$. 


**Formulation: MMSE Equalizer Case**

Introduce matrix variables: \( U_1 = F_1 F_1^\dagger, \quad U_2 = F_2 F_2^\dagger \)

Then the MMSE transceiver design problem becomes

\[
\begin{align*}
\text{minimize}_{U_1, U_2} & \quad \text{tr} \left( (H_1 U_1 H_1^\dagger + H_2 U_2 H_2^\dagger + \rho^2 I)^{-1} \right) \\
\text{subject to} & \quad \text{tr}(U_1) \leq p_1, \quad \text{tr}(U_2) \leq p_2, \\
& \quad U_1 \succeq 0, \quad U_2 \succeq 0.
\end{align*}
\]

Reformulate using the auxiliary matrix variable \( W \):

\[
\begin{align*}
\text{minimize}_{W, U_1, U_2} & \quad \text{tr} \left( W \right) \\
\text{subject to} & \quad \text{tr}(U_1) \leq p_1, \quad \text{tr}(U_2) \leq p_2, \\
& \quad W \succeq (H_1 U_1 H_1^\dagger + H_2 U_2 H_2^\dagger + \rho^2 I)^{-1} \\
& \quad U_1 \succeq 0, \quad U_2 \succeq 0.
\end{align*}
\]
The constraint \( W \succeq (H_1 U_1 H_1^\dagger + H_2 U_2 H_2^\dagger + \rho^2 I)^{-1} \) is equivalent to LMI:

\[
\begin{bmatrix}
W & I \\
I & H_1 U_1 H_1^\dagger + H_2 U_2 H_2^\dagger + \rho^2 I
\end{bmatrix} \succeq 0.
\] (3)

We obtain an SDP formulation:

\[
\begin{array}{l}
\text{minimize}_{W, U_1, U_2} \quad \text{tr} (W) \\
\text{subject to} \quad \text{tr}(U_1) \leq p_1, \quad \text{tr}(U_2) \leq p_2, \\
\quad W \text{ satisfies (3)}, \\
\quad U_1 \succeq 0, \quad U_2 \succeq 0.
\end{array}
\]

Interior point method with arithmetic complexity \( O(n^{6.5} \log(1/\epsilon)) \), \( \epsilon > 0 \) is the solution accuracy.
OFDM: Diagonal Designs are Optimal

If $H_1$ and $H_2$ are diagonal, as in the OFDM systems, then the optimal transmitters are also diagonal.

Implication

The MMSE transceivers for an multi-user OFDM system can be implemented by optimally setting the data rates and allocating power to each subcarrier for all the users.
Linearly Precoded/Power Loaded OFDM

General Linearly Precoded OFDM System

Power-Loaded OFDM System
From SDP to SOCP Formulation

- Restricting to diagonal designs, the SDP becomes SOCP:

$$\min_{\mathbf{w}, \mathbf{u}_1, \mathbf{u}_2} \sum_{i=1}^{n} w_i$$

subject to

$$\sum_{i=1}^{n} u_1(i) \leq p_1, \quad \sum_{i=1}^{n} u_2(i) \leq p_2,$$

$$w_i \left( |h_1(i)|^2 u_1(i) + |h_2(i)|^2 u_2(i) + \rho^2 \right) \geq 1,$$

$$u_1(i) \geq 0, \quad u_2(i) \geq 0, \quad i = 1, 2, \ldots, n$$

- There exist highly efficient (general purpose) interior point methods to solve the above second order cone program.

- Arithmetic complexity $O(n^{3.5} \log(1/\epsilon))$, $\epsilon > 0$ is the accuracy.
Properties of Optimal MMSE Transceiver

Let $\mathbf{u}_1^* \geq 0, \mathbf{u}_2^* \geq 0$ be the optimal transceivers. Define:

\[
\begin{align*}
I_1 &= \{i \mid \mathbf{u}_1^*(i) > 0, \mathbf{u}_2^*(i) = 0\}, \\
I_2 &= \{i \mid \mathbf{u}_1^*(i) = 0, \mathbf{u}_2^*(i) > 0\}, \\
I_s &= \{i \mid \mathbf{u}_1^*(i) > 0, \mathbf{u}_2^*(i) > 0\}, \\
I_u &= \{i \mid \mathbf{u}_1^*(i) = 0, \mathbf{u}_2^*(i) = 0\}.
\end{align*}
\]

$I_1, I_2$: subcarriers allocated to user 1 and user 2; $I_s$ and $I_u$: subcarriers shared and unused;
data rates: $(|I_1| + |I_s|)/n, (|I_2| + |I_s|)/n$

- For each $i \in I_1$ and $j \in I_2$, we have $\frac{|h_1(i)|^2}{|h_2(i)|^2} \geq \frac{|h_1(j)|^2}{|h_2(j)|^2}$.

- For all $i, j \in I_s$, we have $\frac{|h_1(i)|^2}{|h_2(i)|^2} = \frac{|h_1(j)|^2}{|h_2(j)|^2}$.

- For any $i \in I_u$ and any $j \in I_1 \cup I_s$, we have $|h_1(i)|^2 < |h_1(j)|^2$.
  Similarly, for any $i \in I_u$ and any $j \in I_2 \cup I_s$, we have $|h_2(i)|^2 < |h_2(j)|^2$. 
Intuitive Interpretation

• $x = H_1 F_1 s_1 + H_2 F_2 s_2 + \rho n$, with $H_i$, $F_i$ diagonal; 
  $x(i) = h_1(i) f_1(i) s_1(i) + h_2(i) f_2(i) s_2(i) + \rho^2 n(i)$.  

• In a fading environment, the path gains $|h_1(i)|^2$, $|h_2(i)|^2$ are random, 
  ⇒ the probability of having two equal path gains is zero.  
  ⇒ $I_s$ is singleton: *at most one subcarrier should be shared by the two users.*  

• The remaining subcarriers are allocated to the two users according to 
  the path gain ratios: subcarrier $i$ to user 1 and subcarrier $j$ to user 2 only if  
  $$\frac{|h_1(i)|^2}{|h_2(i)|^2} \geq \frac{|h_1(j)|^2}{|h_2(j)|^2}.$$  

• The subcarriers in $I_u$ have small path gains for both users (i.e., both 
  $|h_1(i)|^2$ and $|h_2(i)|^2$ are small), and they should not be used by either 
  user, i.e., they are useless subcarriers.
A Strongly Polynomial Time Algorithm

- The properties of optimal MMSE transceivers can be used to design a combinatorial algorithm.
- Assume
  \[
  \frac{|h_1(1)|^2}{|h_2(1)|^2} > \frac{|h_1(2)|^2}{|h_2(2)|^2} > \cdots > \frac{|h_1(n-1)|^2}{|h_2(n-1)|^2} > \frac{|h_1(n)|^2}{|h_2(n)|^2}.
  \]
- Then \( I_1 \subseteq \{1, \ldots, i\} \) and \( I_2 \subseteq \{i, \ldots, n\} \) for some \( i \).
- Leads to an \( O(n^3) \) strongly polynomial time (combinatorial) algorithm (vs. \( O(n^{3.5} \log 1/\epsilon) \) interior point algorithm for SOC).
Practical Implications

Subcarrier Allocation and Power Loading

Transmitters Channel Matrices Receivers

\[ s_1 \rightarrow F_1 \rightarrow h_{1(1)} \rightarrow h_{1(2)} \rightarrow h_{1(3)} \rightarrow h_{1(4)} \rightarrow G_1 \rightarrow s_1 \]

\[ s_2 \rightarrow F_2 \rightarrow h_{2(1)} \rightarrow h_{2(2)} \rightarrow h_{2(3)} \rightarrow h_{2(4)} \rightarrow G_2 \rightarrow s_2 \]

Noise \( n \):

\[ x = \text{received signal} \]
**General $m$-User Case**

Mathematical model:

$$x = H_1 F_1 s_1 + H_2 F_2 s_2 + \cdots + H_m F_m s_m + \rho n.$$ 

Let $G_i$ be the linear MMSE matrix equalizer at the $i$-th receiver. Then the total MSE is given by

$$\rho^2 tr \left( (H_1 F_1 F_1^\dagger H_1^\dagger + \cdots + H_i F_i F_i^\dagger H_i^\dagger + \rho^2 I)^{-1} \right) + (m-1)n.$$ 

Let $U_i = F_i F_i^\dagger$. Then the power constrained optimal MMSE transmitter design problem can be described as:

$$\text{minimize}_{U_1, \ldots, U_m} \quad \text{tr} \left( (H_1 U_1 H_1^\dagger + \cdots + H_m U_m H_m^\dagger + \rho^2 I)^{-1} \right)$$

subject to

$$\text{tr}(U_i) \leq p_i, \quad U_i \succeq 0, \quad i = 1, \ldots, m.$$
SDP/SOCP Formulation

minimize \( \text{tr}(W) \)
subject to \( \text{tr}(U_i) \leq p_i, \quad U_i \succeq 0, \quad i = 1, 2, \ldots, m, \)
\[
\begin{bmatrix}
W & I \\
I & H_1 U_1 H_1^* + \cdots + H_m U_m H_m^* + \rho^2 I
\end{bmatrix} \succeq 0.
\]

SOCP formulation

minimize_{w,u_1,\ldots,u_m} \sum_{i=1}^n w_i 
subject to \sum_{i=1}^n u_j(i) \leq p_j, \quad j = 1, 2, \ldots, m, \quad w_i \left( |h_1(i)|^2 u_1(i) + \cdots + |h_m(i)|^2 u_m(i) + \rho^2 \right) \geq 1, \quad u_j(i) \geq 0, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m.$$
Simulation Scenario

Uplink with 16 active users and 160 available subcarriers

Three schemes:

- **AMOUR**— No channel knowledge; Each user uses 10 subcarriers, spreads 8 bits over these carriers using a DFT-type spreading.

- **Individually MMSE power-loaded OFDM**— Same subcarrier allocation as AMOUR. Each user sends 1 bit per subcarrier, i.e. 10 bits per block; knows its allocated channels and does MMSE power loading for these bits.

- **Multi-user MMSE power loaded OFDM**— Using the SOCP formulation. In this case the subcarrier allocations and the number of bits per block vary from block to block, but the average number of bits per block remains 10.
Simulation Results

16-user OFDM in a length 3 Rayleigh channel, 160 subcarriers, Coding gain smaller here

Multi-usr MSE power loading via SOCP, 10 bits/blk on ave
AMOUR, no channel knowledge, 8 bits/blk
Single-usr MSE power loaded OFDM with AMOUR subcarrier alloc, 10 bits/blk
Lecture Summary

• Presented various SDP and SOCP formulations and algorithms for the optimal transceiver design problems

• Results provide valuable guidelines and insights for the practical system design

• Possible extensions: other receiver structures, multiuser downlink case, QoS