Important Note: To get the best value out of this precept, come with your calculator or computer that can readily evaluate the formulas for $\mathbf{x}$ for the various cases below. Then in the precept you can work this out numerically not just for the numbers given, but for different values of $W_0$ and $\alpha$, any other asset returns or even asset types for which you may have or can guess the numbers, and so on, as time permits.

Throughout this question you have an initial wealth $W_0$, and are a mean-variance investor trying to maximize

$$E[W] - \frac{1}{2} \alpha V[W]$$

where $W$ denotes your random final wealth.

Suppose the menu of assets available to you includes a riskless asset paying a sure total or gross rate of return $R_0$, and several risky assets $i = 1, 2, \ldots n$ paying random total or gross rates of return $r_i$, with

$$E[r_i] = \mu_i, \quad V[r_i] = V_{ii}, \quad \text{Cov}[r_i, r_j] = V_{ij}.$$ 

Write the column vector of the mean rates of return as $\mu$ and the variance-covariance matrix as $\Omega$.

(a) Write $x_0$ for the amount (number of dollars, not fraction of $W_0$) that you invest in the riskless asset, and $x_i$ for the amount you invest in the risky asset $i$ for $i = 1, 2, \ldots n$. Write $\mathbf{x}$ for the column vector of your investments in the $n$ risky assets. Of course

$$x_0 = W_0 - \sum_{i=1}^{n} x_i.$$ 

Suppose there are no other constraints on your choices of the portfolio allocation, so short sales of risky assets, and borrowing at the riskless rate to invest in the risky assets, are both possible. In this case, show that your optimal investment choice is given by

$$\mathbf{x} = \frac{1}{\alpha} \Omega^{-1} (\mu - R_0 \mathbf{u}).$$ 

where $\mathbf{u}$ is a column “unit vector” all of whose components equal 1. You don’t have to use vector methods to do the calculus. (But you might find it convenient to do so if you have done MAT 203–204.) Just gather together your results into the vector notation at the end.

(b) Now suppose there are four assets, Large capitalization U.S. stocks (Large-Caps for short), Small capitalization U.S. stocks (Small-Caps), International Stocks, and U.S. Treasury Bonds, numbered assets 1 to 4 in this order. The riskless asset is cash, paying zero
interest (so \( R_0 = 1.000 \)). On the risky assets, we have the following data for the period Oct. 2002 – Oct. 2009:

\[
\mu = \begin{bmatrix}
1.0242994 \\
1.0617073 \\
1.0868341 \\
1.072539
\end{bmatrix}
\]

\[
\Omega = \begin{bmatrix}
0.034925757 & 0.039143933 & 0.045862227 & -0.005294927 \\
0.039143933 & 0.04784564 & 0.051731174 & -0.00647438 \\
0.045862227 & 0.051731174 & 0.062572873 & -0.007736565 \\
-0.005294927 & -0.00647438 & -0.007736565 & 0.003283272
\end{bmatrix}
\]

(For those of you who are interested in these things, the data were obtained from the site http://finance.yahoo.com, by getting quotes on historical data, adjusted for distributions and splits, on index funds that track asset classes:

US Large Cap Equities: i-shares S&P 500 Index Fund, IVV
US Small Cap Equities: i-shares S&P Small Cap 600 Index Fund, IJR
International Equities: i-shares MSCI EAFE Index Fund, EFA
Vanguard Long Term Treasury Bonds, VUSTX)

You have \( W_0 = 10,000 \) and \( \alpha = 1/1000 \).

Evaluate the optimal \( x \) of part (a) numerically. (You can do so using Excel, using its built-in “functions” for matrix and vector multiplication, or Mathematica or even your calculators.

(c) Next go back to the algebra, and suppose there is no riskless asset, and you must put all of your wealth into one of the \( n \) risky assets, so there is a constraint

\[
W_0 = \sum_{i=1}^n x_i.
\]

Show that the minimum-variance portfolio is given by

\[
x = W_0 \frac{1}{u'\Omega^{-1}u} \Omega^{-1}u,
\]

(d) Evaluate this numerically for the four assets and values given above.

(e) Finally, a harder question, to be done if time permits. Suppose there is no riskless asset, and you are a mean-variance person with the objective function given above. Show that your optimal investment choice is given by

\[
x = W_0 \frac{1}{u'\Omega^{-1}u} \Omega^{-1}u + \frac{1}{\alpha} \left[ \Omega^{-1} \mu - \frac{u'\Omega^{-1}\mu}{u'\Omega^{-1}u} \Omega^{-1}u \right],
\]

(f) Evaluate this numerically for the four assets and values given above.

(g) Do you think the numerical results in (d) and (f) are reasonable? If not, what changes in the specification might produce more reasonable results?
Part (a):

The expression for the random final wealth is

\[ W = x_0 R + \sum_{i=1}^{n} x_i r_i = W_0 R + \sum_{i=1}^{n} x_i (r_i - R_0). \]

Then

\[ E[W] = W_0 R + \sum_{i=1}^{n} x_i (\mu_i - R_0) \]

and

\[ V[W] = \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} x_i x_j. \]

Then the objective function expressed as a function of \( x \), say \( F(x) \), is

\[ F(x) = E[W] - \frac{1}{2} \alpha V[W] = \sum_{i=1}^{n} x_i (\mu_i - R_0) - \frac{1}{2} \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} x_i x_j \]

using vector-matrix notation, where \( u \) is the “unit vector” all of whose components equal 1.

To optimize this with respect to any one of the components of \( x \), say \( x_1 \), note that \( x_1 \) enters the last term in the first line on the right hand side in several places, namely

\[- \frac{1}{2} \alpha \left[ V_{11} (x_1)^2 + V_{12} x_1 x_2 + V_{13} x_1 x_3 + \ldots + V_{1n} x_1 x_n \right. \]
\[ + \left. V_{21} x_2 x_1 + V_{31} x_3 x_1 + \ldots + V_{n1} x_n x_1 \right] \]

Then

\[ \frac{\partial F}{\partial x_1} = \mu_1 - R_0 - \frac{1}{2} \alpha \left[ 2 V_{11} x_1 + V_{12} x_2 + V_{13} x_3 + \ldots + V_{1n} x_n \right. \]
\[ + \left. V_{21} x_2 + V_{31} x_3 + \ldots + V_{n1} x_n \right], \]

using the symmetry of the \( V_{ij} \)'s. Setting this equal to zero, and similarly collecting the conditions for all the other \( x_j \)'s, we have

\[ \mu_i - R_0 - \alpha \left[ V_{i1} x_1 + V_{i2} x_2 + V_{i3} x_3 + \ldots + V_{in} x_n \right] = 0, \]

or in vector-matrix notation,

\[ \mu - R_0 \mathbf{u} = \alpha \mathbf{\Omega} \mathbf{x}. \]
So the solution is
\[ x = \frac{1}{\alpha} \Omega^{-1} (\mu - R_0 u). \]

If you can do vector calculus directly, you can write the total derivative of the function \( F \) as
\[ \mu - R_0 u - \alpha \Omega x, \]
set this equal to the zero vector, and obtain the same solution.

Part (b):

Numerical solutions (using Mathematica calculations):
\[ x_0 = -29465.01, \quad x_1 = -53829.21, \quad x_2 = 11724.99, \quad x_3 = 36696.06, \quad x_4 = 44873.17. \]

Part (c):

Here we want to minimize
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} x_i x_j \]
subject to the constraint
\[ \sum_{i=1}^{n} x_i = W_0. \]

Setting up the Lagrangian and using the previous result for the derivatives, the first-order conditions are
\[ V_{i1} x_1 + V_{i2} x_2 + V_{i3} x_3 + \ldots + V_{in} x_n = \frac{1}{2} \lambda \]
for all \( i \), where \( \lambda \) is the Lagrange multiplier. That is,
\[ \Omega x = \frac{1}{2} \lambda u \]
or
\[ x = \frac{1}{2} \lambda \Omega^{-1} u. \]

The constraint can be written as \( u' x = W_0 \). So premultiply the equation just above by \( u \) to get
\[ W_0 = u' x = \frac{1}{2} \lambda u' \Omega^{-1} u. \]

Therefore
\[ \frac{1}{2} \lambda = \frac{W_0}{u' \Omega^{-1} u}, \]
and then
\[ x = \frac{W_0}{u' \Omega^{-1} u} \Omega^{-1} u. \]

Part (d):

Solution:
\[ x_1 = 633.06, \quad x_2 = 576.66, \quad x_3 = 380.14, \quad x_4 = 8410.13. \]
Part (e):

Here we want to maximize

\[
F(x) = E[W] - \frac{1}{2} \alpha V[W] = \sum_{i=1}^{n} x_i \mu_i - \frac{1}{2} \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} x_i x_j
\]

subject to the constraint

\[
\sum_{i=1}^{n} x_i = W_0.
\]

Setting up the Lagrangian, the first-order conditions are

\[
\mu_i - \alpha [ V_{i1} x_1 + V_{i2} x_2 + V_{i3} x_3 + \ldots + V_{in} x_n ] - \lambda = 0,
\]

for \( i = 1, 2, \ldots n \), or in vector-matrix notation,

\[
\mu - \alpha \Omega x - \lambda u = 0.
\]

That is,

\[
x = \frac{1}{\alpha} \Omega^{-1} [\mu - \lambda u].
\]

Then

\[
W_0 = u' x = \frac{1}{\alpha} u' \Omega^{-1} [\mu - \lambda u]
\]

This yields

\[
\lambda = \frac{u' \Omega^{-1} \mu - \alpha W_0}{u' \Omega^{-1} u}.
\]

Substituting this in the expression for \( x \) and simplifying, we get the given answer.

Part (f):

Solution:

\[
x_1 = -55694.52, \quad x_2 = 10025.85, \quad x_3 = 35575.97, \quad x_4 = 20092.70.
\]

Part (g):

This is for open discussion and experimentation in class.

But note one thing: Asset 4 has an expected return of a little over 7% with relatively little risk (standard deviation 5.7% per year). No wonder people want to be short in cash (asset 0) to invest a lot in Treasuries (Asset 4) in Part (b), and even short stocks (asset 1) in Part (f). The question is whether anyone will lend money at the riskless rate (here zero) to invest in Treasuries.