Introduction to Fourier Transforms

- Fourier transform as a limit of the Fourier series
- Inverse Fourier transform: The Fourier integral theorem
- Example: the rect and sinc functions
- Cosine and Sine Transforms
- Symmetry properties
- Periodic signals and $\delta$ functions
Fourier Series

Suppose \( x(t) \) is not periodic. We can compute the Fourier series as if \( x \) was periodic with period \( T \) by using the values of \( x(t) \) on the interval \( t \in [-T/2, T/2) \).

\[
a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi k f_0 t} \, dt,
\]

\[
x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t},
\]

where \( f_0 = 1/T \).

The two signals \( x \) and \( x_T \) will match on the interval \( [-T/2, T/2) \) but \( \tilde{x}(t) \) will be periodic.

What happens if we let \( T \) increase?

Rect Example

For example, assume \( x(t) = \text{rect}(t) \), and that we are computing the Fourier series over an interval \( T \),

\[
f(t) = \text{rect}(t)
\]

The fundamental period for the Fourier series in \( T \), and the fundamental frequency is \( f_0 = 1/T \).

The Fourier series coefficients are

\[
a_k = \frac{1}{T} \text{sinc} (kf_0)
\]

where \( \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \).
Rect Example Continued

Take a look at the Fourier series coefficients of the rect function (previous slide). We find them by simply evaluating \( \frac{1}{T} \text{sinc}(f) \) at the points \( f = kf_0 \).

More densely sampled, same sinc() envelope, decreased amplitude.
Fourier Transforms

Given a continuous time signal \( x(t) \), define its Fourier transform as the function of a real \( f \):

\[
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt
\]

This is similar to the expression for the Fourier series coefficients.

Note: Usually \( X(f) \) is written as \( X(i2\pi f) \) or \( X(i\omega) \). This corresponds to the Laplace transform notation which we encountered when discussing transfer functions \( H(s) \).

We can interpret this as the result of expanding \( x(t) \) as a Fourier series in an interval \([-T/2, T/2)\), and then letting \( T \to \infty \).

The Fourier series for \( x(t) \) in the interval \([-T/2, T/2)\):

\[
x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kf_0 t}
\]

where

\[
a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi kf_0 t} \, dt.
\]

Define the truncated Fourier transform:

\[
X_T(f) = \int_{-T/2}^{T/2} x(t) e^{-j2\pi ft} \, dt
\]

so that

\[
a_k = \frac{1}{T} X_T(kf_0) = \frac{1}{T} X_T \left( \frac{k}{T} \right).
\]
The Fourier series is then
\[ x_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X_T(kf_0) e^{j2\pi kf_0 t} \]

The limit of the truncated Fourier transform is
\[ X(f) = \lim_{T \to \infty} X_T(f) \]

The Fourier series converges to a Riemann integral:
\[
\begin{align*}
x(t) &= \lim_{T \to \infty} x_T(t) \\
&= \lim_{T \to \infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} X_T \left( \frac{k}{T} \right) e^{j2\pi \frac{k}{T} t} \\
&= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df.
\end{align*}
\]

**Continuous-time Fourier Transform**

Which yields the *inversion formula* for the Fourier transform, the *Fourier integral theorem*:

\[
\begin{align*}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt, \\
x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df.
\end{align*}
\]
Comments:

- There are usually technical conditions which must be satisfied for the integrals to converge – forms of smoothness or Dirichlet conditions.
- The intuition is that Fourier transforms can be viewed as a limit of Fourier series as the period grows to infinity, and the sum becomes an integral.
- \[ \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df \] is called the inverse Fourier transform of \( X(f) \). Notice that it is identical to the Fourier transform except for the sign in the exponent of the complex exponential.
- If the inverse Fourier transform is integrated with respect to \( \omega \) rather than \( f \), then a scaling factor of \( 1/(2\pi) \) is needed.

Cosine and Sine Transforms

Assume \( x(t) \) is a possibly complex signal.

\[
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \\
= \int_{-\infty}^{\infty} x(t) (\cos(2\pi ft) - j \sin(2\pi ft)) \, dt \\
= \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt.
\]
Fourier Transform Notation

For convenience, we will write the Fourier transform of a signal $x(t)$ as

$$\mathcal{F} [x(t)] = X(f)$$

and the inverse Fourier transform of $X(f)$ as

$$\mathcal{F}^{-1} [X(f)] = x(t).$$

Note that

$$\mathcal{F}^{-1} [\mathcal{F} [x(t)]] = x(t)$$

and at points of continuity of $x(t)$.

Duality

Notice that the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are almost the same.

*Duality Theorem:* If $x(t) \Leftrightarrow X(f)$, then $X(t) \Leftrightarrow x(-f)$.

In other words, $\mathcal{F} [\mathcal{F} [x(t)]] = x(-t)$. 
Example of Duality

- Since \( \text{rect}(t) \Leftrightarrow \text{sinc}(f) \) then

\[
\text{sinc}(t) \Leftrightarrow \text{rect}(-f) = \text{rect}(f)
\]

(Notice that if the function is even then duality is very simple)

Generalized Fourier Transforms: \( \delta \) Functions

A unit impulse \( \delta(t) \) is not a signal in the usual sense (it is a generalized function or distribution). However, if we proceed using the sifting property, we get a result that makes sense:

\[
\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft} \, dt = 1
\]

so

\( \delta(t) \Leftrightarrow 1 \)

This is a generalized Fourier transform. It behaves in most ways like an ordinary FT.
Shifted $\delta$

A shifted delta has the Fourier transform

$$
\mathcal{F}[\delta(t - t_0)] = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi ft} \, dt
= e^{-j2\pi f t_0}
$$

so we have the transform pair

$$
\delta(t - t_0) \Leftrightarrow e^{-j2\pi f t_0}
$$

Constant

Next we would like to find the Fourier transform of a constant signal $x(t) = 1$. However, direct evaluation doesn’t work:

$$
\mathcal{F}[1] = \int_{-\infty}^{\infty} e^{-j2\pi ft} \, dt
= e^{-j2\pi ft} \bigg|_{-\infty}^{\infty}
= e^{-j2\pi f t} \bigg|_{-\infty}^{\infty}
$$

and this doesn’t converge to any obvious value for a particular $f$.

We instead use duality to guess that the answer is a $\delta$ function, which we can easily verify.

$$
\mathcal{F}^{-1} [\delta(f)] = \int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} \, df
= 1.
$$
So we have the transform pair

\[ 1 \Leftrightarrow \delta(f) \]

This also does what we expect – a constant signal in time corresponds to an impulse a zero frequency.

**Sinusoidal Signals**

If the \( \delta \) function is shifted in frequency,

\[
\mathcal{F}^{-1} \left[ \delta(f - f_0) \right] = \int_{-\infty}^{\infty} \delta(f - f_0)e^{j2\pi ft} df = e^{j2\pi f_0 t}
\]

so

\[ e^{j2\pi f_0 t} \Leftrightarrow \delta(f - f_0) \]
Cosine

With Euler’s relations we can find the Fourier transforms of sines and cosines

\[ \mathcal{F} \left[ \cos(2\pi f_0 t) \right] = \mathcal{F} \left[ \frac{1}{2} (e^{i2\pi f_0 t} + e^{-j2\pi f_0 t}) \right] \]
\[ = \frac{1}{2} (\mathcal{F} [e^{i2\pi f_0 t}] + \mathcal{F} [e^{-j2\pi f_0 t}]) \]
\[ = \frac{1}{2} (\delta(f - f_0) + \delta(f + f_0)). \]

so

\[ \cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2} (\delta(f - f_0) + \delta(f + f_0)). \]

Sine

Similarly, since \( \sin(f_0 t) = \frac{1}{2j} (e^{i2\pi f_0 t} - e^{-j2\pi f_0 t}) \) we can show that

\[ \sin(f_0 t) \Leftrightarrow \frac{j}{2} (\delta(f + f_0) - \delta(f - f_0)). \]

The Fourier transform of a sine or cosine at a frequency \( f_0 \) only has energy exactly at \( \pm f_0 \), which is what we would expect.