The distribution was as follows:

<table>
<thead>
<tr>
<th></th>
<th>100</th>
<th>90-99</th>
<th>80-89</th>
<th>70-79</th>
<th>&lt; 70</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>11</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

**Question 1:**

(a) (5 points) $F_1$ being FOSD over $F_2$ is equivalent to

$$\int_a^b \phi(W) f_1(W) \, dW > \int_a^b \phi(W) f_2(W) \, dW,$$

for any increasing function $\phi$, and in particular $\phi(W) = W$ is an increasing function.

(b) (10 points) The converse of the statement in (a) is that if $F_1$ gives a higher mean than $F_2$, then $F_1$ is FOSD over $F_2$. To show that this is not true, we need to find an example where $F_1$ gives a higher mean than $F_2$ but $F_1$ is not FOSD over $F_2$, that is, $F_1(W)$ is not $< F_2(W)$ everywhere (except of course for equality at the end points, and perhaps also at some points in the interior). In other words, we need an example where $F_1$ gives a higher mean than $F_2$, but $F_1(W)$ and $F_2(W)$ intersect. This is easy: take any two CDFs that intersect; one is almost sure to have a higher mean; call it $F_1$. Here is just one example.

Suppose $F_1$ corresponds to $W$ taking on two values, 0 and 2, with probabilities 1/2 each, and $F_2$ is the degenerate distribution where $W = 0$ for sure. Then the means are $E_1(W) = 1 > 0.5 = E_2(W)$, but $F_1$ and $F_2$ intersect: for example in the range (0.1,0.5) we have $F_1(W) = 0.5 > 0 = F_2(W)$, while in (0.5,2) we have $F_1(W) = 0.5 < 1 = F_2(W)$.

**Question 2:**

(a) (6 points) The cumulative probabilities of the discrete outcomes are

\[
\begin{align*}
\Pr[W = 1] &= 0.3 \\
\Pr[W = 1 \text{ or } 2] &= 0.3 + 0.5 = 0.8 \\
\Pr[W = 1 \text{ or } 2 \text{ or } 3] &= 0.3 + 0.5 + 0.2 = 1.0
\end{align*}
\]

Therefore the continuous cumulative distribution function $F(W)$, being the probability that the realization of wealth is less than or equal to $W$, is given by

\[
F(W) = \begin{cases} 
0 & \text{if } W < 1 \\
0.3 & \text{if } 1 \leq W < 2 \\
0.8 & \text{if } 2 \leq W < 3 \\
1.0 & \text{if } 3 \leq W
\end{cases}
\]

Figure 1 shows this.

(b) (3 points) This is the point $A = (0.3, 0.2)$ marked in Figure 2.
Figure 1:

Figure 2:
(c) (8 points) The new cumulative distribution $B$ is similarly defined by

\[ F(W) = \begin{cases} 
0 & \text{if } W < 1 \\
 p_1 & \text{if } 1 \leq W < 2 \\
 p_1 + p_2 & \text{if } 2 \leq W < 3 \\
 p_1 + p_2 + p_3 = 1 & \text{if } 3 \leq W
\end{cases} \]

For $B$ to be FOSD over $A$, we need this to be a shift to the right or down of the old one. That is, we need

\[ p_1 < 0.3, \quad p_1 + p_2 < 0.8 \]

or in terms of the variables $p_1$ and $p_3$ shown in the probability triangle diagram:

\[ p_1 < 0.3, \quad 1 - p_3 < 0.8 \]

or

\[ p_1 < 0.3, \quad p_3 > 0.2 \]

Figure 2 shows this region shaded. This has lower $p_1$ and higher $p_3$ than at the initial point, which confirms the intuition that a rightward shift of the distribution is needed.

(d) (8 points) For SOSD we want the cumulative of the cumulative distribution function to be smaller, that is,

\[ p_1 \leq 0.3 \]

and

\[ p_1 + (p_1 + p_2) \leq 0.3 + (0.3 + 0.5), \quad \text{or} \quad p_1 + (1 - p_3) \leq 1.1 \quad \text{or} \quad p_3 \geq p_1 - 0.1 \]

To ensure equal means, we need the cumulatives of the cumulative distribution functions to be equal at the right hand end-point, so

\[ p_1 + (p_1 + p_2) + (p_1 + p_2 + p_3) = 0.3 + (0.3 + 0.5) + (0.3 + 0.5 + 0.2) \]

or

\[ p_1 + (1 - p_3) + 1 = 1.1 + 1 \quad \text{or} \quad p_3 = p_1 - 0.1 \]

Putting these together, the relevant conditions are

\[ p_1 \leq 0.3 \quad \text{and} \quad p_3 = p_1 - 0.1 \]

Figure 2 shows the values $(p_1, p_3)$ satisfying these as a straight line extending to the southwest from the initial point $(0.3, 0.2)$. This means lower $p_1$ and $p_3$ and higher $p_2$, indicating a shift of the distribution toward the center. Moreover, the reductions in $p_1$ and $p_3$ must be appropriately balanced for the shift to be mean-preserving, that is why we get only a one-dimensional set (the straight line) instead of the whole area to the south-west of the initial point.
Question 3:

(a) (5 points) The individual should be indifferent between getting $W_0 + P_s$ for sure, and the prospect of final wealth $W_0 + G$ with probability $p$ and $W_0 + B$ with probability $1 - p$. Therefore the equation is

$$u(W_0 + P_s) = p u(W_0 + G) + (1 - p) u(W_0 + B)$$

(b) (5 points) The individual should be indifferent between staying with $W_0$ for sure, and the prospect of final wealth $W_0 - P_b + G$ with probability $p$ and $W_0 - P_b + B$ with probability $1 - p$. Therefore the equation is

$$u(W_0) = p u(W_0 - P_b + G) + (1 - p) u(W_0 - P_b + B)$$

(c) (10 points) With the numbers given, the selling price is defined by

$$(10 + P_s)^{1/2} = 0.5 \times (20)^{1/2} + 0.5 \times (10)^{1/2} = 3.8172068$$

Therefore

$$10 + P_s = (3.8172068)^2 = 14.571,$$  \text{ or }  P_s = 4.571

The buying price is defined by

$$(10)^{1/2} = 0.5 \times (20 - P_b)^{1/2} + 0.5 \times (10 - P_b)^{1/2}$$

The simplest way to solve this is to graph the function

$$0.5 \times (20 - X)^{1/2} + 0.5 \times (10 - X)^{1/2} - (10)^{1/2}$$

over $0 < X < 5$ and see where it hits zero. You can get more accurate answers by narrowing the range. Figure 3 shows my final iteration of this, and it yields $P_b = 4.375$.
(d) (10 points) If the $V$ function has constant absolute risk aversion $A$, the equation for the selling price becomes

$$- \exp[-A (W_0 + P_s)] = -p \exp[-A (W_0 + G)] - (1 - p) \exp[-A (W_0 + B)]$$

or

$$\exp[-A P_s] = p \exp[-A G] + (1 - p) \exp[-A B]$$

The equation for the buying price becomes

$$- \exp[-AW_0] = -p \exp[-A (W_0 - P_b + G)] - (1 - p) \exp[-A (W_0 - P_b + B)]$$

or

$$1 = p \exp[-A (-P_b + G)] + (1 - p) \exp[-A (-P_b + B)]$$

or

$$\exp[-A P_b] = p \exp[-A G] + (1 - p) \exp[-A B],$$

which is the same as the equation for the selling price. Therefore $P_b = P_s$.

(Additional information: Observe that for the square root function in part (c) the relative risk aversion is constant. Therefore the absolute risk aversion is decreasing. Therefore the buying price is less than the selling price.)

**Question 4:**

(a) (5 points) The expression for the random final wealth is

$$W = (1 - Y) W_0 R_0 + Y W_0 R = W_0 R_0 + Y W_0 (R - R_0)$$

and expected utility is

$$EU(Y) = \int_{R_L}^{R_H} u(W) f(R) dR$$

(b) (5 points) The first-order condition is

$$EU'(Y) = \int_{R_L}^{R_H} u'(W) W_0 (R - R_0) f(R) dR$$

$$= W_0 \int_{R_L}^{R_H} u'(W) (R - R_0) f(R) dR = 0$$

or

$$\int_{R_L}^{R_H} u'(W) (R - R_0) f(R) dR = 0$$

(c) (5 points) The second-order condition is

$$EU''(Y) = W_0 \int_{R_L}^{R_H} u''(W) W_0 (R - R_0)^2 f(R) dR$$

$$= (W_0)^2 \int_{R_L}^{R_H} u''(W) (R - R_0)^2 f(R) dR < 0$$
which is true since $u'' < 0$ everywhere.

(d) (15 points) Following the general methodology of comparative statics as discussed in the class handout No. 5, pp. 3–4. An increase in $W_0$ will lead to a lower optimal $Y$ if the whole function $EU'(Y)$ shifts downward. The condition for this is found by differentiating the integral with respect to $W_0$. We have

$$\frac{\partial}{\partial W_0} \int_{R_L}^{R_H} u'(W) (R - R_0) f(R) dR < 0$$

or

$$\int_{R_L}^{R_H} u''(W) [R_0 + Y (R - R_0)] (R - R_0) f(R) dR < 0$$

or

$$\int_{R_L}^{R_H} u''(W) \frac{W}{W_0} (R - R_0) f(R) dR < 0$$

This requires

$$\int_{R_L}^{R_H} W u''(W) (R - R_0) f(R) dR < 0$$

Now use the method that was developed in the class. Use

$$W u''(W) = -r(W) u'(W)$$

so the condition we need is

$$\int_{R_L}^{R_H} r(W) u'(W) (R - R_0) f(R) dR > 0$$

Note that when $R = R_0$, $W = W_0 R_0$, and $W$ is an increasing function of $R$. Suppose $r(W)$ is an increasing function of $W$. Then for $R > R_0$, which implies $W > W_0 R_0$, we have $r(W) > r(W_0 R_0)$, and then

$$r(W) u'(W) (R - R_0) f(R) > r(W_0 R_0) u'(W) (R - R_0) f(R)$$

For $R < R_0$, which implies $W < W_0 R_0$, we have $r(W) < r(W_0 R_0)$, and then

$$r(W) u'(W) f(R) < r(W_0 R_0) u'(W) f(R).$$

Multiply by $R - R_0$, which is negative in this range. So the multiplication reverses the direction of the inequality, giving

$$r(W) u'(W) (R - R_0) f(R) > r(W_0 R_0) u'(W) (R - R_0) f(R)$$

again. So we can integrate over the whole range $[R_L, R_H]$:

$$\int_{R_L}^{R_H} r(W) u'(W) (R - R_0) f(R) > \int_{R_L}^{R_H} r(W_0 R_0) u'(W) (R - R_0) f(R)$$

$$= r(W_0 R_0) \int_{R_L}^{R_H} u'(W) (R - R_0) f(R)$$
and the last integral is zero by the first-order condition at the initial point.

Thus we have shown that if \( r(W) \) is an increasing function, then a small increase in \( W_0 \) makes \( EU'(Y) \) negative at the old optimum \( Y \). But by the second-order condition, \( EU''(Y) \) is a decreasing function at the optimum. Therefore to restore the first-order condition, \( Y \) must decrease. That is, when \( W_0 \) increases slightly, the new optimal \( Y \) is less than the old.

I recommend that you re-read this in parallel with the corresponding treatment, on pp. 5-6 of Lecture Note No. 5, of absolute risk aversion and total amounts invested in the risky asset, to reinforce your understanding, and note the mathematical similarities and the differences in assumptions about the coefficients of absolute and relative risk aversion and their implications.