There are $S$ states of the world labeled $s = 1, 2, \ldots, S$, and $H$ traders labeled $h = 1, 2, \ldots, H$. Each trader is a price-taker. The probability of state $s$ occurring is $\pi_s$. There is one physical good, and the endowment of trader $h$ in state $s$ is denoted by $C_{0,h}^s$. There are complete markets for Arrow-Debreu securities (contingent claims); a security for state $s$ is a promise to deliver a unit of the physical good if state $s$ occurs, and nothing in any other state. Each trader is a price-taker in the markets for contingent claims. Denote the price of a state-$s$ Arrow-Debreu security by $P_s$. The price in the ex ante market for a promise to pay one unit of the physical good regardless of which state occurs is 1, so

$$\sum_{s=1}^{S} P_s = 1.$$ 

(a) Write down the budget constraint for trader $h$.

(b) Each trader has a utility-of-consequences function $V(C) = C^{1/2}$.

(i) Write down the expression for trader $h$’s expected utility.

(ii) Find expressions for trader $h$’s optimal choices $C_h^s$ of the contingent claims.

(iii) Write down the equilibrium conditions for contingent claim markets.

(iv) Hence show that in equilibrium,

$$P_s = \frac{\pi_s \left( \hat{C}_s \right)^{-1/2}}{\sum_t \pi_t \left( \hat{C}_t \right)^{-1/2}},$$

where the index of summation $t$ is used to denote states in the denominator, so as to distinguish it from the particular state $s$, and

$$\hat{C}_s = \frac{1}{H} \sum_{h=1}^{H} C_{0,h}^s.$$ 

is the average endowment of the good in state $s$. Obtain an economic explanation for this result.

(c) Now suppose that each trader has a linear-quadratic utility-of-consequences function $V(C) = aC - \frac{1}{2}bC^2$, where $a$ and $b$ are positive constants. Assume that the values of $a$ and $b$ are such that every trader’s $V$ function is increasing for all $C$ in the range that is relevant in this analysis. Following the same steps as in part (b) it can be shown (this will be supplied in the solutions to be posted) that in equilibrium,

$$P_s = \frac{\pi_s \left( a - b \hat{C}_s \right)}{\sum_t \pi_t \left[ a - b \hat{C}_t \right]},$$

where $t$ and $\hat{C}_s$ have the same interpretation as in part (b).

Do these pricing equations – the one you derived in (b) and the one stated in (c) – suggest any general themes about efficient risk-bearing?
(a) Denoting trader \( h \)'s consumption in state \( s \) by \( C^h_s \), the budget constraint is
\[
\sum_s P_s C^h_s \leq \sum_s P_s C^0,h_s \equiv W^{0,h}
\]
where I have introduced the abbreviation \( W^{0,h} \) for the right hand side of the constraint.

(b) (i) His expected utility is
\[
\sum_s \pi_s (C^h_s)^{1/2}
\]
(ii) Therefore the first-order conditions for the optimal choice are, for all states \( s \):
\[
\frac{1}{2} \pi_s (C^h_s)^{-1/2} = \lambda_h P_s
\]
or
\[
C^h_s = \left( \frac{\pi_s}{2 \lambda_h P_s} \right)^2 = \frac{1}{4 \lambda_h^2} (\pi_s)^2 (P_s)^{-2}
\]
Then
\[
P_s C^h_s = \frac{1}{4 \lambda_h^2} (\pi_s)^2 (P_s)^{-1}
\]
Adding over states,
\[
W^{0,h} = \frac{1}{4 \lambda_h^2} \sum_s (\pi_s)^2 (P_s)^{-1}
\]
Eliminating the \( \lambda_h \) term,
\[
C^h_s = \frac{(\pi_s)^2 (P_s)^{-2}}{\sum_t (\pi_t)^2 (P_t)^{-1}} W^{0,h} \tag{1}
\]
where I have used the index of summation \( t \) for states in the denominator to distinguish it from the particular state \( s \).

(iii) Adding over traders, the equilibrium condition is
\[
\sum_h C^h_s = \sum_h C^0,h_s \equiv C^0_s,
\]
using the abbreviation defined by the equivalence sign \( \equiv \).
Therefore
\[
C^0_s = \frac{(\pi_s)^2 (P_s)^{-2}}{\sum_t (\pi_t)^2 (P_t)^{-1}} \sum_h W^{0,h}
\]
(iv) Write the complicated denominator as \( D \), and \( W^0 = \sum_h W^{0,h} \), for brevity. Then
\[
(P_s)^2 = (\pi_s)^2 (C^0_s)^{-1} W^0 / D
\]

or

\[ P_s = \pi_s \left( C_s^0 \right)^{-1/2} \left( W^0 / D \right)^{1/2} \]

Summing over states,

\[ 1 = \sum_s \pi_s \left( C_s^0 \right)^{-1/2} \left( W^0 / D \right)^{1/2} \]

Therefore

\[ P_s = \frac{\pi_s \left( C_s^0 \right)^{-1/2}}{\sum_t \pi_t \left( C_t^0 \right)^{-1/2}} \]

Finally, using the definition

\[ \hat{C}_s = \frac{1}{H} \sum_{h=1}^{H} C_{s,h}^0 = \frac{1}{H} C_s^0, \]

this easily converts to

\[ P_s = \frac{\pi_s \left( \hat{C}_s \right)^{-1/2}}{\sum_t \pi_t \left( \hat{C}_t \right)^{-1/2}} \]

Observe carefully the sequence of logical steps in the argument. They follow the standard logic of microeconomics: [1] choice, [2] equilibrium. [1] First we must find each individual’s demand functions – quantity demanded on the left, and prices and endowments on the right. In deriving this, we have to solve the first-order conditions we get from the Lagrangian, and the budget constraint. If you failed to use the budget constraint, you must have missed something. [2] Having found the individual demand functions, we can sum them to get the aggregate demands. If you did not have this step, again you must have missed something. [3] Then we must have the equilibrium conditions: set the aggregate quantities demanded equal to the aggregate quantities supplied for each good. [4] Finally, solve the equilibrium conditions for the prices.

There is another way to think about this. Each individual’s demand functions for state-by-state consumption (1) are proportional to that individual’s aggregate value of state-by-state endowments \( W_{0,h} \), and the constant of proportionality is the same for all individuals. Therefore we can regard the whole group just like one individual who has the sum total of everyone’s endowments, or an average or representative individual who has the average endowment. Then the equilibrium prices must be such that the average individual is happy to go on holding his average endowment, and not wish to engage in any trade. This is exactly what Robert Lucas did in his famous paper on asset prices (Econometrica November 1978), which was part of the foundation of the modern approach to rational expectations in macroeconomics.

(c) (i) Investor \( h \)'s expected utility is

\[ \sum_s \pi_s \left[ a C_s^h - \frac{1}{2} b \left( C_s^h \right)^2 \right] \]

(ii) Therefore the first-order conditions for the optimal choice are

\[ \pi_s \left[ a - b C_s^h \right] = \lambda_h P_s \]
or

\[ C_s^h = \frac{a}{b} - \frac{\lambda_h}{b} \frac{P_s}{\pi_s} \]

Then

\[ P_s C_s^h = \frac{a}{b} P_s - \frac{\lambda_h}{b} \frac{(P_s)^2}{\pi_s} \]

Adding over states,

\[ W^{0,h} = \frac{a}{b} \sum_s P_s - \frac{\lambda_h}{b} \frac{\sum_s (P_s)^2}{\pi_s} \]

Note that \( \sum_s P_s = 1 \), and abbreviate \( \sum_s (P_s)^2/\pi_s = Z \). Then

\[ \lambda_h = \left( a - b W^{0,h} \right)/Z. \]

and then

\[ C_s^h = \frac{a}{b} - \frac{a - b W^{0,h}}{bZ} \frac{P_s}{\pi_s} \]

(iii) Adding over investors and using the equilibrium condition

\[ C_s^0 = \frac{a}{b} H - \frac{a H - b W^0}{bZ} \frac{P_s}{\pi_s} \]

(iv) Also note that \( \hat{C}_s = C_s^0/H. \) Then

\[ \frac{a - b W^0/H}{Z} P_s = \pi_s \left[ a - b \hat{C}_s \right] \]

Add over states again:

\[ \frac{a - b W^0/H}{Z} = \sum_s \pi_s \left[ a - b \hat{C}_s \right] \]

Hence

\[ P_s = \frac{\pi_s \left[ a - b \hat{C}_s \right]}{\sum_t \pi_t \left[ a - b \hat{C}_t \right]} \]

To derive a general principle, observe that in (b), \( (\hat{C}_s)^{-1/2} \) is the marginal utility of the average consumption (equals average endowment) in state s. In (c), \( a - b \hat{C}_s \) has the same interpretation. Hence the general idea is that prices of Arrow-Debreu securities will be proportional to the product of the probabilities of the respective states and the marginal utilities of consumption in them, i.e. the expected marginal valuations of wealth in those states.

Here is some further elaboration. With a general utility-of-consequences function \( V \), we have first-order conditions

\[ \pi_s V'(C_s^h) = \lambda_h P_s \]

for investor h’s optimal choice of final consumption of contingent claims in the different states. So the prices will be proportional to a suitable average of these marginal utilities,
the average being taken across investors. But the average of the marginal utilities does not
in general have to equal the marginal utility of average consumption. That requires some
special aggregation properties that happen to hold in these examples. If marginal utility is a
linear function of consumption as in part (c), the evidently the average of marginal utilities is
the marginal utility of the average. The case in (b) is somewhat more complicated, involving
aggregation when preferences are identical and homothetic.