Validity of the Mean-Variance Approach

Constant absolute risk aversion (CARA):

\[ u(W) = - \exp(-\alpha W) \]

Final wealth \( W \) will be a random variable, whose distribution is affected by the allocation choices. Assume normal distribution: mean \( E[W] \), Variance \( V[W] \). These are functions of the allocation choices.

\[ EU = - E[\exp(-\alpha W)] = - \exp\{-\alpha E[W] + \frac{1}{2} \alpha^2 V[W]\} \]

So maximizing \( EU \) is equivalent to

minimizing \( - \alpha E[W] + \frac{1}{2} \alpha^2 V[W] \)

or

maximizing \( E[W] - \frac{1}{2} \alpha V[W] \)
One Riskless, One Risky Asset

Safe asset: gross return rate $R$ (1 plus interest rate)
Risky asset: random gross return rate $r$

Mean $\mu = \mathbb{E}[r] > R$, Variance $\sigma^2 = \mathbb{V}[r]$

Initial wealth $W_0$. If $x$ in risky asset,
final wealth $W = (W_0 - x) R + x r = R W_0 + (r - R) x$

$$
\mathbb{E}[W] = W_0 R + x (\mu - R) \\
\mathbb{V}[W] = x^2 \sigma^2; \quad \text{Std. Dev.} = x \sigma
$$

Choose $x$ to maximize $W_0 R + x (\mu - R) - \frac{1}{2} \alpha x^2 \sigma^2 $

FOC $\mu - R - \alpha x \sigma^2 = 0$, therefore optimum

$$
x = \frac{\mu - R}{\alpha \sigma^2}
$$

Observe $x$ independent of $W_0$. CARA-Normal model under uncertainty
is like quasi-linear utility in ordinary demand theory.
As $x$ varies, straight line in (Mean, Std.Dev.) figure.

\[ P_s = (0, W_0 R) \text{ safe}; \quad P_r = (W_0 \sigma, W_0 \mu) \text{ risky}; \]

Beyond $P_r$ possible if leveraged borrowing OK
(In dotted line as shown if borrowing rate = safe rate $R$;
with kink if borrowing rate $>\text{ safe rate}$.)

$P^*$ is optimal portfolio
Two Risky Assets

$W_0 = 1$; Random gross return rates $r_1, r_2$

Means $\mu_1 > \mu_2$; Std. Devs. $\sigma_1, \sigma_2$, Corr. Coefft. $\rho$

Portfolio $(x, 1 - x)$. Final $W = x r_1 + (1 - x) r_2$

\[
E[W] = x \mu_1 + (1 - x) \mu_2 = \mu_2 + x (\mu_1 - \mu_2)
\]

\[
V[W] = x^2 \sigma_1^2 + (1 - x)^2 \sigma_2^2 + 2 x (1 - x) \rho \sigma_1 \sigma_2
\]

\[
= (\sigma_2^2 - 2 x [(\sigma_2^2 - \rho \sigma_1 \sigma_2] + x^2 [(\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + (\sigma_2^2)]
\]

\[
\frac{\partial V[W]}{\partial x} = \begin{cases} 
- 2 [(\sigma_2^2 - \rho \sigma_1 \sigma_2] & \text{at } x = 0 \\
2 [(\sigma_1^2 - \rho \sigma_1 \sigma_2] & \text{at } x = 1
\end{cases}
\]

So diversification can reduce variance if $\rho < \min [\sigma_1/\sigma_2, \sigma_2/\sigma_1]$

To minimize variance, $x = \frac{\sigma_2^2 - \rho \sigma_1 \sigma_2}{(\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + (\sigma_2^2]}$
Optimum: \[ x = \frac{\mu_1 - \mu_2}{\alpha} + (\sigma_2)^2 - \rho \sigma_1 \sigma_2 \]
\[ \frac{(\sigma_1)^2}{2 \rho \sigma_1 \sigma_2 + (\sigma_2)^2} \]

\[ P_1, P_2 \] points for each asset; \( P_m \) minimum-variance portfolio, \( P^* \) optimum

Portion \( P_2 \) \( P_m \) dominated; \( P_m, P_1 \) efficient frontier

Continuation past \( P_1 \) if short sales of 2 OK
One Riskless, Two Risky Assets

First combine two riskies; this gets all points like $P_h$ on all lines like $P_s P_r$

Then mix with riskless; this gets

Efficient frontier $P_s P_F$ tangential to risky combination curve

Then along curve segment $P_F P_1$ if no leveraged borrowing; continue straight line $P_s P_F$ if leveraged borrowing OK
With preferences as shown, optimum $P^*$ mixes safe asset with particular risky combination $P_F$.

“Mutual fund” $P_F$ is the same for all investors regardless of risk-aversion (so long as optimum in $P_s P_F$).

Investors who are even less risk-averse may go beyond $P_F$ including corner solution at $P_1$ or tangency past $P_1$ if can sell 2 short to buy more 1.
Capital Asset Pricing Model

Individual investors take the rates of return as given
but these must be determined in equilibrium

Suppose one safe and two risky assets

Investor $h$ with initial wealth $W_h$

Invests $x^h_1$ dollars in the shares of firm 1,
$x^h_2$ dollars in the shares of firm 2,
and $(W_h - x^h_1 - x^h_2)$ in the safe asset.

Expression for random final wealth $W =$

$$(W_h - x^h_1 - x^h_2) R + x^h_1 r_1 + x^h_2 r_2 = W_h R + x^h_1 (r_1 - R) + x^h_2 (r_2 - R),$$

Maximizes

$$E[W] - \frac{1}{2} \alpha_h V[W]$$

where

$$E[W] = W_h R + x^h_1 (E[r_1] - R) + x^h_2 (E[r_2] - R)$$

$$V[W] = (x^h_1)^2 V[r_1] + 2 x^h_1 x^h_2 \text{Cov}[r_1, r_2] + (x^h_2)^2 V[r_2]$$
FOCs for optimal portfolio choice (allowing short sales etc. if necessary)

\begin{align*}
E[r_1] - R & = \alpha_h \{ x_1^h V[r_1] + x_2^h \text{Cov}[r_1, r_2] \} \\
E[r_2] - R & = \alpha_h \{ x_1^h \text{Cov}[r_1, r_2] + x_2^h V[r_2] \}
\end{align*}

like “inverse demand functions”.

Rewrite these equations as

\begin{align*}
\tau_h \{ E[r_1] - R \} & = x_1^h V[r_1] + x_2^h \text{Cov}[r_1, r_2] \\
\tau_h \{ E[r_2] - R \} & = x_1^h \text{Cov}[r_1, r_2] + x_2^h V[r_2]
\end{align*}

where \( \tau_h = \frac{1}{\alpha_h} \) is the investor’s risk-tolerance.

Sum these across all investors. Impose equilibrium condition:

Total dollars invested = total values of the firms \( F_1, F_2 \).

Take \( F_1, F_2 \) as given here; related to firms’ profits in Note 6.

\begin{align*}
T\{ E[r_1] - R \} & = F_1 V[r_1] + F_2 \text{Cov}[r_1, r_2] \quad (1) \\
T\{ E[r_2] - R \} & = F_1 \text{Cov}[r_1, r_2] + F_2 V[r_2] \quad (2)
\end{align*}

where \( T = \text{sum of } \tau_h\text{s is the market's risk tolerance.} \)
The market rate of return $r_m$ is weighted average

$$r_m = \frac{r_1 F_1 + r_2 F_2}{F_1 + F_2}$$

Then multiply (1) by $F_1$, (2) by $F_2$ and add:

$$T (F_1 + F_2) \{ E[r_m] - R \}$$

$$= (F_1)^2 V[r_1] + 2 F_1 F_2 \text{Cov}[r_1, r_2] + (F_2)^2 V[r_2]$$

$$= V[r_1 F_1 + r_2 F_2] = (F_1 + F_2)^2 V[r_m]$$

or

$$E[r_m] - R = \frac{F_1 + F_2}{T} V[r_m]$$

Risk premium on the market as a whole is

$\sim$ variance of the market rate of return, and

$\sim 1 / \text{market's risk tolerance}$

Factor $(F_1 + F_2)/T$ is the market price of risk

It is endogenous in the whole equilibrium.
Similar work with FOC for asset 1 yields:

\[
E[r_1] - R = \frac{F_1 + F_2}{T} \text{Cov}[r_1, r_m] \\
= \frac{\text{Cov}[r_1, r_m]}{\text{V}[r_m]} \{ E[r_m] - R \}
\]

This gives two important conclusions

\[
E[r_1] - R = \frac{\text{Cov}[r_1, r_m]}{\text{V}[r_m]} \{ E[r_m] - R \}
\]

Risk premium on firm-1 stock depends on its

*systematic* risk (correlation with whole market) only,

not *idiosyncratic* risk (part uncorrelated with market)

Coefficient is *beta* of firm-1 stock
The risk premium in the market on any one stock depends on the covariance of returns between the stock and the market not on variance of the stock itself.

The “idiosyncratic” risk in one stock (the part that is not correlated with the market) can be diversified away, so investor not paid for bearing it.

The risk in the whole market must be borne by the collectivity of investors, so this earns a risk premium proportional to their collective risk aversion $1/T$. 